

The Mathematical Association of Victoria
SPECIALIST MATHEMATICS 2012
Trial Written Examination 1 - SOLUTIONS

Question 1

Solution 1: Take the **upwards** direction as positive.

Data: $u = 20 \text{ m/s}$

$$a = -g \text{ m/s}^2$$

$$s = -60 \text{ m}$$

$$t = ?$$

Correct data [M1]

Substitute the above data into $s = ut + \frac{1}{2}at^2$:

$$-60 = 20t - \frac{1}{2}gt^2 \quad \text{[M1]}$$

$$\Rightarrow -120 = 40t - gt^2$$

$$\Rightarrow gt^2 - 40t - 120 = 0$$

where $b = -40$ and $c = -120$.

Total 2 marks

Solution 2: Take the **downwards** direction as positive.

Data: $u = -20 \text{ m/s}$

$$a = g \text{ m/s}^2$$

$$s = 60 \text{ m}$$

$$t = ?$$

Correct data [M1]

Substitute the above data into $s = ut + \frac{1}{2}at^2$:

$$60 = -20t + \frac{1}{2}gt^2 \quad \text{[M1]}$$

$$\Rightarrow 120 = -40t + gt^2$$

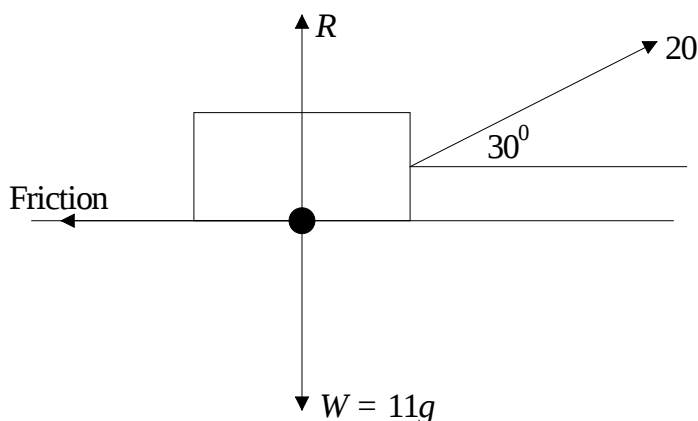
$$\Rightarrow gt^2 - 40t - 120 = 0$$

where $b = -40$ and $c = -120$.

Total 2 marks

Question 2**a.**Normal reaction force R .Weight force $W = 11g$.

Friction force.

Pulling force of 20 Newton acting in a direction 30° to the horizontal.

All forces labeled [A1]

Do NOT deduct 1 mark if the friction force is labeled as μR . This (incorrect) assumption gets penalised in **part b**.**b.**Despite knowing that $\mu = 0.2$ it is NOT known whether or not the object is on the point of sliding. Therefore the friction force CANNOT be assumed to be equal to μR .

Horizontal component of pulling force: $20 \cos(30^\circ) = 20 \left(\frac{\sqrt{3}}{2} \right) = 10\sqrt{3}$. [M1]

Net force in **horizontal direction**:
$$\left. \begin{aligned} F_{net} &= ma = 0 \\ F_{net} &= \text{Friction} - 10\sqrt{3} \end{aligned} \right\} \Rightarrow 0 = \text{Friction} - 10\sqrt{3}$$
$$\Rightarrow \text{Friction} = 10\sqrt{3}.$$
 [A1]

Total 3 marks**NOTE:**This answer is consistent with $0 \leq \text{Friction} = \mu R$ and shows that the object is not on the point of sliding:

Vertical component of pulling force: $20 \sin(30^\circ) = 20 \left(\frac{1}{2} \right) = 10$.

Net force in **vertical direction**:
$$\left. \begin{aligned} F_{net} &= ma = 0 \\ F_{net} &= R + 10 - 11g \end{aligned} \right\} \Rightarrow 0 = R + 10 - 11g$$
$$\Rightarrow R = 11g - 10$$
$$\Rightarrow \mu R = 0.2(11g - 10) = 2.2g - 2 > 10\sqrt{3}.$$

Question 3**a.**

$$\text{Let } y = \tan^{-1}\left(\frac{2}{\sqrt{x+1}}\right).$$

$$\text{Chain rule: Let } u = \frac{2}{\sqrt{x+1}} \Rightarrow y = \tan^{-1}(u).$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{1+u^2} \times \left(-\frac{1}{(x+1)^{3/2}} \right) \quad [\text{M1}]$$

$$= \frac{1}{1+\frac{4}{x+1}} \times \left(-\frac{1}{(x+1)^{3/2}} \right)$$

$$= \frac{x+1}{x+5} \times \left(-\frac{1}{(x+1)^{3/2}} \right)$$

$$= \frac{1}{x+5} \times \left(-\frac{1}{(x+1)^{1/2}} \right)$$

$$= \frac{-1}{(x+5)(x+1)^{1/2}} \quad [\text{A1}]$$

$$\text{where } a = -1, b = 5 \text{ and } c = \frac{1}{2}.$$

b.

$$\tan^{-1}\left(\frac{2}{\sqrt{x+1}}\right) = \frac{\pi}{3}$$

$$\Rightarrow \frac{2}{\sqrt{x+1}} = \sqrt{3}$$

$$\Rightarrow \frac{4}{x+1} = 3$$

$$\Rightarrow x = \frac{1}{3}. \quad [\text{A1}]$$

Total 3 marks

Question 4**a.**

$$w^3 + 4 - i4\sqrt{3} = 0$$

$$\Rightarrow w^3 = -4 + i4\sqrt{3}.$$

Therefore the required numbers are the cube roots of $-4 + i4\sqrt{3}$.

$$\text{Polar form: } -4 + i4\sqrt{3} = 8\text{cis}\left(\frac{2\pi}{3}\right) \quad [\text{M1}]$$

$$= 8\text{cis}\left(\frac{2\pi}{3} + 2n\pi\right) \text{ where } n \in \mathbb{Z}.$$

$$\text{Let } w = r\text{cis}(\theta) \Rightarrow w^3 = r^3\text{cis}(3\theta).$$

$$\text{Therefore } r^3\text{cis}(3\theta) = 8\text{cis}\left(\frac{2\pi}{3} + 2n\pi\right).$$

Equate modulus and argument:

$$r^3 = 8 \Rightarrow r = 2.$$

$$3\theta = \frac{2\pi}{3} + 2n\pi \Rightarrow \theta = \frac{2\pi}{9} + \frac{2n\pi}{3}.$$

$$n = 0: \quad w = 2\text{cis}\left(\frac{2\pi}{9}\right). \quad [\text{A1}]$$

$$n = 1: \quad w = 2\text{cis}\left(\frac{8\pi}{9}\right).$$

$$n = -1: \quad w = 2\text{cis}\left(\frac{-4\pi}{9}\right). \quad \text{Both of the remaining values of } w \quad [\text{A1}]$$

b.

$$u^3 = -4 - i4\sqrt{3}$$

$$\Rightarrow \overline{u^3} = -4 + i4\sqrt{3}$$

$$\Rightarrow \overline{u}^3 = -4 + i4\sqrt{3}.$$

Therefore $\overline{u} = w \Rightarrow u = \overline{w}$:

$$u = 2\text{cis}\left(\frac{-2\pi}{9}\right), \quad 2\text{cis}\left(\frac{-8\pi}{9}\right), \quad w = 2\text{cis}\left(\frac{4\pi}{9}\right).$$

Complex conjugate of answers to **part a**.

[A1]

Total 4 marks

Question 5

$$\frac{d\mathbf{r}}{dt} = 2\sin\left(\frac{t}{2}\right)\mathbf{i} + \cos(t)\mathbf{j} + 2t\mathbf{k}$$

$$\Rightarrow \mathbf{r} = \int 2\sin\left(\frac{t}{2}\right)\mathbf{i} + \cos(t)\mathbf{j} + 2t\mathbf{k} \, dt$$

$$= -4\cos\left(\frac{t}{2}\right)\mathbf{i} + \sin(t)\mathbf{j} + t^2\mathbf{k} + \mathbf{C}.$$

[M1]

Substitute $\mathbf{r} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$ at $t = \pi$:

$$-\mathbf{i} - \mathbf{j} + \mathbf{k} = -4\cos\left(\frac{\pi}{2}\right)\mathbf{i} + \sin(\pi)\mathbf{j} + \pi^2\mathbf{k} + \mathbf{C}$$

$$\Rightarrow -\mathbf{i} - \mathbf{j} + \mathbf{k} = \pi^2\mathbf{k} + \mathbf{C}$$

$$\Rightarrow \mathbf{C} = -\mathbf{i} - \mathbf{j} + (1 - \pi^2)\mathbf{k}.$$

Therefore:

$$\mathbf{r} = -4\cos\left(\frac{t}{2}\right)\mathbf{i} + \sin(t)\mathbf{j} + t^2\mathbf{k} + (-\mathbf{i} - \mathbf{j} + (1 - \pi^2)\mathbf{k}) = \left(-4\cos\left(\frac{t}{2}\right) - 1\right)\mathbf{i} + (\sin(t) - 1)\mathbf{j} + (t^2 + 1 - \pi^2)\mathbf{k}.$$

[M1]

Substitute $t = \frac{3\pi}{2}$:

$$\mathbf{r} = \left(-4\cos\left(\frac{3\pi}{4}\right) - 1\right)\mathbf{i} + \left(\sin\left(\frac{3\pi}{4}\right) - 1\right)\mathbf{j} + \left(\frac{9\pi^2}{4} + 1 - \pi^2\right)\mathbf{k}$$

$$= \left(\frac{4}{\sqrt{2}} - 1\right)\mathbf{i} + \left(\frac{1}{\sqrt{2}} - 1\right)\mathbf{j} + \left(\frac{5\pi^2}{4} + 1\right)\mathbf{k} = (2\sqrt{2} - 1)\mathbf{i} + \left(\frac{\sqrt{2} - 2}{2}\right)\mathbf{j} + \left(\frac{5\pi^2 + 4}{4}\right)\mathbf{k}.$$

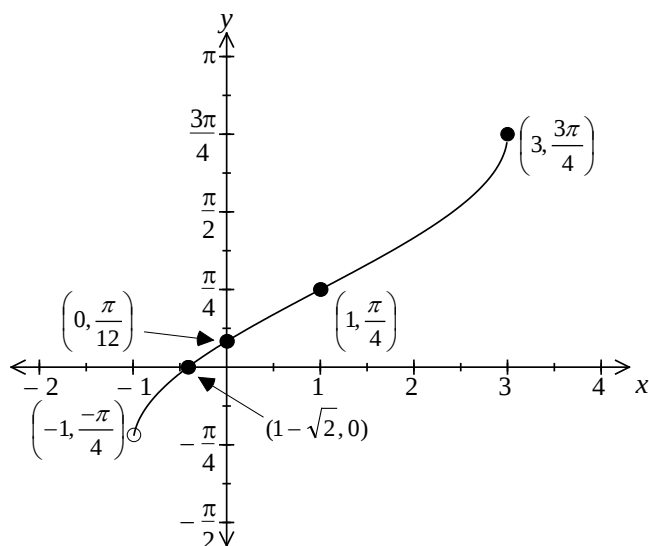
[A1]

Total 3 marks

Question 6**a.**

$$f(x) = \frac{3\pi}{4} - \cos^{-1}\left(\frac{x-1}{2}\right) = -\cos^{-1}\left(\frac{1}{2}(x-1)\right) + \frac{3\pi}{4}.$$

The graph of $y = \cos^{-1}(x)$ is dilated from the y -axis by a factor of 2, translated along the x -axis by 1 unit, reflected in the x -axis and translated along the y -axis by $\frac{3\pi}{4}$ units:



x -intercept: $0 = \frac{3\pi}{4} - \cos^{-1}\left(\frac{x-1}{2}\right)$

$$\Rightarrow \frac{3\pi}{4} = \cos^{-1}\left(\frac{x-1}{2}\right)$$

$$\Rightarrow \frac{-1}{\sqrt{2}} = \frac{x-1}{2}$$

$$\Rightarrow x = 1 - \sqrt{2}.$$

[A1]

y -intercept: $y = \frac{3\pi}{4} - \cos^{-1}\left(\frac{-1}{2}\right) = \frac{3\pi}{4} - \frac{2\pi}{3} = \frac{\pi}{12}.$

[A $\frac{1}{2}$]Endpoints **[A $\frac{1}{2}$]**Shape **[A $\frac{1}{2}$]**Inflection point **[A $\frac{1}{2}$]****Marks totaled and rounded down.**

b. i.

$$V = \pi \int_a^b x^2 dy.$$

$$x = 1: \quad y = \frac{3\pi}{4} - \cos^{-1}(0) = \frac{3\pi}{4} - \frac{\pi}{2} = \frac{\pi}{4}.$$

$$x = 1 + \sqrt{2}: \quad y = \frac{3\pi}{4} - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4} - \frac{\pi}{4} = \frac{\pi}{2}.$$

Both values of y [M1]

$$y = \frac{3\pi}{4} - \cos^{-1}\left(\frac{x-1}{2}\right) \Rightarrow \frac{3\pi}{4} - y = \cos^{-1}\left(\frac{x-1}{2}\right) \Rightarrow \cos\left(\frac{3\pi}{4} - y\right) = \frac{x-1}{2}$$

$$\Rightarrow x = 2 \cos\left(\frac{3\pi}{4} - y\right) + 1.$$

$$\text{Therefore: } V = \pi \int_{\pi/4}^{\pi/2} \left(2 \cos\left(\frac{3\pi}{4} - y\right) + 1\right)^2 dy.$$

[A1]

b. ii.**Solution 1:**

$$\text{Substitute } u = \frac{3\pi}{4} - y:$$

$$V = -\pi \int_{\pi/2}^{\pi/4} (2 \cos(u) + 1)^2 du = \pi \int_{\pi/4}^{\pi/2} (2 \cos(u) + 1)^2 du.$$

[M1]

Expand:

$$V = \pi \int_{\pi/4}^{\pi/2} 4 \cos^2(u) + 4 \cos(u) + 1 du$$

$$= \pi \int_{\pi/4}^{\pi/2} 4 \cos^2(u) du + \pi \int_{\pi/4}^{\pi/2} 4 \cos(u) + 1 du$$

$$= \pi \int_{\pi/4}^{\pi/2} 2 \cos(2u) + 2 du + \pi \int_{\pi/4}^{\pi/2} 4 \cos(u) + 1 du$$

[M1]

$$= \pi \int_{\pi/4}^{\pi/2} 2\cos(2u) + 3 + 4\cos(u) \, du$$

$$= \pi \left[\sin(2u) + 3u + 4\sin(u) \right]_{\pi/4}^{\pi/2} \quad [\text{M1}]$$

$$= \pi \left[\left(0 + \frac{3\pi}{2} + 4 \right) - \left(1 + \frac{3\pi}{4} + 2\sqrt{2} \right) \right]$$

$$= \pi \left[\frac{3\pi}{4} + 3 - 2\sqrt{2} \right] = \pi \left[\frac{12 + 3\pi - 8\sqrt{2}}{4} \right] \text{ cubic units.} \quad [\text{A1}]$$

Total 9 marks**Solution 2:**

Substitute $\cos\left(\frac{3\pi}{4} - y\right) = \cos\left(-\left(y - \frac{3\pi}{4}\right)\right) = \cos\left(y - \frac{3\pi}{4}\right)$:

$$V = \pi \int_{\pi/4}^{\pi/2} \left(2\cos\left(y - \frac{3\pi}{4}\right) + 1 \right)^2 dy.$$

Substitute $u = y - \frac{3\pi}{4}$:

$$V = \pi \int_{-\pi/2}^{-\pi/4} (2\cos(u) + 1)^2 du. \quad [\text{M1}]$$

Expand:

$$V = \pi \int_{-\pi/2}^{-\pi/4} 4\cos^2(u) + 4\cos(u) + 1 \, du$$

$$= \pi \int_{-\pi/2}^{-\pi/4} 4\cos^2(u) \, du + \pi \int_{-\pi/2}^{-\pi/4} 4\cos(u) + 1 \, du$$

$$= \pi \int_{-\pi/2}^{-\pi/4} 2\cos(2u) + 2 \, du + \pi \int_{-\pi/2}^{-\pi/4} 4\cos(u) + 1 \, du \quad [\text{M1}]$$

where the first integral follows from using $4\cos^2(u) = 2\cos(2u) + 2$

$$= \pi \int_{-\pi/2}^{-\pi/4} 2 \cos(2u) + 3 + 4 \cos(u) \, du$$

$$= \pi \left[\sin(2u) + 3u + 4 \sin(u) \right]_{-\pi/2}^{-\pi/4} \quad [\text{M1}]$$

$$= \pi \left[\left(-1 - \frac{3\pi}{4} - 2\sqrt{2} \right) - \left(0 - \frac{3\pi}{2} - 4 \right) \right]$$

$$= \pi \left[\frac{3\pi}{4} + 3 - 2\sqrt{2} \right] = \pi \left[\frac{12 + 3\pi - 8\sqrt{2}}{4} \right] \text{ cubic units.} \quad [\text{A1}]$$

Total 9 marks**Question 7**

Vertical asymptote at $x = -1$: $(-1)^2 + 2a(-1) + b = 0 \Rightarrow 2a - b = 1. \quad \dots (1) \quad [\text{M1}]$

Range of $(-\infty, 0) \cup \left[\frac{1}{4}, +\infty \right) \Rightarrow y$ -coordinate of the turning point is $y = \frac{1}{4}$.

The x -coordinate of the turning point of the reciprocal quadratic function graph is the same as the x -coordinate of the turning point of $g(x) = x^2 + 2ax + b$:

$$x = -a.$$

Therefore the y -coordinate of the turning point is $y = \frac{a}{(-a)^2 + 2a(-a) + b} = \frac{a}{b - a^2}$: [M1]

$$\frac{1}{4} = \frac{a}{b - a^2} \Rightarrow b - a^2 = 4a. \quad \dots (2) \quad [\text{M1}]$$

Solve equations (1) and (2) simultaneously for a and b . Substitute equation (1) into equation (2):

$$2a - 1 - a^2 = 4a$$

$$\Rightarrow a^2 + 2a + 1 = 0$$

$$\Rightarrow (a + 1)^2 = 0$$

$$\Rightarrow a = -1.$$

Substitute $a = -1$ into equation (1): $b = -3$.

Therefore $a = -1$ and $b = -3$. [A1]

Total 4 marks

Question 8**Solution 1:**

$$-\hat{i} + 2\hat{j} + \hat{k} = \alpha(2m\hat{i} - \hat{j} + 3\hat{k}) + \beta(5m\hat{i} - 11\hat{j} + 5\hat{k}) \quad \text{where } \alpha, \beta \in \mathbb{R}$$

$$= (2m\alpha + 5m\beta)\hat{i} + (-\alpha - 11\beta)\hat{j} + (3\alpha + 5\beta)\hat{k}. \quad [\text{M1}]$$

Equate components:

$$\hat{i}\text{-component:} \quad -1 = 2m\alpha + 5m\beta \quad \dots (1)$$

$$\hat{j}\text{-component:} \quad 2 = -\alpha - 11\beta \quad \dots (2)$$

$$\hat{k}\text{-component:} \quad 1 = 3\alpha + 5\beta \quad \dots (3)$$

Solve equation (2) and equation (3) simultaneously for α and β :

$$\alpha = \frac{3}{4} \quad \text{and} \quad \beta = \frac{1}{4}. \quad [\text{M1}]$$

Substitute into equation (1) and solve for m :

$$m = -4. \quad [\text{A1}]$$

Total 3 marks

Solution 2:

$$2m\hat{i} - \hat{j} + 3\hat{k} = \mu(-\hat{i} + 2\hat{j} + \hat{k}) + \lambda(5m\hat{i} - 11\hat{j} + 5\hat{k}) \quad \text{where } \mu, \lambda \in \mathbb{R}$$

$$= (-\mu + 5m\lambda)\hat{i} + (2\mu - 11\lambda)\hat{j} + (\mu + 5\lambda)\hat{k}. \quad [\text{M1}]$$

$$\mu = \frac{4}{3} \quad \text{and} \quad \lambda = \frac{1}{3}. \quad [\text{M1}]$$

$$m = -4. \quad [\text{A1}]$$

Total 3 marks

Solution 3:

$$5m\hat{i} - 11\hat{j} + 5\hat{k} = \gamma(-\hat{i} + 2\hat{j} + \hat{k}) + \delta(2m\hat{i} - \hat{j} + 3\hat{k}) \quad \text{where } \gamma, \delta \in \mathbb{R}$$

$$= (-\gamma + 2m\delta)\hat{i} + (2\gamma - \delta)\hat{j} + (\gamma + 3\delta)\hat{k}. \quad [\text{M1}]$$

$$\gamma = -4 \quad \text{and} \quad \delta = 3. \quad [\text{M1}]$$

$$m = -4. \quad [\text{A1}]$$

Total 3 marks

Question 9**a.****Solution 1:**

$$a = v \frac{dv}{dx} = \frac{1}{v+3} \quad [\text{M1}]$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{v(v+3)}$$

$$\Rightarrow \frac{dx}{dv} = v(v+3) = v^2 + 3v$$

$$\Rightarrow x = \int v^2 + 3v \, dv$$

$$= \frac{1}{3}v^3 + \frac{3}{2}v^2 + C. \quad [\text{M1}]$$

Substitute $v = 0$ and $x = 0$ when $t = 0$: $C = 0$.

$$\text{Therefore } x = \frac{1}{3}v^3 + \frac{3}{2}v^2.$$

Substitute $v = 1$:

$$x = \frac{1}{3} + \frac{3}{2} = \frac{11}{6}. \quad [\text{A1}]$$

Solution 2:From **part b.** $t = \frac{1}{2}v^2 + 3v$. Substitute $v = 1$:

$$t = \frac{7}{2}. \quad [\text{M1}]$$

$$\text{From part b.: } v = -3 + \sqrt{9+2t}$$

$$\Rightarrow \frac{dx}{dt} = -3 + \sqrt{9+2t}$$

$$\Rightarrow x = \int -3 + \sqrt{9+2t} \, dt$$

$$\Rightarrow x = -3t + \frac{1}{3}(9+2t)^{3/2} + C. \quad [\text{M1}]$$

Substitute $x = 0$ when $t = 0$: $0 = 9 + C \Rightarrow C = -9$.

$$\text{Therefore } x = -3t + \frac{1}{3}(9+2t)^{3/2} - 9.$$

Substitute $t = \frac{7}{2}$:

$$x = \frac{-21}{2} + \frac{1}{3}(16)^{3/2} - 9 = \frac{-21}{2} + \frac{64}{3} - 9$$

$$= \frac{11}{6}.$$

[A1]

b.

$$a = \frac{dv}{dt} = \frac{1}{v+3}$$

$$\frac{dt}{dv} = v+3$$

[M1]

$$\Rightarrow t = \int v+3 \, dv$$

$$= \frac{1}{2}v^2 + 3v + K.$$

Substitute $v=0$ when $t=0$: $K=0$.

$$\text{Therefore } t = \frac{1}{2}v^2 + 3v.$$

[M1]

Re-arrange into standard quadratic equation form:

$$v^2 + 6v - 2t = 0.$$

Solve for v :

$$v = \frac{-6 \pm \sqrt{6^2 - 4(1)(-2t)}}{2} = \frac{-6 \pm \sqrt{36 + 8t}}{2} = -3 \pm \sqrt{9 + 2t}.$$

But $v=0$ when $t=0$ and so the negative root solution is rejected.

Expression for v and reason for rejection of negative root solution [M1]

$$\text{Therefore } v = -3 + \sqrt{9 + 2t} \text{ where } b = 9.$$

Total 6 marks

Question 10**Solution 1:**

Let $y = f^{-1}(x)$. Then $x = ye^y$.

Substitute $x = e$: $e = ye^y$

$$\Rightarrow y = 1 \text{ (by inspection).}$$

$$x = ye^y \text{ and } y = 1 \quad [\text{M1}]$$

Implicit differentiation: $x = ye^y$

$$\Rightarrow 1 = \frac{dy}{dx}e^y + ye^y \frac{dy}{dx}. \quad [\text{M1}]$$

Substitute $y = 1$: $1 = \frac{dy}{dx}e + e \frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2e}.$$

Therefore $m_{\text{normal}} = -2e$.

[A1]

Total 3 marks

Solution 2:

Let $y = f^{-1}(x)$. Then $x = ye^y$.

Substitute $x = e$: $e = ye^y$

$$\Rightarrow y = 1 \text{ (by inspection).}$$

$$x = ye^y \text{ and } y = 1 \quad [\text{M1}]$$

Differentiate: $x = ye^y$

$$\Rightarrow \frac{dx}{dy} = e^y + ye^y$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{e^y + ye^y}. \quad [\text{M1}]$$

Substitute $y = 1$: $\frac{dy}{dx} = \frac{1}{2e}$.

Therefore $m_{\text{normal}} = -2e$.

[A1]

Total 3 marks