

Year 2012
VCE
Specialist Mathematics
Solutions
Trial Examination 2



KILBAHA MULTIMEDIA PUBLISHING
PO BOX 2227
KEW VIC 3101
AUSTRALIA

TEL: (03) 9817 5374
FAX: (03) 9817 4334
kilbaha@gmail.com
<http://kilbaha.com.au>

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SECTION 1

ANSWERS

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E
12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E
21	A	B	C	D	E
22	A	B	C	D	E

SECTION 1

Question 1

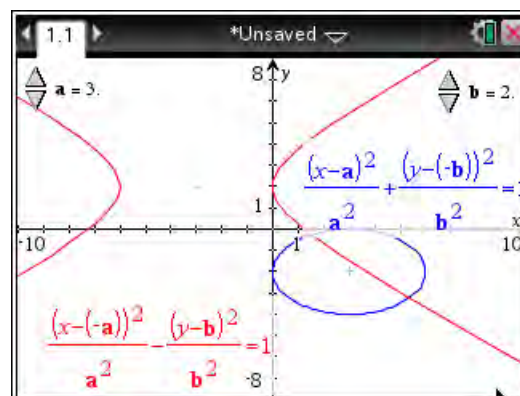
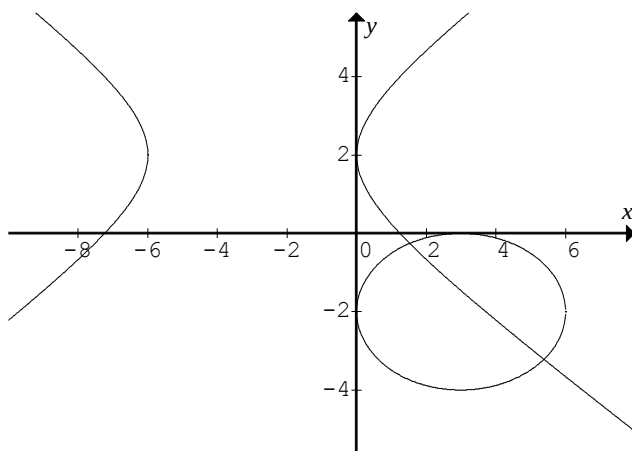
Answer C

Consider the case when $a = 3$ and $b = 2$,

the graphs of $\frac{(x-3)^2}{9} + \frac{(y+2)^2}{4} = 1$

and $\frac{(x+3)^2}{9} - \frac{(y-2)^2}{4} = 1$, are shown,

and intersect at two distinct points.



Question 2

Answer E

Since $|\underline{a}| = \sqrt{2}$ and $|\underline{b}| = \sqrt{2}$ the vectors have the same length

$$\underline{a} \cdot \underline{b} = 1 \quad \cos(\theta) = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|} = \frac{1}{2} \Rightarrow \theta = 60^\circ$$

$\underline{a} + \underline{c} = 3\underline{i} - 3\underline{k}$ is parallel to the vector $\underline{a} - \underline{b} = \underline{i} - \underline{k}$

$\underline{c} = 2\underline{a} - 3\underline{b}$, the vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ are linearly dependent.

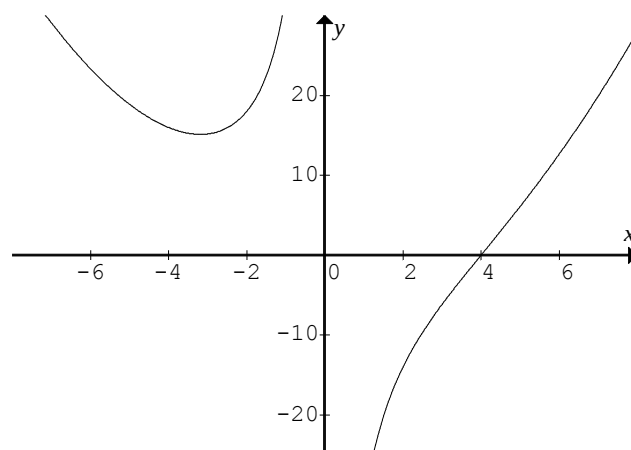
A. B. C. and D. are all true, E. is false.

Question 3

Answer D

The only possibility is when $n = 3$, the graph of

$$y = \frac{x^3 - 64}{2x} \text{ is shown.}$$



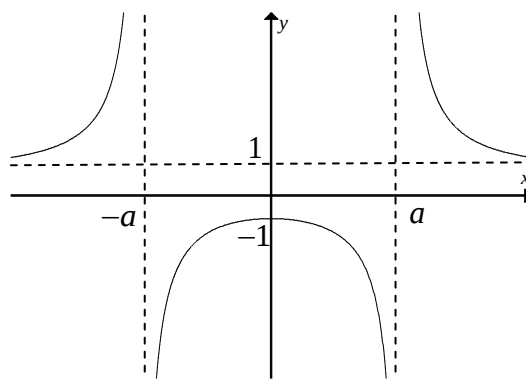
Question 4**Answer E**

$$y = \frac{x^2 + a^2}{x^2 - a^2} = 1 + \frac{2a^2}{x^2 - a^2}$$

the maximal domain is $R \setminus \{\pm a\}$,

the range is $(-\infty, -1] \cup (1, \infty)$.

The graph has vertical asymptotes at $x = \pm a$ and a horizontal asymptote at $y = 1$. The graph crosses the y -axis at $y = -1$ and this point $(0, -1)$ is a maximum turning point.

**Question 5****Answer A**

The asymptotes intersect at the centre of the hyperbola

$-2x = 2x - 4k \Rightarrow 4x = 4k \quad x = k$ and $y = -2k$, the centre is $(k, -2k)$ and the gradient

of the asymptotes is ± 2 , the equation of the hyperbola is $\frac{(x-k)^2}{k^2} - \frac{(y+2k)^2}{4k^2} = 1$

Question 6**Answer B**

The domain and range of $y = \cos^{-1}(x)$ are $|x| \leq 1$ and $[0, \pi]$ respectively, the domain of

$y = \frac{2a}{\pi} \cos^{-1}\left(\frac{x}{a} - 1\right)$ is $\left|\frac{x}{a} - 1\right| \leq 1$ or $-1 \leq \frac{x}{a} - 1 \leq 1 \Rightarrow 0 \leq \frac{x}{a} \leq 2$ or $[0, 2a]$

and the range is $[0, \pi] \times \frac{2a}{\pi} = [0, 2a]$

Question 7**Answer C**

$y = \cot(px) = \frac{\cos(px)}{\sin(px)}$. The graph crosses the x -axis, when $y = 0$ or $\cos(px) = 0$

$\Rightarrow px = (2n+1)\frac{\pi}{2}$ or $x = \frac{(2n+1)\pi}{2p}$. The graph has vertical asymptotes when

$\sin(px) = 0 \Rightarrow px = n\pi$ or $x = \frac{n\pi}{p}$.

Question 8**Answer B**

$P(z)$ is a fourth degree polynomial

$P(z) = (z - ki)(z + ki)(z - 2k)(z + k)$ expanding

$P(z) = (z^2 + k^2)(z^2 - kz - 2k^2)$

$P(z) = z^4 - kz^3 - k^2z^2 - k^3z - 2k^4$

Question 9**Answer A**

$$\{z: |z-a|^2 - |z-bi|^2 = a^2 + b^2\} \text{ let } z = x + yi$$

$$|(x-a+yi)|^2 - |x+(y-b)i|^2 = a^2 + b^2$$

$$(x-a)^2 + y^2 - (x^2 + (y-b)^2) = a^2 + b^2$$

$$x^2 - 2xa + a^2 + y^2 - (x^2 + y^2 - 2by + b^2) = a^2 + b^2$$

$$2by - 2xa = 2b^2$$

$$y = \frac{xa}{b} + b \text{ or } z = x + yi \quad \text{Im}(z) = \frac{a}{b} \text{Re}(z) + b$$

this represents a straight line in the argand plane.

Question 10**Answer D**

$$u = 4\text{cis}(\theta), \quad v = 2\text{cis}\left(\frac{-4\pi}{5}\right)$$

$$\frac{u}{v} = 2\text{cis}\left(\theta + \frac{4\pi}{5}\right) = -2i = 2\text{cis}\left(-\frac{\pi}{2}\right)$$

$$\theta + \frac{4\pi}{5} = -\frac{\pi}{2} \Rightarrow \theta = -\frac{4\pi}{5} - \frac{\pi}{2} = -\frac{13\pi}{10}$$

$$\text{Now } \theta = -\frac{13\pi}{10} + 2\pi = \frac{7\pi}{10}$$

Question 11**Answer D**

$$|b| = \sqrt{4+t^2+1} = \sqrt{5+t^2} = 3 \text{ if } t = \pm 2 \quad \mathbf{A.} \text{ is true.}$$

$$\text{the angle between } \underline{a} \text{ and } \underline{b} \text{ is } \cos(\theta) = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}| |\underline{b}|}$$

$$\cos(\theta) = \frac{-8-t+1}{\sqrt{18} \times \sqrt{5+t^2}} = -\frac{1}{\sqrt{2}} \text{ when } t = 2 \Rightarrow \theta = -135^\circ \text{ so } \mathbf{B.} \text{ is true.}$$

$$\underline{a} \cdot \underline{b} = -8-t+1 = 0 \text{ when } t = -7 \text{ so } \mathbf{C.} \text{ is true.}$$

D. is false, there is no value of t , for which the vector $\underline{a}$ is parallel to the vector $\underline{b}$.

The scalar resolute of $\underline{b}$ in the direction of $\underline{a}$ equals

$$\underline{b} \cdot \hat{\underline{a}} = \frac{\underline{b} \cdot \underline{a}}{|\underline{a}|} = \frac{-8-t+1}{\sqrt{18}} = \frac{1}{\sqrt{2}} \text{ when } t = -10, \text{ so } \mathbf{E.} \text{ is true.}$$

Question 12**Answer B**

$$\dot{\mathbf{r}}(t) = 4\cos(2t)\mathbf{i} + 2\sin(2t)\mathbf{j} \text{ for } t \geq 0$$

$$\mathbf{r}(t) = \int 4\cos(2t)dt \mathbf{i} + \int 2\sin(2t)dt \mathbf{j} = 2\sin(2t)\mathbf{i} - \cos(2t)\mathbf{j} + \mathbf{c}$$

$$\text{Now } \mathbf{r}(0) = \mathbf{0} \Rightarrow -\mathbf{j} + \mathbf{c} = \mathbf{0} \Rightarrow \mathbf{c} = \mathbf{j}$$

$$\mathbf{r}(t) = 2\sin(2t)\mathbf{i} + (1 - \cos(2t))\mathbf{j}$$

$$x = 2\sin(2t) \text{ and } y = 1 - \cos(2t)$$

$$\sin^2(2t) + \cos^2(2t) = 1$$

$$\frac{x^2}{4} + (y-1)^2 = 1$$

an ellipse with centre at (0,1).

Question 13**Answer C**

$$\text{let } u = 9 - 4x^2 \quad \frac{du}{dx} = -8x$$

$$\text{terminals when } x = \frac{3}{2} \quad u = 0 \text{ and when } x = 0 \quad u = 9$$

$$x^2 = \frac{1}{4}(9-u) \quad dx = -\frac{1}{8x} du$$

$$\int_0^{\frac{3}{2}} \frac{x^3}{\sqrt{9-4x^2}} dx = \int_0^{\frac{3}{2}} \frac{x x^2}{\sqrt{9-4x^2}} dx = \int_9^0 -\frac{1}{32} \left(\frac{9-u}{\sqrt{u}} \right) du \quad \text{swap terminals}$$

$$= \frac{1}{32} \int_0^9 \frac{9-u}{\sqrt{u}} du$$

Question 14**Answer B**

$$v(x) = e^{2x} - e^{-2x}$$

$$\text{alternatively } v^2(x) = (e^{2x} - e^{-2x})^2 = e^{4x} - 2 + e^{-4x}$$

$$\frac{dv}{dx} = 2(e^{2x} + e^{-2x})$$

$$\frac{1}{2}v^2 = \frac{1}{2}e^{4x} - 1 + \frac{1}{2}e^{-4x}$$

$$a = v \frac{dv}{dx} = 2(e^{2x} - e^{-2x})(e^{2x} + e^{-2x})$$

$$a(x) = \frac{d}{dx} \left(\frac{1}{2}v^2 \right) = 2(e^{4x} - e^{-4x})$$

$$a(x) = 2(e^{4x} - e^{-4x})$$

Question 15**Answer E**

$$\mathbf{r}(t) = 25t \cos(40^\circ) \mathbf{i} + \left(25t \sin(40^\circ) - \frac{1}{2}gt^2 \right) \mathbf{j}$$

$$\mathbf{r}(t) = Vt \cos(\alpha) \mathbf{i} + \left(Vt \sin(\alpha) - \frac{1}{2}gt^2 \right) \mathbf{j} \quad \text{so that } V = 25 \quad \alpha = 40^\circ$$

$$\text{time of flight } T = \frac{2V \sin(\alpha)}{g} = \frac{2 \times 25 \sin(40^\circ)}{9.8} = 3.28 \text{ seconds}$$

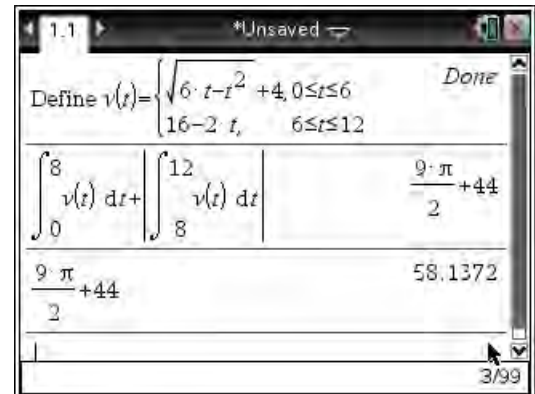
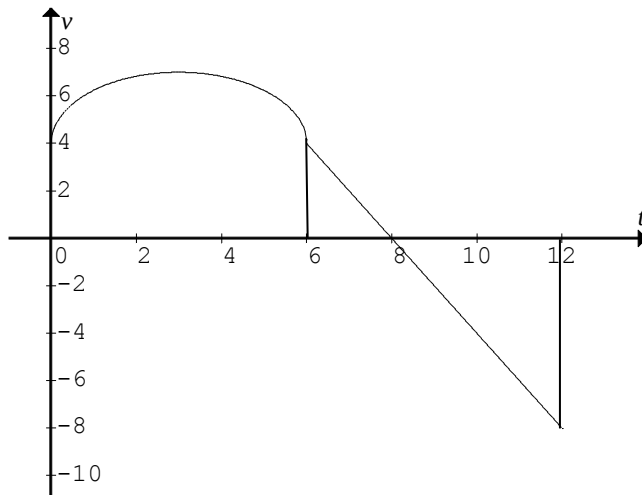
$$\text{maximum height } H = \frac{V^2 \sin^2(\alpha)}{2g} = \frac{25^2 \sin^2(40^\circ)}{2 \times 9.8} = 13.175 \text{ metres}$$

$$\text{the range } R = \frac{V^2 \sin(2\alpha)}{g} = \frac{25^2 \sin(80^\circ)}{9.8} = 62.807 \text{ metres}$$

The golf ball travels in a parabolic path, all of **i. ii. iii. iv** and **v.** are correct.

Question 16**Answer C**

$$v(t) = \begin{cases} \sqrt{6t - t^2} + 4 & \text{for } 0 \leq t \leq 6 \\ 16 - 2t & \text{for } 6 \leq t \leq 12 \end{cases}$$



The distance travelled is half the area of a circle of radius 3, plus the area of a rectangle, plus the area of a triangle above the t -axis, plus the area of a triangle below the t -axis.

$$\frac{9\pi}{2} + 6 \times 4 + \frac{1}{2} \times 2 \times 4 + \frac{1}{2} \times 4 \times 8 \approx 58 \text{ metres.}$$

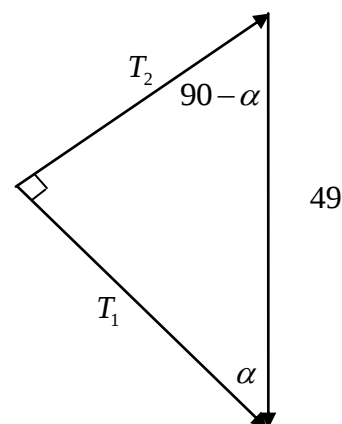
Question 17**Answer E**

The forces are in newtons, the weight force is

$$mg = 5 \times 9.8 = 49 \text{ newtons.}$$

By Lami's theorem, $\frac{T_1}{\sin(90 - \alpha)} = \frac{T_2}{\sin(\alpha)} = \frac{49}{\sin(90^\circ)}$

$$T_1 = 49 \cos(\alpha) \text{ and } T_2 = 49 \sin(\alpha)$$

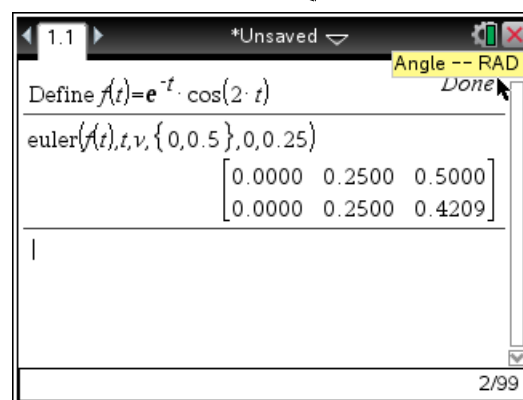
**Question 18****Answer D**

$$\frac{dv}{dt} = f(t) = e^{-t} \cos(2t)$$

$$v(0) = 0 \quad h = 0.25 \quad t_0 = 0 \quad t_1 = 0.25$$

$$v_1 = v_0 + hf(t_0) = 0 + 0.25 \times e^0 \cos(0) = 0.25$$

$$v_2 = v_1 + hf(t_1) = 0.25 + 0.25 \times e^{-0.25} \cos(0.5) = 0.42$$

**Question 19****Answer C**

the differential equation is $a = \frac{dv}{dt} = 2 \cos(2t)$

Question 20**Answer D**

$$F = 10 \text{ newtons} \quad m = 5 \text{ kg} \quad a = \frac{F}{m} \quad a = 2 \text{ ms}^{-2} \quad u = 1 \text{ ms}^{-1} \quad s = 20$$

$$v^2 = u^2 + 2as \Rightarrow v^2 = 1 + 2 \times 2 \times 20 = 81 \text{ so } v = 9 \text{ so } p = mv = 5 \times 9 = 45 \text{ kgms}^{-1}$$

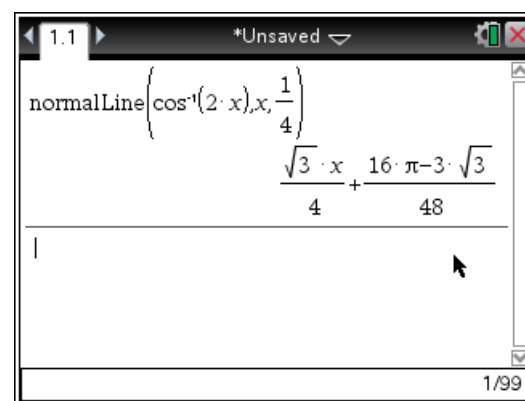
Question 21**Answer A**

$$y = \cos^{-1}(2x) \text{ when } x = \frac{1}{4} \quad y = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\frac{dy}{dx} = \frac{-2}{\sqrt{1-4x^2}} \text{ when } x = \frac{1}{4} \quad m_T = -\frac{4}{\sqrt{3}}$$

$$\Rightarrow m_N = \frac{\sqrt{3}}{4} \text{ equation of the normal is}$$

$$y - \frac{\pi}{3} = \frac{\sqrt{3}}{4} \left(x - \frac{1}{4}\right) \text{ or } y = \frac{\sqrt{3}x}{4} - \frac{\sqrt{3}}{16} + \frac{\pi}{3}$$

**Question 22****Answer A**

by Newton's law of cooling, the temperature of surroundings is 200°C , and the initial

temperature is 3°C , $\frac{dT}{dt} = -k(T - 200) \quad T(0) = 3$

END OF SECTION 1 SUGGESTED ANSWERS

SECTION 2

Question 1

a. $\overrightarrow{OA} = -\hat{i} + 3\hat{j} - 2\hat{k}$ $|\overrightarrow{OA}| = \sqrt{(-1)^2 + 3^2 + (-2)^2} = \sqrt{14}$
 $\overrightarrow{OB} = 2\hat{i} + \hat{j} - 3\hat{k}$ $|\overrightarrow{OB}| = \sqrt{2^2 + 1^2 + (-3)^2} = \sqrt{14}$ M1
 $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 3\hat{i} - 2\hat{j} - \hat{k}$ $|\overrightarrow{AB}| = \sqrt{3^2 + (-2)^2 + (-1)^2} = \sqrt{14}$
 since $|\overrightarrow{OA}| = |\overrightarrow{OB}| = |\overrightarrow{AB}| = \sqrt{14}$ therefore OAB is an equilateral triangle. A1

b. $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$
 $\overrightarrow{OM} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$
 $\overrightarrow{OM} = \overrightarrow{OA} + \frac{1}{2}(\overrightarrow{OB} - \overrightarrow{OA})$ M1
 $\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$
 $\overrightarrow{OM} = \frac{1}{2}(\hat{i} + 4\hat{j} - 5\hat{k})$
 $\overrightarrow{OG} = \frac{2}{3}\overrightarrow{OM} = \frac{1}{3}(\hat{i} + 4\hat{j} - 5\hat{k})$ A1

c. $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$
 $\overrightarrow{GP} = \overrightarrow{OP} - \overrightarrow{OG} = (x\hat{i} + y\hat{j} + z\hat{k}) - \frac{1}{3}(\hat{i} + 4\hat{j} - 5\hat{k})$
 $\overrightarrow{GP} = \frac{1}{3}((3x-1)\hat{i} + (3y-4)\hat{j} + (3z+5)\hat{k})$ M1
 Since $\overrightarrow{OG}$ is perpendicular to $\overrightarrow{GP}$
 $\overrightarrow{OG} \cdot \overrightarrow{GP} = 0 \Rightarrow \frac{1}{9}(1(3x-1) + 4(3y-4) - 5(3z+5)) = 0$ M1
 (1) $3x + 12y - 15z = 42$

- d. $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2} = 3\sqrt{5}$
 (2) $x^2 + y^2 + z^2 = 45$ A1
 $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (x+1)\underline{i} + (y-3)\underline{j} + (z+2)\underline{k}$
 $|\overrightarrow{AP}| = \sqrt{(x+1)^2 + (y-3)^2 + (z+2)^2} = 3\sqrt{5}$ expanding M1
 $x^2 + 2x + 1 + y^2 - 6y + 9 + z^2 + 4z + 4 = 45$ from (2)
 (3) $2x - 6y + 4z + 14 = 0$
 similarly
 $\overrightarrow{BP} = \overrightarrow{OP} - \overrightarrow{OB} = (x-2)\underline{i} + (y-1)\underline{j} + (z+3)\underline{k}$
 $|\overrightarrow{BP}| = \sqrt{(x-2)^2 + (y-1)^2 + (z+3)^2} = 3\sqrt{5}$ expanding M1
 $x^2 - 4x + 4 + y^2 - 2y + 1 + z^2 + 6z + 9 = 45$ from (2)
 (4) $-4x - 2y + 6z + 14 = 0$
- e. Solving (1) (2) (3) and (4)
 gives $x = 4$ $y = 5$ and $z = 2$
 or $x = -\frac{10}{3}$ $y = -\frac{7}{3}$ and $z = -\frac{16}{3}$ A1
 both set of answers are acceptable, see f.

$eq1: 3 \cdot x + 12 \cdot y - 15 \cdot z = 42$	$3 \cdot x + 12 \cdot y - 15 \cdot z = 42$
$eq2: x^2 + y^2 + z^2 = 45$	$x^2 + y^2 + z^2 = 45$
$eq3: 2 \cdot x - 6 \cdot y + 4 \cdot z + 14 = 0$	$2 \cdot x - 6 \cdot y + 4 \cdot z + 14 = 0$
$eq4: -4 \cdot x - 2 \cdot y + 6 \cdot z + 14 = 0$	$-4 \cdot x - 2 \cdot y + 6 \cdot z + 14 = 0$
$\text{solve} \left\{ \begin{array}{l} eq1 \\ eq2 \\ eq3 \\ eq4 \end{array}, \{x, y, z\} \right\}$	$x = -\frac{10}{3} \text{ and } y = -\frac{7}{3} \text{ and } z = -\frac{16}{3} \text{ or } x = 4 \text{ and } y = 5 \text{ and } z = 2$

- f. $\overrightarrow{GP} = \frac{11}{3}(\underline{i} + \underline{j} + \underline{k})$ if $x = 4$ $y = 5$ and $z = 2$ M1
 $\overrightarrow{GP} = -\frac{11}{3}(\underline{i} + \underline{j} + \underline{k})$ if $x = -\frac{10}{3}$ $y = -\frac{7}{3}$ and $z = -\frac{16}{3}$
 The height of the pyramid in both cases is $|\overrightarrow{GP}| = \frac{11\sqrt{3}}{3}$ A1

Question 2

a. $\underline{r}(0) = (6\sin(0) - 3\sin(0))\underline{i} + (6\cos(0) + 3\cos(0))\underline{j}$

$$\underline{r}(0) = 0\underline{i} + 9\underline{j}$$

at (0,9)

A1

b. $\underline{r}(\pi) = (6\sin(2\pi) - 3\sin(4\pi))\underline{i} + (6\cos(2\pi) + 3\cos(4\pi))\underline{j}$

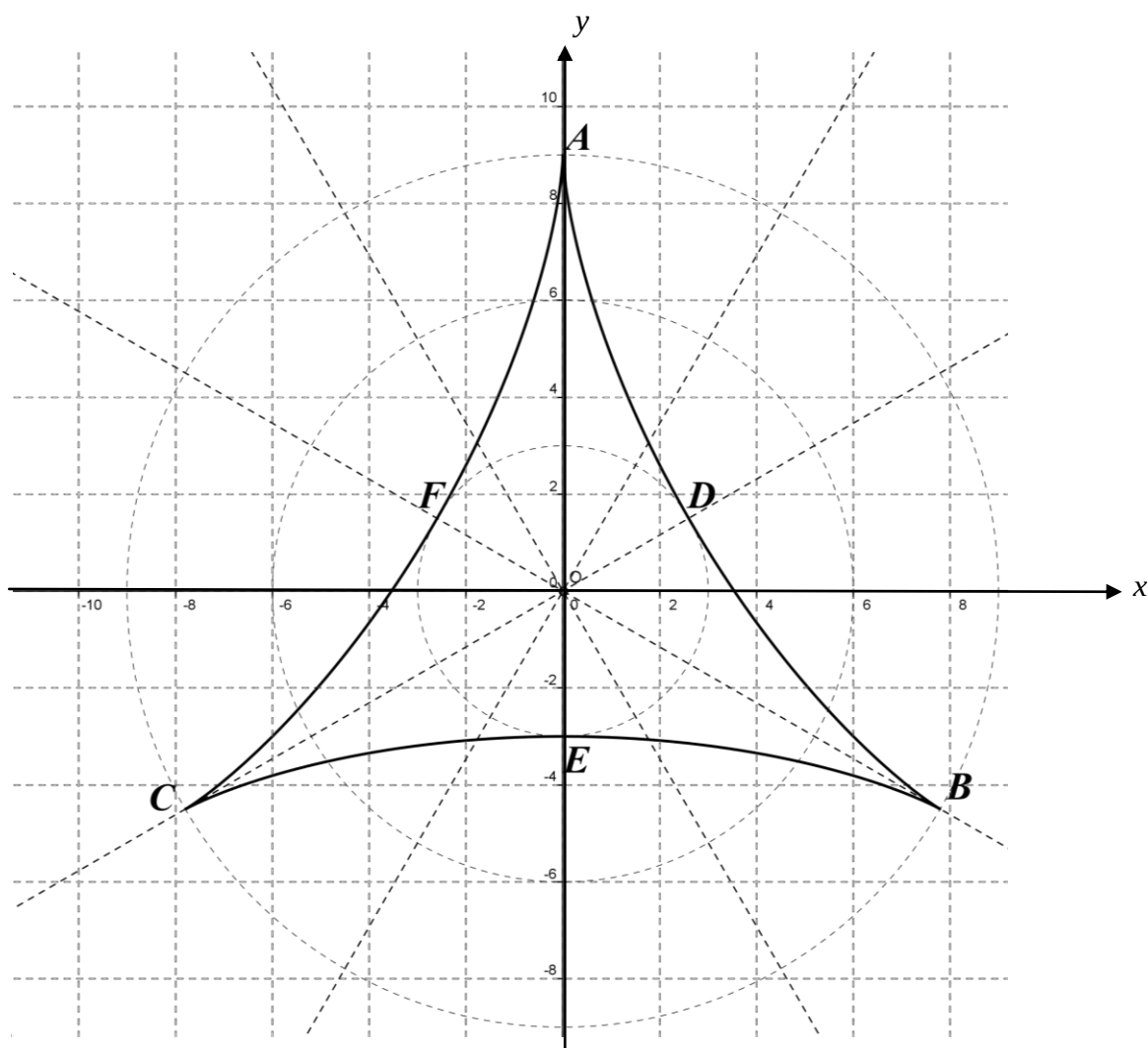
$$\underline{r}(\pi) = \underline{r}(0) = 0\underline{i} + 9\underline{j} \quad \text{at } (0,9) \text{ again}$$

after π seconds.

A1

c. correct shape and scale factors

A2

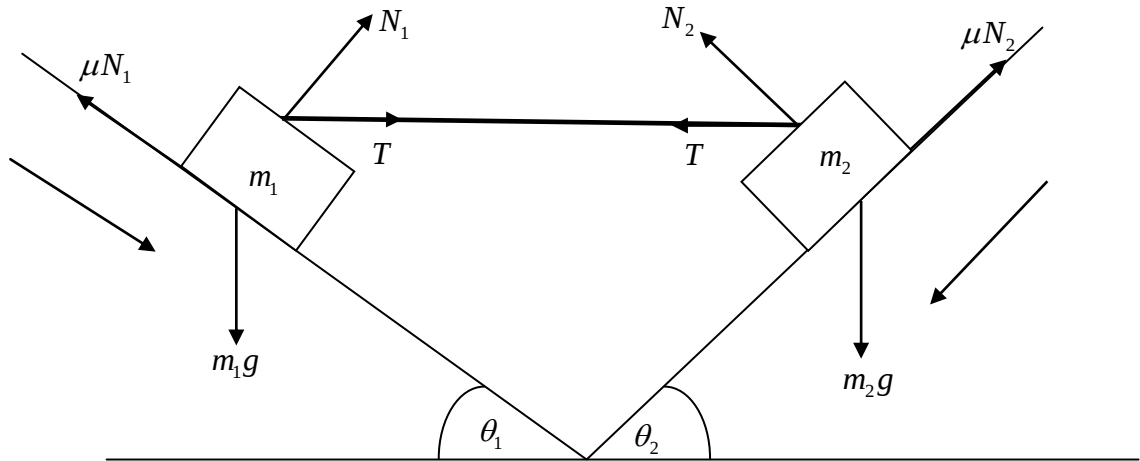


- d.** $\dot{\mathbf{r}}(t) = (12 \cos(2t) - 12 \cos(4t))\mathbf{i} - (12 \sin(2t) + 12 \sin(4t))\mathbf{j}$
 $|\dot{\mathbf{r}}(t)| = \sqrt{(12(\cos(2t) - \cos(4t)))^2 + (-12(\sin(2t) + \sin(4t)))^2}$ M1
 $|\dot{\mathbf{r}}(t)| = \sqrt{144(\cos^2(2t) - 2\cos(2t)\cos(4t) + \cos^2(4t) + \sin^2(2t) + 2\sin(2t)\sin(4t) + \sin^2(4t))}$
 $|\dot{\mathbf{r}}(t)| = \sqrt{144(2 - 2(\cos(2t)\cos(4t) - \sin(2t)\sin(4t)))}$
 $|\dot{\mathbf{r}}(t)| = 12\sqrt{2(1 - \cos(6t))}$ M1
 $|\dot{\mathbf{r}}(t)| = 12\sqrt{4\sin^2(3t)}$
 $|\dot{\mathbf{r}}(t)| = 24|\sin(3t)|$ so that $a = 3$ A1
- e.** at rest $|\dot{\mathbf{r}}(t)| = 0 \Rightarrow \sin(3t) = 0$
 $3t = 0, \pi, 2\pi$
 $t = 0, \frac{\pi}{3}, \frac{2\pi}{3}$ A1
 $\mathbf{r}(0) = 0\mathbf{i} + 9\mathbf{j}$ at $(0, 9)$ point A
 $\mathbf{r}\left(\frac{\pi}{3}\right) = \frac{9}{2}(\sqrt{3}\mathbf{i} - \mathbf{j})$ at $\left(\frac{9\sqrt{3}}{2}, -\frac{9}{2}\right)$ point B
 $\mathbf{r}\left(\frac{2\pi}{3}\right) = -\frac{9}{2}(\sqrt{3}\mathbf{i} + \mathbf{j})$ at $\left(-\frac{9\sqrt{3}}{2}, -\frac{9}{2}\right)$ point C
correct points and labelled on the diagram in **c.** A1
- f.** maximum speed $|\dot{\mathbf{r}}(t)| = 24 \text{ ms}^{-1}$ and occurs when $|\sin(3t)| = 1$ A1
 $3t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$
 $t = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ A1
 $\mathbf{r}\left(\frac{\pi}{6}\right) = \frac{3}{2}(\sqrt{3}\mathbf{i} + \mathbf{j})$ at $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$ point D
 $\mathbf{r}\left(\frac{\pi}{2}\right) = -3\mathbf{j}$ at $(0, -3)$ point E
 $\mathbf{r}\left(\frac{5\pi}{6}\right) = \frac{3}{2}(-\sqrt{3}\mathbf{i} + \mathbf{j})$ at $\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$ point F
correct points and labelled on the diagram in **c.** A1
- g.** $\int_0^{\pi} 24|\sin(3t)| dt = 3 \int_0^{\frac{\pi}{3}} 24 \sin(3t) dt = 48 \text{ metres}$ A1

Define $r(t) = [6 \cdot \sin(2 \cdot t) - 3 \cdot \sin(4 \cdot t) \quad 6 \cdot \cos(2 \cdot t) + 3 \cdot \cos(4 \cdot t)]$	Done
$r(0)$	$[0 \quad 9]$
$r(\pi)$	$[0 \quad 9]$
Define $v(t) = \frac{d}{dt}(r(t))$	Done
$tCollect(\text{norm}(v(t)))$	$12 \cdot \sqrt{-2 \cdot (\cos(6 \cdot t) - 1)}$
$r\left(\frac{\pi}{3}\right)$	$\begin{bmatrix} \frac{9 \cdot \sqrt{3}}{2} & -9 \\ 2 & 2 \end{bmatrix}$
$r\left(\frac{2 \cdot \pi}{3}\right)$	$\begin{bmatrix} -9 \cdot \sqrt{3} & -9 \\ 2 & 2 \end{bmatrix}$
$r\left(\frac{\pi}{6}\right)$	$\begin{bmatrix} \frac{3 \cdot \sqrt{3}}{2} & \frac{3}{2} \\ 2 & 2 \end{bmatrix}$
$r\left(\frac{\pi}{2}\right)$	$[0 \quad -3]$
$r\left(\frac{5 \cdot \pi}{6}\right)$	$\begin{bmatrix} -\frac{3 \cdot \sqrt{3}}{2} & \frac{3}{2} \\ 2 & 2 \end{bmatrix}$
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Question 3

a.



The forces all given in newtons, acting on the mass m_1 are the weight force m_1g , the normal reaction N_1 , the tension in the rod T , and the frictional force μN_1 . The forces acting on the mass m_2 are the weight force m_2g , the normal reaction N_2 , the tension in the rod T , and the frictional force μN_2 . A1

b. For the particle of mass m_1 on the slope inclined at θ_1 to the horizontal,

since m_1 is on the point of moving down the plane,

resolving perpendicular to the slope

$$(1) N_1 + T \sin(\theta_1) - m_1g \cos(\theta_1) = 0 \quad A1$$

resolving downwards parallel to the slope

$$(2) m_1g \sin(\theta_1) + T \cos(\theta_1) - \mu N_1 = 0 \quad A1$$

For the particle of mass m_2 on the slope inclined at θ_2 to the horizontal,

since m_2 is on the point of moving down the plane,

resolving perpendicular to the slope

$$(3) N_2 + T \sin(\theta_2) - m_2g \cos(\theta_2) = 0$$

resolving downwards parallel to the slope

$$(4) m_2g \sin(\theta_2) + T \cos(\theta_2) - \mu N_2 = 0$$

eliminating the normal reaction,

M1

from (1) $N_1 = m_1 g \cos(\theta_1) - T \sin(\theta_1)$ substitute into (2)

$$(2) \quad m_1 g \sin(\theta_1) + T \cos(\theta_1) - \mu(m_1 g \cos(\theta_1) - T \sin(\theta_1)) = 0$$

$$\Rightarrow T(\cos(\theta_1) + \mu \sin(\theta_1)) = m_1 g(\mu \cos(\theta_1) - \sin(\theta_1))$$

$$\Rightarrow T = \frac{m_1 g(\mu \cos(\theta_1) - \sin(\theta_1))}{\cos(\theta_1) + \mu \sin(\theta_1)}$$

M1

similarly from (3) $N_2 = m_2 g \cos(\theta_2) - T \sin(\theta_2)$ substitute into (4)

$$(4) \quad m_2 g \sin(\theta_2) + T \cos(\theta_2) - \mu(m_2 g \cos(\theta_2) - T \sin(\theta_2)) = 0$$

$$\Rightarrow T(\cos(\theta_2) + \mu \sin(\theta_2)) = m_2 g(\mu \cos(\theta_2) - \sin(\theta_2))$$

$$\Rightarrow T = \frac{m_2 g(\mu \cos(\theta_2) - \sin(\theta_2))}{\cos(\theta_2) + \mu \sin(\theta_2)}$$

now eliminating T , gives

$$\frac{m_1 g(\mu \cos(\theta_1) - \sin(\theta_1))}{\cos(\theta_1) + \mu \sin(\theta_1)} = \frac{m_2 g(\mu \cos(\theta_2) - \sin(\theta_2))}{\cos(\theta_2) + \mu \sin(\theta_2)}$$

$$\frac{m_1}{m_2} = \frac{(\mu \cos(\theta_2) - \sin(\theta_2))(\cos(\theta_1) + \mu \sin(\theta_1))}{(\mu \cos(\theta_1) - \sin(\theta_1))(\cos(\theta_2) + \mu \sin(\theta_2))}$$

now divide both numerator and denominator by $-\cos(\theta_1)\cos(\theta_2)$

M1

$$\frac{m_1}{m_2} = \frac{(\tan(\theta_2) - \mu)(1 + \mu \tan(\theta_1))}{(\tan(\theta_1) - \mu)(1 + \mu \tan(\theta_2))}$$

c. resolving around m_1 perpendicular to the slope

$$(1) N_1 - m_1 g \cos(\theta_1) = 0$$

resolving downwards parallel to the slope

$$(2) m_1 g \sin(\theta_1) - \mu N_1 = 0$$

$$m_1 g \sin(\theta_1) = \mu m_1 g \cos(\theta_1)$$

$$\Rightarrow \mu = \tan(\theta_1)$$

M1

resolving around m_2 perpendicular to the slope

$$(3) N_2 - m_2 g \cos(\theta_2) = 0$$

resolving downwards parallel to the slope

$$(4) m_2 g \sin(\theta_2) - \mu N_2 = m_2 a$$

from (3) $N_2 = m_2 g \cos(\theta_2)$ substitute into (4)

$$m_2 a = m_2 g \sin(\theta_2) - \mu m_2 g \cos(\theta_2)$$

$$a = g(\sin(\theta_2) - \mu \cos(\theta_2))$$

A1

$$u = 0 \quad s = D \quad t = T \quad \mu = \tan(\theta_1) \text{ and } \theta_2 = 2\theta_1$$

$$\text{using } s = ut + \frac{1}{2}at^2$$

$$D = \frac{g}{2}(\sin(\theta_2) - \mu \cos(\theta_2))T^2 \text{ but } \mu = \tan(\theta_1) \text{ and } \theta_2 = 2\theta_1$$

$$\frac{2D}{gT^2} = \sin(2\theta_1) - \tan(\theta_1)\cos(2\theta_1)$$

M1

$$\frac{2D}{gT^2} = \sin(2\theta_1) - \frac{\sin(\theta_1)\cos(2\theta_1)}{\cos(\theta_1)}$$

$$\frac{2D}{gT^2} = \frac{\sin(2\theta_1)\cos(\theta_1) - \sin(\theta_1)\cos(2\theta_1)}{\cos(\theta_1)}$$

$$\frac{2D}{gT^2} = \frac{\sin(2\theta_1 - \theta_1)}{\cos(\theta_1)} = \frac{\sin(\theta_1)}{\cos(\theta_1)} = \tan(\theta_1)$$

A1

$$\theta_1 = \tan^{-1}\left(\frac{2D}{gT^2}\right)$$

Question 4

a. $|z - c| = 1$, $c = -2 + i$ let $z = x + yi$

$$|(x+2) + (y-1)i| = 1$$

$$\sqrt{(x+2)^2 + (y-1)^2} = 1$$

$$(x+2)^2 + (y-1)^2 = 1$$

a circle with centre $(-2, 1)$ and radius 1.

A1

b. $\left| z - \frac{1}{5}(-2 + 11i) \right| = |z - c|$, $c = -2 + i$ let $z = x + yi$

$$\left| \left(x + \frac{2}{5} \right) + \left(y - \frac{11}{5} \right) i \right| = |(x+2) + (y-1)i|$$

M1

$$\sqrt{\left(x + \frac{2}{5} \right)^2 + \left(y - \frac{11}{5} \right)^2} = \sqrt{(x+2)^2 + (y-1)^2}$$

$$x^2 + \frac{4x}{5} + \frac{4}{25} + y^2 - \frac{22y}{5} + \frac{121}{25} = x^2 + 4x + 4 + y^2 - 2y + 1$$

$$12y + 16x = 0, T \text{ is the line } y = -\frac{4x}{3}$$

$$3y + 4x = 0 \quad z = x + yi \quad \operatorname{Re}(z) = x \text{ and } \operatorname{Im}(z) = y$$

$$p = 3 \text{ and } q = 4$$

A1

c. from **b.** $y = -\frac{4x}{3}$ substitute into $(x+2)^2 + (y-1)^2 = 1$

$$(x+2)^2 + \left(-\frac{4x}{3} - 1 \right)^2 = 1$$

$$x^2 + 4x + 4 + \frac{16x^2}{9} + \frac{8x}{3} + 1 = 1$$

M1

$$25x^2 + 60x + 36 = 0$$

$$(5x+6)^2 = 0$$

$$x = -\frac{6}{5} \Rightarrow y = \frac{8}{5}$$

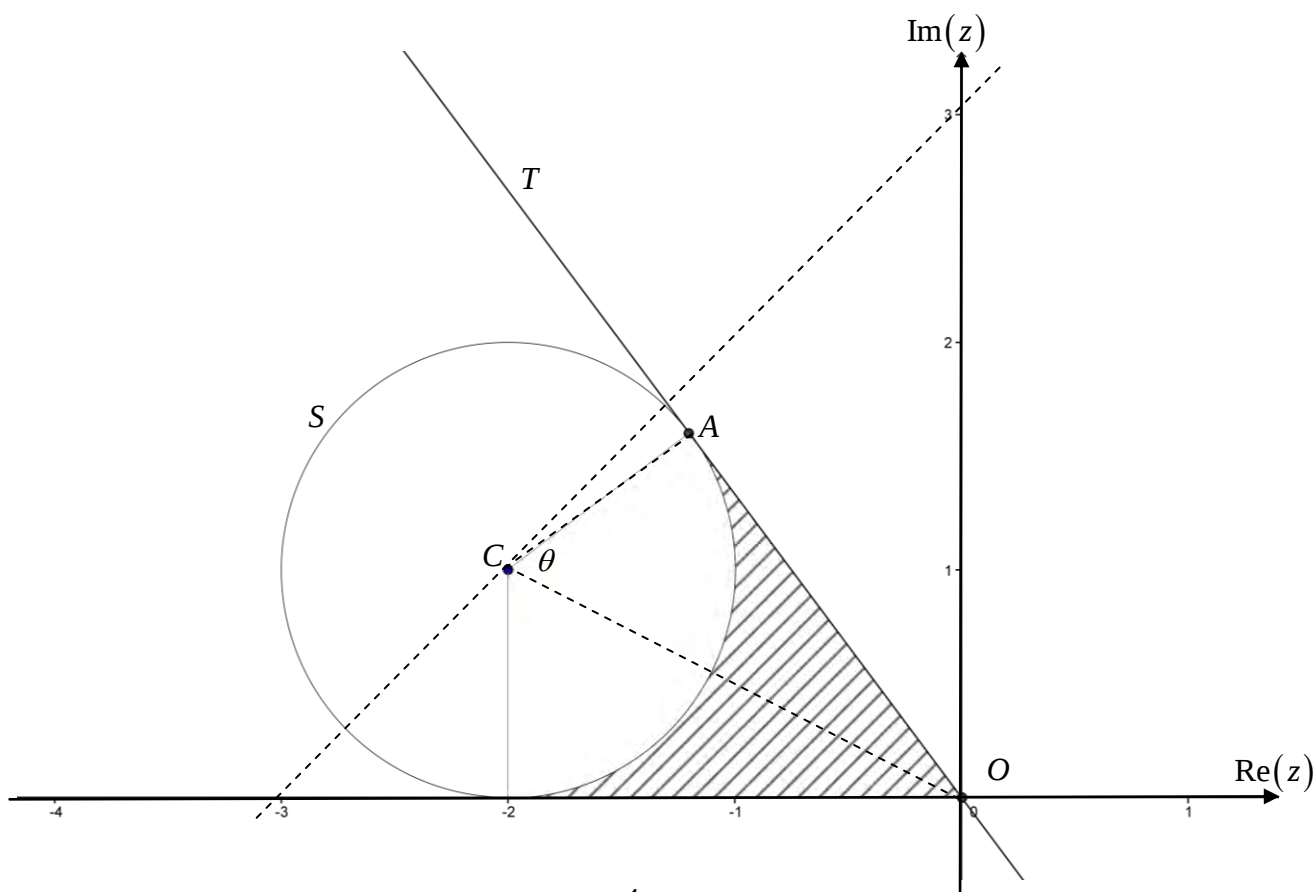
$$A\left(-\frac{6}{5}, \frac{8}{5}\right)$$

A1

d. S is the circle with centre $C(-2, 1)$ and radius 1, and T is the line $y = -\frac{4x}{3}$

$$\text{intersecting at the point } A\left(-\frac{6}{5}, \frac{8}{5}\right)$$

A2



- e. since the gradient of the line T is $m = -\frac{4}{3}$, α is the angle T makes with the positive real axis. $\text{Arg}(z) = \alpha = \pi - \tan^{-1}\left(\frac{4}{3}\right)$ A1
- f. the maximum value of $|z| \in S$, is the furthest point on S , from the origin, since $OC = \sqrt{5}$ and $CA = 1$ the radius of the circle, then $z \in S \quad |z|_{\max} = \sqrt{5} + 1$ A1
- g. The shaded area is twice the area of the triangle OAC , minus the area of the sector of the circle. Now $\angle OAC = 90^\circ$, since OA is a tangent to the circle, also $OA = 2 \quad AC = 1 \quad OC = \sqrt{5}$, let $\theta = \angle ACO$, M1
 $\sin(\theta) = \frac{2}{\sqrt{5}} \quad \cos(\theta) = \frac{1}{\sqrt{5}} \quad \text{and} \quad \tan(\theta) = 2$
 $\text{Area} = 2\left(\frac{1}{2} \times 2 \times 1 - \frac{1}{2} \times 1^2 \times \theta\right) = 2 - \theta$
 $\text{Area} = 2 - \tan^{-1}(2) \quad \text{or alternatively} \quad \text{A1}$
 $\text{Area} = 2 - \sin^{-1}\left(\frac{2}{\sqrt{5}}\right) = 2 - \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$

Question 5

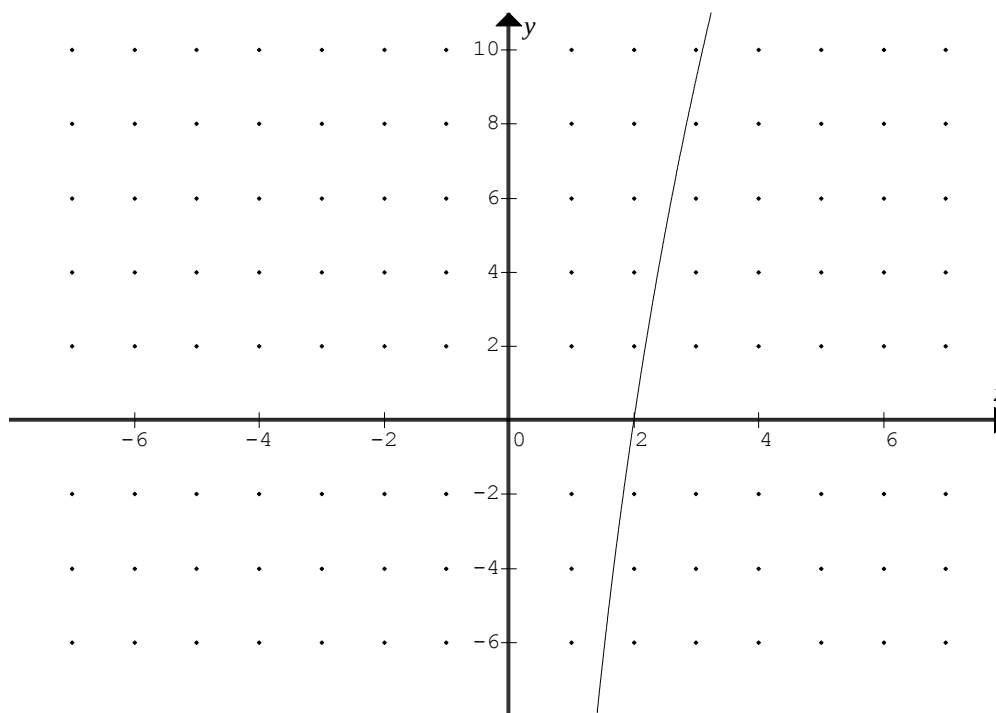
a. $y = \frac{16(x-2)}{\sqrt{x}}$

$\frac{dy}{dx} = \frac{8(x+2)}{\sqrt{x^3}}$ for turning points $\frac{dy}{dx} = 0 \Rightarrow x = -2$ A1

but the domain of the function is $x > 0$ so there are no turning points. A1

b. crosses the x -axis at $x = 2$, $(2, 0)$

and the y -axis, $x = 0$ is a vertical asymptote. A1



c. $y = \frac{16(x-2)}{\sqrt{x}}$

$16x - 32 = y\sqrt{x}$

$16x - y\sqrt{x} - 32 = 0$ A1

using the quadratic formulae,

$\sqrt{x} = \frac{y \pm \sqrt{y^2 - 4 \times 16 \times -32}}{32}$ since $\sqrt{x} > 0$ must take the positive A1

$\sqrt{x} = \frac{y + \sqrt{y^2 + 2048}}{32}$

$x = \frac{\left(y + \sqrt{y^2 + 2048}\right)^2}{1024}$

d. solving $y = \frac{16(x-2)}{\sqrt{x}} = 9$ gives $x = \frac{9\sqrt{2129} + 1105}{512}$ as the radius

the diameter is $\frac{9\sqrt{2129} + 1105}{256}$ cm A1

e.i. $V = \pi \int_a^b x^2 dy$

$$V = \pi \int_0^9 \frac{\left(y + \sqrt{y^2 + 2048}\right)^4}{1048576} dy$$
 A1

ii. $V = 172.54 \text{ cm}^3$ A1

f. solving $V = \pi \int_0^h \frac{\left(y + \sqrt{y^2 + 2048}\right)^4}{1048576} dy = 150$

gives $h = 8.16 \text{ cm}$ A1

g. $V = \pi \int_0^h \frac{\left(y + \sqrt{y^2 + 2048}\right)^4}{1048576} dy$

$$\left. \frac{dV}{dh} \right|_{h=4} = \frac{\pi(65\sqrt{129} + 2177)}{512}$$
 A1

given $\frac{dV}{dt} = 2 \text{ cm}^3/\text{sec}$ and $\frac{dh}{dt} = \frac{dh}{dV} \frac{dV}{dt}$ M1

$$\frac{dh}{dt} = \frac{1024}{\pi(65\sqrt{129} + 2177)} \text{ cm/s}$$
 A1

Define $f1(x) = \frac{16 \cdot (x-2)}{\sqrt{x}}$	Done
$\frac{d}{dx}(f1(x))$	$\frac{8 \cdot (x+2)}{x^2}$
solve($f1(x)=9, x$)	$x = \frac{9 \cdot \sqrt{2129 + 1105}}{512}$
Define $v(h) = \pi \cdot \int_0^h \frac{(y + \sqrt{y^2 + 2048})^4}{1048576} dy$	Done
$v(9)$	172.537
solve($v(h)=150, h$)	$h = 8.15617$
$\frac{d}{dh}(v(h)) _{h=4}$	$\frac{(65 \cdot \sqrt{129 + 2177}) \cdot \pi}{512}$
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END OF SECTION 2 SUGGESTED ANSWERS