

2012 VCAA Specialist Math Exam 2 Solutions

© 2012 itute.com

Free download from www.itute.com

Note: Some steps can be done by CAS to save time

SECTION 1

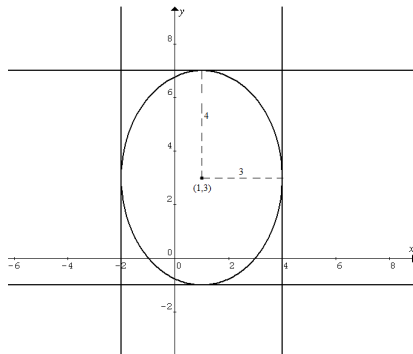
1	2	3	4	5	6	7	8	9	10	11
D	C	E	D	C	C	B	E	B	A	E

12	13	14	15	16	17	18	19	20	21	22
A	B	B	D	A	D	C	E	C	E	D

Q1 $y = \frac{1}{2x^2 - x - 6} = \frac{1}{(2x+3)(x-2)} = \frac{1}{2(x+\frac{3}{2})(x-2)}$

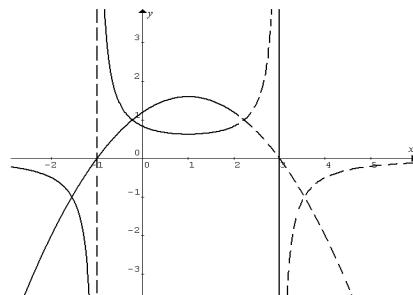
Straight line asymptotes are: $y = 0$, $x = -\frac{3}{2}$ and $x = 2$. D

Q2



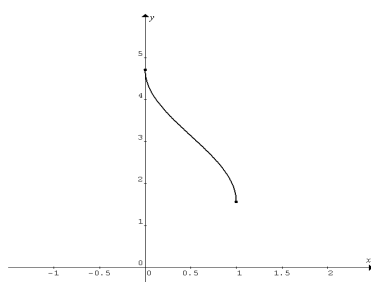
Equation of the ellipse: $\frac{(x-1)^2}{9} + \frac{(y-3)^2}{16} = 1$

Q3 See graph below.



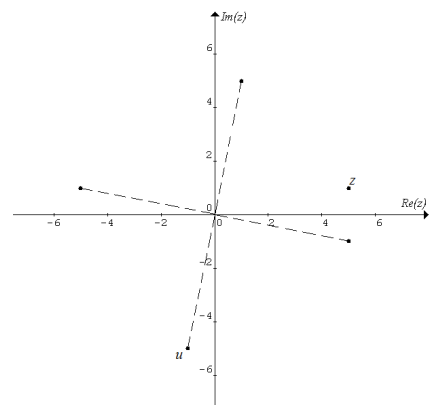
Q4 Find the domain and range from the graph of

$f(x) = \arccos(2x-1) + \frac{\pi}{2}$.



Q5 $z = \sqrt{2} \operatorname{cis}\left(-\frac{4\pi}{5}\right)$, $w = z^9 = (\sqrt{2})^9 \operatorname{cis}\left(-\frac{4\pi}{5} \times 9\right)$
 $= 16\sqrt{2} \operatorname{cis}\left(\frac{4\pi}{5} - 8\pi\right) = 16\sqrt{2} \operatorname{cis}\left(\frac{4\pi}{5}\right)$ C

Q6 $u = i^3 \bar{z}$ is the reflection of z about the x -axis followed by an anticlockwise rotation through $3 \times \frac{\pi}{2}$ about the origin. The result is the same as reflecting z about the y -axis and then rotating anticlockwise through $\frac{\pi}{2}$ about the origin. C



Q7 $|z + 2i| = |z|$, $|z - (-2i)| = |z|$ defines a set of points such that each point is equidistant from $(0, -2)$ and $(0, 0)$. This set of points forms a straight line which is the perpendicular bisector of the line segment joining $(0, -2)$ and $(0, 0)$.

C

The equation is $y = -1$, i.e. $\operatorname{Im}(z) = -1$.

E

B

Q8 $(z - \bar{z})(z + \bar{z}) = 2bi \times 2a = 4abi$

E

Q9

$x_0 = 0$ $y_0 = 1$ $\frac{dy}{dx} = \cos 0 = 1$

$x_1 = 0.1$ $y_1 \approx 1 + 0.1 \times 1 \approx 1.1$ $\frac{dy}{dx} = \cos 0.1 \approx 0.995$

$x_2 = 0.2$ $y_2 \approx 1.1 + 0.1 \times 0.995 \approx 1.1995$

$\frac{dy}{dx}$ is a decreasing function in the region under consideration,

D

$\therefore 1.1995$ is an overestimate of y at $x = 0.2$.

B

Q10 $\frac{dy}{dx} = xy$

At $(1, 1)$, $\frac{dy}{dx} = 1$; at $(1, -1)$, $\frac{dy}{dx} = -1$; at $(-1, 1)$, $\frac{dy}{dx} = -1$;

at $(-1, -1)$, $\frac{dy}{dx} = 1$; at $(0, 1)$, $\frac{dy}{dx} = 0$; at $(1, 0.3)$, $\frac{dy}{dx} = 0.3$ A

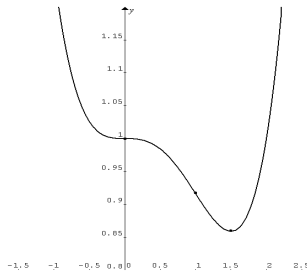
Q11 Let $\frac{d^2y}{dx^2} = x^2 - x = 0$, $x(x-1) = 0$, $\therefore$ a possible point of inflection at $x = 0$ or $x = 1$.

$$\frac{d^2y}{dx^2} = x^2 - x, \therefore \frac{dy}{dx} = \frac{x^3}{3} - \frac{x^2}{2} + c = x^2 \left(\frac{x}{3} - \frac{1}{2} \right) + c$$

Given $\frac{dy}{dx} = 0$ at $x = 0$, $\therefore c = 0$ and $\frac{dy}{dx} = 0$ at $x = \frac{3}{2}$ also.

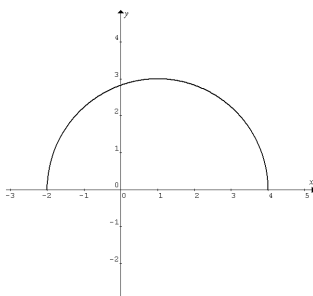
x	< 0	0	$0 < x < \frac{3}{2}$	$\frac{3}{2}$	> 0
$\frac{dy}{dx}$	-	0	-	0	+
Nature		point of inflection		local minimum	

There must be another point of inflection between $x = 0$ and $x = \frac{3}{2}$. See graph below.



Q12 $y = \sqrt{9 - (x-1)^2}$ is the positive half of the circle $(x-1)^2 + y^2 = 9$ which has a radius of 3 units. A sphere is formed when it is rotated about the x -axis. See graph below.

$$\text{Volume} = \frac{4}{3} \pi \times 3^3 = 4\pi(3)^2$$



$$\text{Q13 } \int_0^{\frac{\pi}{3}} \sin^3 x \cos^4 x dx = \int_0^{\frac{\pi}{3}} \sin^2 x \cos^4 x \sin x dx$$

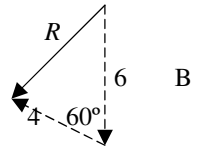
$$= \int_0^{\frac{\pi}{3}} (1 - \cos^2 x) \cos^4 x \sin x dx$$

$$= \int_1^{\frac{1}{2}} (u^6 - u^4) du$$

Let $u = \cos x$,
 $-\frac{du}{dx} = \sin x$.
 When $x = 0$, $u = 1$.
 When $x = \frac{\pi}{3}$, $u = \frac{1}{2}$.

B

Q14 Resultant force R
 $= \sqrt{4^2 + 6^2 - 2(4)(6)\cos 60^\circ} = 2\sqrt{7}$ N



Q15 Let $\tilde{a}\tilde{b} = 0$
 $(2\tilde{i} + m\tilde{j} - 3\tilde{k})(m^2\tilde{i} - \tilde{j} + \tilde{k}) = 0$
 $\therefore 2m^2 - m - 3 = 0$, $(2m-3)(m+1) = 0$,
 $\therefore m = \frac{3}{2}, -1$

D

Q16 Distance between $P(-2, 4, 3)$ and $Q(1, -2, 1)$
 $= \sqrt{(1 - (-2))^2 + (-2 - 4)^2 + (1 - 3)^2} = 7$

A

Q17 $\tilde{u} = 2\tilde{i} - 2\tilde{j} + \tilde{k}$, $\tilde{v} = 3\tilde{i} - 6\tilde{j} + 2\tilde{k}$
 $|\tilde{u}| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$, $\therefore \hat{u} = \frac{1}{3}(2\tilde{i} - 2\tilde{j} + \tilde{k})$
 Scalar resolute of $\tilde{v}$ in the direction of $\tilde{u} = \tilde{v} \cdot \hat{u}$
 $= (3\tilde{i} - 6\tilde{j} + 2\tilde{k}) \cdot \frac{1}{3}(2\tilde{i} - 2\tilde{j} + \tilde{k}) = \frac{1}{3}(6 + 12 + 2) = \frac{20}{3}$
 Vector resolute of $\tilde{v}$ in the direction of $\tilde{u} = (\tilde{v} \cdot \hat{u})\hat{u}$
 $= \frac{20}{3} \times \frac{1}{3}(2\tilde{i} - 2\tilde{j} + \tilde{k}) = \frac{20}{9}(2\tilde{i} - 2\tilde{j} + \tilde{k})$

D

Q18 x_0 is a red herring.

C

Q19 Given $v = x$, $x = -1$ when $t = 3$.

$$\frac{dx}{dt} = x, \frac{dt}{dx} = \frac{1}{x}, t = \int_{-1}^x \frac{1}{x} dx + 3 = [\log_e |x|]_{-1}^x + 3 = \log_e |x| + 3$$

$$\therefore |x| = e^{t-3}, \therefore x = -e^{t-3} \text{ to satisfy } x = -1 \text{ when } t = 3.$$

E

Q20 Resultant force = mass $\times$ acceleration

$$mg - 3g = (m+3) \times 4.9, mg - 3g = (m+3) \times \frac{g}{2}$$

$$\therefore m - 3 = \frac{m+3}{2}, \therefore m = 9.0$$

C

Q21 $v = x^2$, $\frac{1}{2}v^2 = \frac{1}{2}x^4$, $a = \frac{d}{dx} \left(\frac{1}{2}v^2 \right) = \frac{d}{dx} \left(\frac{1}{2}x^4 \right) = 2x^3$

When $x = 2$, $a = 16$ and $F = ma = 3 \times 16 = 48$ N

E

Q22 $2T \sin 60^\circ - 12g = 0$, $\therefore \sqrt{3}T - 12g = 0$,

$$T = \frac{12g}{\sqrt{3}} = 4\sqrt{3}g$$

D

SECTION 2

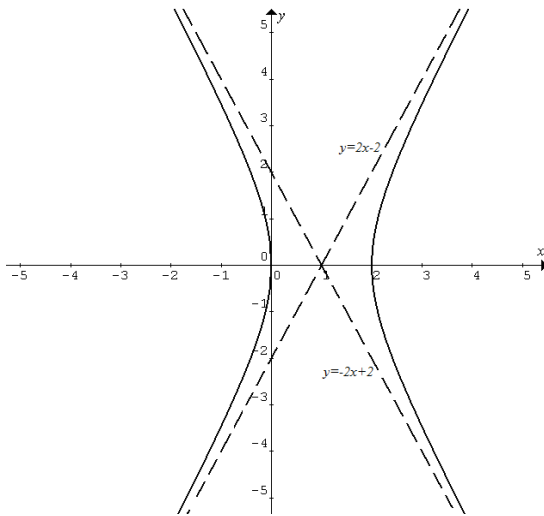
Q1a Given $x = \operatorname{cosec} \theta + 1$, $y = 2 \cot \theta$

$$1 - \cos^2 \theta = \sin^2 \theta, \frac{1 - \cos^2 \theta}{\sin^2 \theta} = 1, \frac{1}{\sin^2 \theta} - \frac{\cos^2 \theta}{\sin^2 \theta} = 1,$$

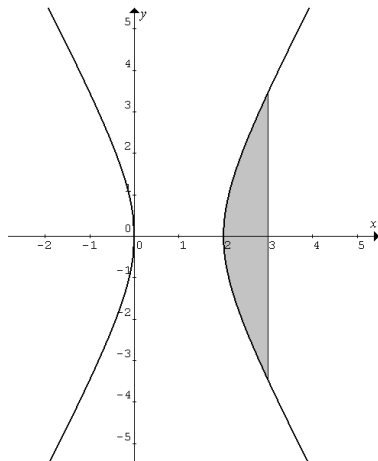
$$\therefore \operatorname{cosec}^2 \theta - \cot^2 \theta = 1, \therefore (x-1)^2 - \left(\frac{y}{2}\right)^2 = 1$$

$$\therefore (x-1)^2 - \frac{y^2}{4} = 1$$

Q1b



Q1c



$$V_{\text{solid}} = \pi \int_2^3 y^2 dx = 4\pi \int_2^3 [(x-1)^2 - 1] dx = 4\pi \left[\frac{(x-1)^3}{3} - x \right]_2^3 = \frac{16\pi}{3}$$

Q1d At $\theta = \frac{7\pi}{6}$, $x = \operatorname{cosec} \frac{7\pi}{6} + 1 = -1$, $y = 2 \cot \frac{7\pi}{6} = 2\sqrt{3}$

By implicit differentiation of $(x-1)^2 - \frac{y^2}{4} = 1$,

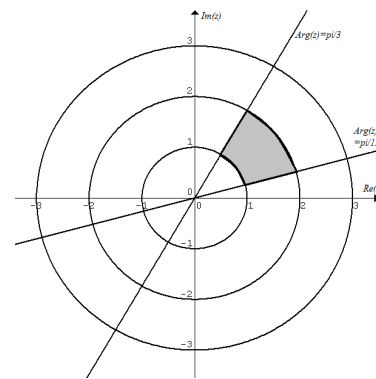
$$2(x-1) - \frac{y}{2} \times \frac{dy}{dx} = 0, \therefore \frac{dy}{dx} = \frac{4(x-1)}{y} = -\frac{4}{\sqrt{3}} = -\frac{4\sqrt{3}}{3}.$$

$$\begin{aligned} \text{Q2a } \sin \frac{\pi}{12} &= \sqrt{1 - \cos^2 \frac{\pi}{12}} = \sqrt{1 - \left(\frac{\sqrt{\sqrt{3}+2}}{2}\right)^2} \\ &= \sqrt{1 - \frac{\sqrt{3}+2}{4}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2} \end{aligned}$$

Q2bi $z_1 = \cos \frac{\pi}{12} + i \sin \frac{\pi}{12}$ or simply $z_1 = \operatorname{cis} \frac{\pi}{12}$

Q2bii $z_1^4 = \operatorname{cis} \left(4 \times \frac{\pi}{12} \right) = \operatorname{cis} \frac{\pi}{3}$

Q2c



Q2d $\text{Area} = (\pi \times 2^2 - \pi \times 1^2) \times \frac{\frac{\pi}{3} - \frac{\pi}{12}}{2\pi} = 3\pi \times \frac{1}{8} = \frac{3\pi}{8}$

Q2ei $z_1 = \cos \frac{\pi}{12} + i \sin \frac{\pi}{12}$, $z_1^n = \cos \frac{n\pi}{12} + i \sin \frac{n\pi}{12}$

$\operatorname{Re}(z_1^n) = 0$, $\cos \frac{n\pi}{12} = 0$

$\therefore \frac{n\pi}{12} = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots$, i.e. $\pm$ odd integral multiple of

$\frac{\pi}{2}$, which can be expressed as $\frac{n\pi}{12} = (2k+1)\frac{\pi}{2}$ where $k \in \mathbb{Z}$.

$\therefore n = (2k+1)6$.

Q2eii For the values of n found in part i, $\operatorname{Re}(z_1^n) = 0$,

$\therefore z_1^n = i \sin \frac{n\pi}{12} = i \sin \frac{(2k+1)\pi}{2} = i(\pm 1) = \pm i$ for $k \in \mathbb{Z}$.

Q3a $V = 17 \tan^{-1} \frac{\pi T}{6}$, $T \geq 0$

As $T \rightarrow \infty$, $\tan^{-1} \frac{\pi T}{6} \rightarrow \frac{\pi}{2}$, $\therefore V \rightarrow \frac{17\pi}{2}$ (≈ 26.7) m s^{-1}

Q3b $a = \frac{dV}{dT} = 17 \times \frac{\frac{\pi}{6}}{1 + \left(\frac{\pi T}{6}\right)^2} \approx 0.3 \text{ m s}^{-2}$ when $T = 10$

Q3c When $V = 25$, $17 \tan^{-1} \frac{\pi T}{6} = 25$, $\frac{\pi T}{6} = \tan^{-1} \frac{25}{17}$, $T \approx 19$ s

Q3di $\frac{dv}{dt} = -\frac{1}{100}(145 - 2t)$, $v = 25$ when $t = 0$

$$\therefore v = \int_0^t -\frac{1}{100}(145 - 2t)dt + 25 = \left[-\frac{1}{100}(145t - t^2) \right]_0^t + 25$$

$$= 0.01t^2 - 1.45t + 25$$

Q3dii When $v = 0$, $0.01t^2 - 1.45t + 25 = 0$, $t \geq 0$
 $\therefore t = 20$ s

Q3ei Distance (m) travelled during each stage:

Stage 1 distance $= \int_0^{19} \left(17 \tan^{-1} \frac{\pi T}{6} \right) dT \approx 400$ by CAS

Stage 2 distance $= 25 \times 120 = 3000$

Stage 3 distance $= \int_0^{20} (0.01t^2 - 1.45t + 25) dt \approx 237$ by CAS

Q3eii Total distance ≈ 3637 m

Q4a Let $\theta = \pi(1.3t - 0.1)$ for $0 \leq t \leq 0.154$ hours.

$$\tilde{r} = (6800 \sin(\pi(1.3t - 0.1)))\tilde{i} + (6800 \cos(\pi(1.3t - 0.1)) - 6400)\tilde{j}$$

$$\therefore \tilde{r} = (6800 \sin \theta)\tilde{i} + (6800 \cos \theta - 6400)\tilde{j}$$

At point P, the $\tilde{i}$ component is zero, $6800 \sin \theta = 0$, $\theta = 0$.
 $\therefore h = 6800 \cos 0 - 6400 = 400$ km

Q4b $\tilde{v} = \frac{d\tilde{r}}{dt} = \frac{d\tilde{r}}{d\theta} \times \frac{d\theta}{dt} = 6800 \times 1.3\pi((\cos \theta)\tilde{i} - (\sin \theta)\tilde{j})$

$$\tilde{a} = \frac{d\tilde{v}}{dt} = \frac{d\tilde{v}}{d\theta} \times \frac{d\theta}{dt} = 6800 \times (1.3\pi)^2((- \sin \theta)\tilde{i} - (\cos \theta)\tilde{j})$$

$$\therefore \tilde{a}(t) = -6800 \times (1.3\pi)^2((\sin(\pi(1.3t - 0.1)))\tilde{i} + (\cos(\pi(1.3t - 0.1)))\tilde{j})$$

$$\hat{v} \cdot \hat{a} = ((\cos \theta)\tilde{i} - (\sin \theta)\tilde{j}) \cdot ((- \sin \theta)\tilde{i} - (\cos \theta)\tilde{j})$$

$$= -\sin \theta \cos \theta + \sin \theta \cos \theta = 0$$

$$\therefore \tilde{a} \perp \tilde{v}$$

Q4c $\tilde{v} = 6800 \times 1.3\pi((\cos \theta)\tilde{i} - (\sin \theta)\tilde{j})$

$$\text{Speed} = |\tilde{v}| = 6800 \times 1.3\pi \sqrt{\cos^2 \theta + (-\sin \theta)^2}$$

$$= 6800 \times 1.3\pi \approx 27772 \text{ km/h}$$

Q4d Parametric equations:

$$x = 6800 \sin \theta, y = 6800 \cos \theta - 6400$$

Since $\sin^2 \theta + \cos^2 \theta = 1$, $\therefore$ the cartesian equation of the path is

$$\left(\frac{x}{6800} \right)^2 + \left(\frac{y + 6400}{6800} \right)^2 = 1, \text{ i.e. } x^2 + (y + 6400)^2 = 6800^2.$$

It is a circular path of radius 6800 km centred at $(0, -6400)$ km.

Q4e $\tilde{r} = (6800 \sin \theta)\tilde{i} + (6800 \cos \theta - 6400)\tilde{j}$

$$|\tilde{r}| = \sqrt{(6800 \sin \theta)^2 + (6800 \cos \theta - 6400)^2} = 1000$$

$$\therefore (6800 \sin \theta)^2 + (6800 \cos \theta - 6400)^2 = 1000^2 \text{ and can be}$$

$$\text{simplified to } (\sin \theta)^2 + \left(\cos \theta - \frac{16}{17} \right)^2 = \left(\frac{5}{34} \right)^2$$

$$\therefore \sin^2 \theta + \cos^2 \theta - \frac{32}{17} \cos \theta + \frac{256}{289} = \frac{25}{1156}$$

$$1 - \frac{32}{17} \cos \theta + \frac{256}{289} = \frac{25}{1156}, \therefore \cos \theta \approx 0.99035$$

$$\therefore \theta \approx -0.139 \text{ or } 0.139 \text{ by CAS}$$

$$\therefore \pi(1.3t - 0.1) \approx -0.139 \text{ or } 0.139, \therefore t \approx 0.04 \text{ or } 0.11 \text{ hours}$$

Q5a $u = 0$, $s = 10$, $v = 6$, find t .

$$s = \frac{1}{2}(u + v)t, t = \frac{10}{3} \text{ s}$$

Q5b $s = -6$, $u = 10$, $a = -9.8$, find t .

$$s = ut + \frac{1}{2}at^2, t \approx 2.5 \text{ s}$$

Q5c Assume that the end of the slide is at ground level.

$$\text{Total time of travel allowed for the chocolate} = \frac{10}{3} + 4 = \frac{22}{3} \text{ s}$$

$$a = -9.8, s = -6, t = \frac{22}{3}, \text{ find } u.$$

$$s = ut + \frac{1}{2}at^2, u = 35.1 \text{ m s}^{-1}$$

Q5di $a = -\frac{1}{10}\sqrt{196 - v^2}$,

$$\frac{dv}{dt} = -\frac{1}{10}\sqrt{196 - v^2}, \therefore \frac{dt}{dv} = 10 \times \frac{-1}{\sqrt{196 - v^2}},$$

$$\frac{t}{10} = \int \frac{-1}{\sqrt{14^2 - v^2}} dv + c, \frac{t}{10} = \cos^{-1} \left(\frac{v}{14} \right) + c$$

When $t = 0$, $v = 7$, $\therefore 0 = \cos^{-1} \left(\frac{7}{14} \right) + c$, $c = -\frac{\pi}{3}$

$$\therefore \frac{t}{10} = \cos^{-1} \left(\frac{v}{14} \right) - \frac{\pi}{3}, \therefore v = 14 \cos \left(\frac{t}{10} + \frac{\pi}{3} \right)$$

Q5dii When $v = 0$, $\frac{t}{10} = \cos^{-1}(0) - \frac{\pi}{3} = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$, $\therefore t = \frac{5\pi}{3}$.

Q5diii $v = 14 \cos \left(\frac{t}{10} + \frac{\pi}{3} \right)$, $\frac{dx}{dt} = 14 \cos \left(\frac{t}{10} + \frac{\pi}{3} \right)$

$$x = \int_0^{\frac{5\pi}{3}} 14 \cos \left(\frac{t}{10} + \frac{\pi}{3} \right) dt \approx 18.8 \text{ m by CAS}$$

$$\text{Distance required} \approx 18.8 \text{ m}$$

Please inform mathline@itute.com re conceptual and/or mathematical errors