2012 VCAA Specialist Mathematics Exam 1 Solutions

Q1 $\int \frac{6+x}{x^2+4}dx = \int \left(\frac{6}{x^2+4}+\frac{x}{x^2+4}\right)dx$

$\int \frac{6}{x^2+4}dx + \int \frac{x}{x^2+4}dx = 3\int \frac{2}{x^2+4}dx + \frac{1}{2}\int \frac{1}{u}du$ $\boxed{u = x^2+4}$

$= 3\tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}\log_e\left(x^2+4\right) + 0$

Q2 $2\cos x = \sqrt{3}\cot x$, $2\cos x - \sqrt{3}\cot x = 0$

$2\cos x - \sqrt{3}\frac{\cos x}{\sin x} = 0$, $\left(2 - \frac{\sqrt{3}}{\sin x}\right)\cos x = 0$

$\therefore\ 2 - \frac{\sqrt{3}}{\sin x} = 0$, i.e. $\sin x = \frac{\sqrt{3}}{2}$, $x = \left(2n + \frac{1}{2} \mp \frac{1}{6}\right)\pi$

OR $\cos x = 0$, i.e. $x = \left(n + \frac{1}{2}\right)\pi$, where $n = 0, \pm 1, \pm 2, \ldots\ldots$

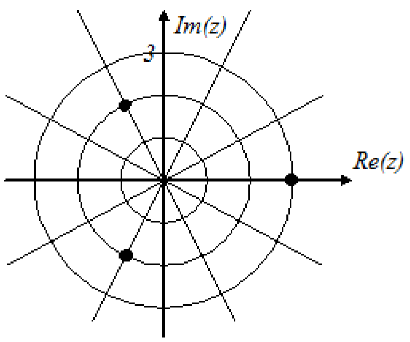
Q3a Given $z = 2cis\left(\frac{2\pi}{3}\right) = -1 + i\sqrt{3}$ is a root of the equation $z^3 - z^2 - 2z - 12 = 0$ which has real coefficients,

$\therefore\ z = -1 - i\sqrt{3}$ is also a root.

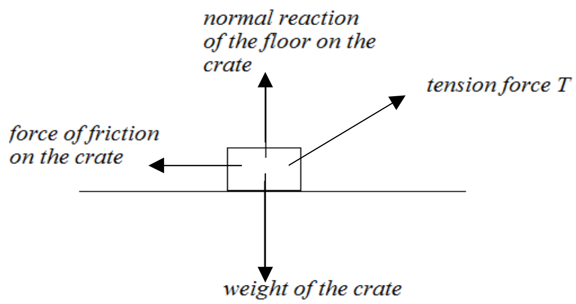
Since $(z-\alpha)(z-\beta)(z-\gamma) = z^3 - z^2 - 2z - 12$ where $\alpha = -1 + i\sqrt{3}$ and $\beta = -1 - i\sqrt{3}$ and γ is the third root,

$\therefore\ \alpha\beta\gamma = 12$, $\left(-1+i\sqrt{3}\right)\left(-1-i\sqrt{3}\right)\gamma = 12$, $\therefore\ 4\gamma = 12$, $\gamma = 3$.

Q3b



Q4a



Q4b Without the crate leaving the floor, maximum tension $T_{\max}$ occurs when the normal reaction force $N \to 0$,

$T_{\max}\sin 30° - 50g = 0$, $\therefore\ T_{\max} = 100g = 980\,\text{N}$

Q4c On the point of moving:

$T\sin 30° + N - 50g = 0$ …… (1)

and $T\cos 30° - \mu N = 0$ …… (2)

From (2), $N = \frac{T\cos 30°}{\mu} = \frac{5\sqrt{3}T}{2}$ …… (3)

Substitute (3) in (1): $\frac{T}{2} + \frac{5\sqrt{3}T}{2} - 50g = 0$, $\therefore\ \left(1+5\sqrt{3}\right)T = 100g$

$T = \frac{100g}{1+5\sqrt{3}}\,\text{N}$

Q5 $y = \tan^{-1}(2x)$, $\frac{dy}{dx} = \frac{2}{1+(2x)^2} = \frac{2}{1+4x^2} = 2\left(1+4x^2\right)^{-1}$

$\frac{d^2y}{dx^2} = -2\left(1+4x^2\right)^{-2}(8x) = -\frac{16x}{\left(1+4x^2\right)^2} = -4x\left(\frac{2}{1+4x^2}\right)^2$

Comparing with $\frac{d^2y}{dx^2} = ax\left(\frac{2}{1+4x^2}\right)^2$, $a = -4$

Q6 $xy^2 + y + \left(\log_e(x-2)\right)^2 = 14$

Implicit differentiation: $\frac{d}{dx}\left(xy^2\right) + \frac{dy}{dx} + \frac{d}{dx}\left(\log_e(x-2)\right)^2 = 0$

$\therefore\ y^2 + 2xy\frac{dy}{dx} + \frac{dy}{dx} + \frac{2\log_e(x-2)}{x-2} = 0$

At $(3,2)$, $4 + 13\frac{dy}{dx} = 0$, $\therefore\ \frac{dy}{dx} = -\frac{4}{13}$

Q7 $y = (x-1)\sqrt{2-x}$, $1 \le x \le 2$

Let $y = 0$ to find the x-intercepts: $(x-1)\sqrt{2-x} = 0$, $x = 1, 2$

$y > 0$ for $1 \le x \le 2$

Area of the region enclosed by the curve and the x-axis

$= \int_1^2 (x-1)\sqrt{2-x}\,dx$

$= \int_1^0 -(1-u)u^{\frac{1}{2}}du$

$= \int_0^1 (1-u)u^{\frac{1}{2}}du$

$= \int_0^1 \left(u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)du$

$= \left[\frac{2u^{\frac{3}{2}}}{3} - \frac{2u^{\frac{5}{2}}}{5}\right]_0^1 = \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$

Let $u = 2 - x$, $\therefore\ x - 1 = 1 - u$ and $\frac{du}{dx} = -1$

When $x = 1$, $u = 1$.

When $x = 2$, $u = 0$.

Q8 $v=\frac{2x}{\sqrt{1+x^2}}$, $\frac{1}{2}v^2=\frac{2x^2}{1+x^2}$

$a=\frac{d}{dx}\left(\frac{1}{2}v^2\right)=\frac{d}{dx}\left(\frac{2x^2}{1+x^2}\right)=\frac{(1+x^2)(4x)-(2x^2)(2x)}{(1+x^2)^2}$

$=\frac{4x}{(1+x^2)^2}$

Q9a $\tilde{r}(t)=\left(2\sqrt{t^2+2}-t^2\right)\tilde{i}+\left(2\sqrt{t^2+2}+2t\right)\tilde{j}$, $t\geq 0$

$\tilde{v}(t)=\frac{d\tilde{r}}{dt}=\left(\frac{2t}{\sqrt{t^2+2}}-2t\right)\tilde{i}+\left(\frac{2t}{\sqrt{t^2+2}}+2\right)\tilde{j}$

Q9b At $t=1$, $\tilde{v}=\left(\frac{2}{\sqrt{3}}-2\right)\tilde{i}+\left(\frac{2}{\sqrt{3}}+2\right)\tilde{j}$,

and the speed $|\tilde{v}|=\sqrt{\left(\frac{2}{\sqrt{3}}-2\right)^2+\left(\frac{2}{\sqrt{3}}+2\right)^2}=\frac{4\sqrt{6}}{3}$

Q9c $\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$, .: $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{2}{\sqrt{3}}+2}{\frac{2}{\sqrt{3}}-2}=\frac{1+\sqrt{3}}{1-\sqrt{3}}$ at $t=1$

Q9d At time $t=0$, $\tilde{r}=2\sqrt{2}\tilde{i}+2\sqrt{2}\tilde{j}$ and makes an angle of $\frac{\pi}{4}$ with the positive *x*-axis, whilst vector $-\sqrt{3}\tilde{i}+\tilde{j}$ makes an angle of $\frac{5\pi}{6}$ with the positive *x*-axis.

.: *angle between the two vectors* $=\frac{5\pi}{6}-\frac{\pi}{4}=\frac{7\pi}{12}$.

Q10ai $-1\leq\frac{x}{2}\leq 1$, .: $-2\leq x\leq 2$. The maximal domain of $f_1(x)=\sin^{-1}\left(\frac{x}{2}\right)$ is $[-2,2]$.

Q10aii $25x^2-1>0$, .: $x<-\frac{1}{5}$ or $x>\frac{1}{5}$. The maximal domain of $f_2(x)=\frac{3}{\sqrt{25x^2-1}}$ is $\left(-\infty,-\frac{1}{5}\right)\cup\left(\frac{1}{5},\infty\right)$.

Q10aiii $f(x)=\sin^{-1}\left(\frac{x}{2}\right)+\frac{3}{\sqrt{25x^2-1}}$ is defined over the intersection of the maximal domains of $f_1(x)$ and $f_2(x)$, i.e. $\left[-2,-\frac{1}{5}\right)\cup\left(\frac{1}{5},2\right]$.

Q10b $h(x)=\sin^{-1}\left(\frac{x}{2}\right)+\sin^{-1}(3x)$

Let $\theta=h\left(\frac{1}{4}\right)=\sin^{-1}\left(\frac{1}{8}\right)+\sin^{-1}\left(\frac{3}{4}\right)=\alpha+\beta$ where $\alpha=\sin^{-1}\left(\frac{1}{8}\right)$ and $\beta=\sin^{-1}\left(\frac{3}{4}\right)$.

.: $\sin\alpha=\frac{1}{8}$ and $\sin\beta=\frac{3}{4}$

.: $\cos\alpha=\frac{3\sqrt{7}}{8}$ and $\cos\beta=\frac{\sqrt{7}}{4}$ by the identity $\sin^2 A+\cos^2 A=1$.

.: $\sin\theta=\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$=\frac{1}{8}\times\frac{\sqrt{7}}{4}+\frac{3\sqrt{7}}{8}\times\frac{3}{4}=\frac{5\sqrt{7}}{16}$

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