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Unit 3 and 4 Specialist Mathematics: Exam 2

Practice Exam Solutions

Stop!

Don't look at these solutions until you have attempted the exam.

Found a mistake?

Check the Engage Education website for updated solutions, and then email practiceexams@ee.org.au.

Section A

Question 1

The correct answer is A.

Question 2

The correct answer is A.

Looking at the answers, should group $\cos(x - y + z)$ as both $\cos[(x - y) + z]$ and $\cos[x - (y + z)]$

Using trigonometric identities:

$\cos[(x - y) + z] = \cos(x - y)\cos(y) - \sin(x - y)\sin(y)$, which is not a selectable answer.

$\cos[x - (y + z)] = \cos(x)\cos(y + z) + \sin(x)\sin(y + z)$, which is answer A.

Question 3

The correct answer is B.

$$\begin{aligned} & \int \frac{1}{2} \sin(2x) \sqrt{1 - \cos x} \, dx \\ &= \int \sin(x) \cos(x) \sqrt{1 - \cos x} \, dx \end{aligned}$$

Notice how all the answers have $1 - \cos x$, so let $u = 1 - \cos x$, $\frac{du}{dx} = \sin(x)$, $1 - u = \cos(x)$

$$\begin{aligned} &= \int (1 - u) \sqrt{u} \, du \\ &= \int u^{0.5} - u^{1.5} \, du \\ &= \frac{2}{3} u^{1.5} - \frac{2}{5} u^{2.5} + c \\ &= \frac{2}{3} (1 - \cos x)^{1.5} - \frac{2}{5} (1 - \cos x)^{2.5} + c \end{aligned}$$

Now c could be zero, hence B is an anti-derivative.

Question 4

The correct answer is A.

$$f(x) = \tan^{-1}(x)$$

$$f'(x) = \frac{1}{x^2 + 1}$$

$$f''(x) = -\frac{2x}{(x^2 + 1)^2}$$

$$\text{Let } f'(x) = f''(x)$$

$$\frac{1}{x^2 + 1} = -\frac{2x}{(x^2 + 1)^2}$$

$$(x^2 + 1)^2 = -2x(x^2 + 1)$$

$$(x^2 + 1) = -2x$$

$$x^2 + 2x + 1 = 0$$

$$(x + 1)^2 = 0$$

When $f'(x) = f''(x)$, $x = -1$

Question 5

The correct answer is C.

Using linear approximation with step size of -1, $y(-2) = y(-1) - \frac{dy}{dx}(-1)$, $\frac{dy}{dx}(-1) \approx 0.79$

$$y(-2) \approx c - 0.79$$

Question 6

The correct answer is D.

Both D and E cannot be true.

Logically, 3 or more 2 dimensional vectors are dependant.

Alternatively,

$$\frac{1}{7}(-\mathbf{a} + 2\mathbf{b}) = \mathbf{j} \text{ and } \frac{1}{7}(3\mathbf{a} + \mathbf{b}) = \mathbf{i}$$

$$\mathbf{c} = 3\mathbf{a} - 2\mathbf{b}$$

Hence $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are dependent, and so D is the correct answer.

Question 7

The correct answer is D.

Graph is of the form $y = ax^3 + c$, with a and c as constants.

Question 8

The correct answer is B.

$$\frac{d(\mathbf{r}(t))}{dt} = \sec^2(t) \mathbf{i} + \tan(t) \sec^2(t) \mathbf{j}$$

$$\mathbf{v}\left(\frac{3\pi}{4}\right) = 2\mathbf{i} - 4\mathbf{j}$$

Question 9

The correct answer is A.

Question 10

The correct answer is D.

$$\text{area} = \int_0^{\frac{\pi}{2}} 2 \cos^{-1}(2x) dx = 1$$

Question 11

The correct answer is A.

$$\int \frac{e^{2x} - 1}{e^{2x} + 1} - \frac{e^{2x} + 1}{e^{2x} - 1} dx$$

$$= - \int \frac{4e^{2x}}{e^{4x} - 1} dx$$

$$\text{Let } u = e^{2x}, \frac{du}{dx} = 2e^{2x}$$

$$= - \int \frac{2}{u^2 - 1} du$$

$$= \int \frac{1}{u + 1} du - \int \frac{1}{u - 1} du$$

$$= \log_e(u + 1) - \log_e(u - 1)$$

$$= \log_e\left(\frac{u + 1}{u - 1}\right) + c$$

$$= \log_e\left(\frac{e^{2x} + 1}{e^{2x} - 1}\right) + c$$

Question 12

The correct answer is C.

$$\text{Let equation 1 be } \frac{x^2}{2} + y^2 = 1$$

$$\text{Let equation 2 be } \frac{x^2}{2} + y = c$$

$$\text{equation 1} - \text{equation 2: } y^2 - y = 1 - c$$

$$y^2 - y + c - 1 = 0$$

Using the quadratic formula:

$$y = \frac{1 \pm \sqrt{1 - 4c + 4}}{2}$$

For there to be real solutions for P,

$$1 - 4c + 4 \geq 0$$

$$c \leq \frac{5}{4}$$

Question 13

The correct answer is C.

Question 14

The correct answer is E.

Question 15

The correct answer is E.

Visually, $|z|$ is the distance from the origin to the point z . Using Pythagoras's rule, $\text{distance} = \sqrt{a^2 + b^2}$.

Question 16

The correct answer is E.

The xz -plane has normal vector $\hat{\mathbf{k}}$. The angle, α , between $\hat{\mathbf{k}}$ and $\mathbf{p}$ is given by:

$$\hat{\mathbf{k}} \cdot \mathbf{p} = |\hat{\mathbf{k}}||\mathbf{p}| \cos(\alpha)$$

$$\cos(\alpha) = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

The angle, θ , between the xz -plane and the vector $\mathbf{p}$ equals $90 - \alpha$.

$$\therefore \alpha = 90 - \theta$$

$$\therefore \cos(90 - \theta) = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

$$\cos(90 - \theta) = \sin(\theta)$$

$$\text{Hence } \theta = \sin^{-1}\left(\frac{c}{\sqrt{a^2 + b^2 + c^2}}\right)$$

Question 17

The correct answer is D.

$$(\mathbf{a} + \mathbf{b})(\mathbf{c} + \mathbf{d}) = \mathbf{0}, \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{d} = \mathbf{0} \quad (1)$$

$$(\mathbf{b} + \mathbf{c})(\mathbf{a} + \mathbf{d}) = \mathbf{0}, \mathbf{b} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{d} + \mathbf{c} \cdot \mathbf{a} = \mathbf{0} \quad (2)$$

$$(1) - (2): \mathbf{b} \cdot \mathbf{c} - \mathbf{c} \cdot \mathbf{d} - \mathbf{b} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{d} = \mathbf{0}$$

$$(\mathbf{c} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{d}) = \mathbf{0}$$

Since $\mathbf{c} - \mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} - \mathbf{d} \neq \mathbf{0}$

$\mathbf{c} - \mathbf{a}$ and $\mathbf{b} - \mathbf{d}$ are perpendicular.

Question 18

The correct answer is A.

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = -2(x-3)^3$$

$$\frac{1}{2}v^2 = -\frac{(x-3)^4}{2} + c$$

$$v^2 = -(x-3)^4 + 2c$$

At $x = 3 + \sqrt{2}$, $v = 0$, thus $c = 2$.

$$v^2 = 4 - (x-3)^4$$

Maximum velocity occurs when $(x - 3)^4 = 0, v = 2$

Minimum displacement from O when $v = 0, x = 3$. To see this more clearly, rearrange the expression to make x the subject:

$$(x - 3)^4 = 4 - v^2$$

$$x = 3 + \sqrt[4]{4 - v^2}$$

Question 19

The correct answer is B.

$$\text{Velocity} = \frac{\text{momentum}}{\text{mass}}$$

Hence written in $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ form, the change in velocity is from $(15, -5, 5)$ to $(5, 0, -5)$, where the magnitude of each of these vectors is the speed.

$$V_1 = \sqrt{15^2 + (-5)^2 + 5^2} = \sqrt{275} \approx 16.6$$

$$V_2 = \sqrt{5^2 + 0^2 + (-5)^2} = \sqrt{50} \approx 7.1$$

$V_2 - V_1$ is closest to -10.

Question 20

The correct answer is E.

$1.3 = \sqrt{1.2^2 + 0.5^2}$, hence it is a right angle triangle.

$$\text{Thus } \frac{T_s}{T_c} = \frac{1.2}{0.5} \approx 0.42$$

Question 21

The correct answer is B.

$$z + \bar{z} = a + bi + (a - bi) = 2a$$

Question 22

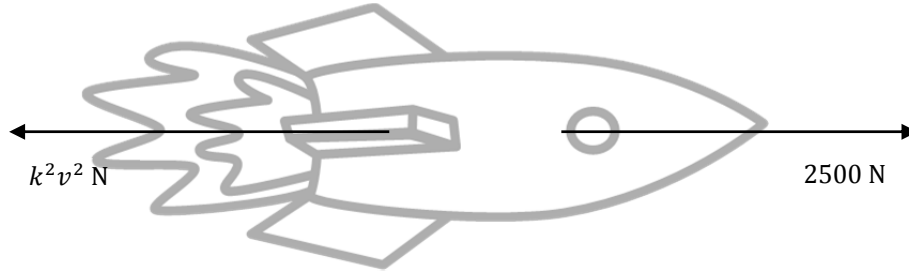
The correct answer is D.

Let $z = a \operatorname{cis}(b)$. Then $iz = \operatorname{cis}\left(\frac{\pi}{2}\right) \times a \operatorname{cis}(b) = a \operatorname{cis}\left(b + \frac{\pi}{2}\right)$, which is (geometrically speaking) z rotated 90° around the origin.

Section B

Marks allocated are indicated by a number in square brackets, for example, [1] indicates that the line is worth one mark.

Question 1a



[1 mark for both arrows in correct direction and with correct values]

Question 1b i

$$a = \frac{F}{m} = \frac{1}{1000}(2500 - k^2 v^2) \quad [1]$$

Question 1b ii

$$a = \frac{dv}{dt} = \frac{1}{1000}(2500 - k^2 v^2)$$

$$\frac{dt}{dv} = \frac{1000}{2500 - k^2 v^2}$$

$$t = \int \frac{1000}{2500 - k^2 v^2} dv + c \quad [1]$$

Use partial fractions:

$$\frac{1000}{2500 - k^2 v^2} = \frac{A}{50 - kv} + \frac{B}{50 + kv}$$

:

$$A = B = 10 \quad [1]$$

$$\therefore t = \int \frac{10}{50 - kv} dv + \int \frac{10}{50 + kv} dv + c$$

$$t = -\frac{10}{k} \log_e |50 - kv| + \frac{10}{k} \log_e |50 + kv| + c$$

$$t = \frac{10}{k} \log_e \left| \frac{50 + kv}{50 - kv} \right| + c \quad [1]$$

$$\text{When } t = 0, v = 0: 0 = \frac{10}{k} \log_e |1| + c$$

$$\therefore c = 0 \quad [1]$$

$$t = \frac{10}{k} \log_e \left| \frac{50 + kv}{50 - kv} \right|$$

$$e^{\frac{tk}{10}} = \frac{50 + kv}{50 - kv} \quad (\text{can remove the absolute value since } e^a > 0 \text{ for all } a \in \mathbb{R})$$

$$50e^{\frac{tk}{10}} - kve^{\frac{tk}{10}} = 50 + kv$$

$$50\left(e^{\frac{tk}{10}} - 1\right) = kv\left(1 + e^{\frac{tk}{10}}\right)$$

$$v = \frac{50\left(e^{\frac{tk}{10}} - 1\right)}{k\left(e^{\frac{tk}{10}} + 1\right)} = \frac{50(\alpha - 1)}{k(\alpha + 1)} [1]$$

Question 1b iii

$$\text{When } t = \frac{10}{k}, v = 50, \alpha = e^{\frac{10k}{10k}} = e$$

$$\therefore 50 = \frac{50(e - 1)}{k(e + 1)}$$

$$k = \frac{e-1}{e+1} [1]$$

Question 1c

$$t = \frac{10}{k} = \frac{10(e+1)}{e-1} \approx 22 \text{ s} [1]$$

Question 1d

$$v_m = \frac{50(e+1)}{e-1} [1]$$

Question 2a

$$\text{area} = \int_0^5 \left(\frac{1}{10}x^2 + 1\right) dx + \int_5^6 3.5 dx [1]$$

$$= \left[\frac{1}{30}x^3 + x\right]_0^5 + [3.5x]_5^6$$

$$= \frac{1}{30} \times 125 + 5 + 3.5 \times 6 - 3.5 \times 5$$

$$= \frac{38}{5} [1]$$

Question 2b

$$x^2 = 10(y - 1) [1]$$

$$V = \pi \int_1^{3.5} x^2 dy [1]$$

$$= 10\pi \int_1^{3.5} (y - 1) dy$$

$$= 10\pi \left[\frac{y^2}{2} - y\right]_1^{3.5}$$

$$= 10\pi \left(\frac{3.5^2}{2} - 3.5 - \frac{1}{2} + 1\right)$$

$$= \frac{250\pi}{8} = \frac{125\pi}{4} \text{ m}^2 [1]$$

Question 2c

$$\frac{dV}{dt} = \frac{4}{\pi} \text{ m}^3/\text{minute}$$

$$\therefore \text{it will take } \frac{125\pi}{4} \times \frac{4}{\pi} = 125 \text{ minutes to fill [1]}$$

During this time, the water level rises from 0 to 2.5 m.

$$\text{Average rate at which water rises} = \frac{2.5 \text{ m}}{125 \text{ min}} = \frac{2500 \text{ mm}}{125 \times 60 \text{ seconds}} = \frac{1}{3} \text{ mm/s [1]}$$

Question 2d i

$$V(h) = 10\pi \int_1^{1+h} (y+1) dy, 0 \leq h \leq 2.5 \text{ [1]}$$

Question 2d ii

$$V(h) = 10\pi \left[\frac{y^2}{2} - y \right]_1^{1+h}$$

$$= 10\pi \left(\frac{1+2h+h^2}{2} - h - \frac{1}{2} \right)$$

$$= 10\pi \left(\frac{(2h+h^2)}{2} - \frac{2h}{2} \right)$$

$$= 10\pi h^2 \text{ [1]}$$

$$\therefore \frac{dV}{dh} = 20\pi h \text{ [1]}$$

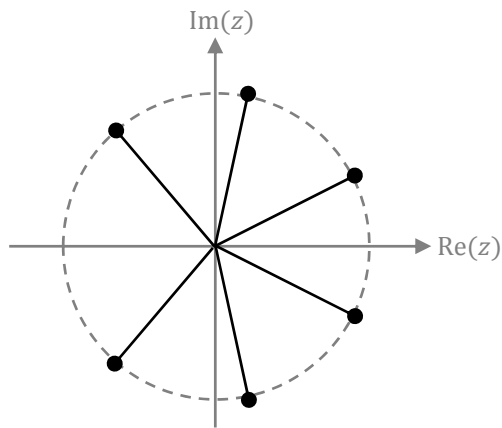
$$\frac{dh}{dt} = \frac{dh}{dV} \frac{dV}{dt} \text{ [1]}$$

$$= \frac{1}{20\pi h} \times \frac{\pi}{4} = \frac{1}{80h} \text{ [1]}$$

Question 3a

Use long division or CAS calculator to find: $P(z) = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$ [1]

Then read off values: $A = C = E = G = 1, B = D = F = -1$ [1]

Question 3b

Note that $z = -1$ due to the $z + 1$ term in $P(z)$.

[2 for roots]

Question 3c

$$\frac{z^7 + 1}{z + 1} = 0$$

$$z^7 = -1 = \text{cis}(-\pi) \quad [1]$$

$$z = \text{cis}\left(\frac{-\pi + 2n\pi}{7}\right), n = -2, -1, 0, 1, 2, 3 \text{ (note that } n = -3 \text{ is excluded due to the } z + 1 \text{ term in } P(z)) \quad [1]$$

$$z = \text{cis}\left(-\frac{5\pi}{7}\right), \text{cis}\left(-\frac{3\pi}{7}\right), \text{cis}\left(-\frac{\pi}{7}\right), \text{cis}\left(\frac{\pi}{7}\right), \text{cis}\left(\frac{3\pi}{7}\right), \text{cis}\left(\frac{5\pi}{7}\right) \quad [1]$$

Question 4a

$f = 0$ since the object is at rest [1]

Question 4b

Using resolution of forces:

Perpendicular to the plank: $N - mg \cos(30^\circ) = 0$, so $N = 21.22 \text{ N}$ [1]

Parallel to the plank: $f - mg \sin(30^\circ) = 0$, so $f = 12.25 \text{ N}$ (which is less than the maximum possible friction force, $\mu N = 14.85 \text{ N}$) [1]

Question 4c

Again, using resolution of forces:

Perpendicular to the plank: $N = mg \cos(40^\circ)$ [1]

Net force is parallel to the plank:

$$ma = f - mg \sin(40^\circ)$$

$$ma = \mu N - mg \sin(40^\circ)$$

$$ma = 0.7 \times 2.5 \times 9.8 \times \cos(40^\circ) - 2.5 \times 9.8 \times \sin(40^\circ)$$

$$a = -\frac{2.61}{2.5} = -1.04 \text{ m/s}^2 \text{ (ie. acceleration down the plank) [1 mark, must include direction]}$$

Question 4d

Use constant acceleration equation $v^2 = u^2 + 2as$ with $u = 0$, $a = 1.04 \text{ m/s}^2$, $d = 3 \text{ m}$ and v unknown:

$$v = \sqrt{2 \times 1.04 \times 3} = 2.50 \text{ m/s}^2 \text{ [1]}$$

$$p = mv = 2.5 \times 2.50 = 6.25 \text{ kg m/s [1]}$$

Question 4e

Use resolution of forces:

Perpendicular to plank:

$$\text{normal reaction force} - \text{gravity component} + \text{pulling force component} = 0$$

$$N - mg \cos(40^\circ) + 20 \sin(30^\circ) = 0 \text{ (30^\circ is the angle between the plank and the pulling force)}$$

$$N = 2.5 \times 9.8 \times \cos(40^\circ) - 20 \sin(30^\circ) = 8.77 \text{ N [1]}$$

Parallel to the plank, ignoring friction:

$$\text{pulling force component} - \text{gravity component} = 20 \cos(30^\circ) - mg \sin(40^\circ)$$

$$= 1.57 \text{ N up the plane [1]}$$

Therefore, friction will act to oppose the motion of the box up the plane (ie. it will act down the plane)

$$f_{\max} = \mu N = 0.7 \times 8.77 \text{ N [1]}$$

Since the maximum frictional force is greater than the resolved force, the net force is 0. [1]

Question 4f

It will remain stationary. [1]

Question 5a

$$AC = \mathbf{c} - \mathbf{a}, BC = \mathbf{c} - \mathbf{b}, BA = \mathbf{a} - \mathbf{b} \text{ [2]}$$

Question 5b

$$OM = \frac{1}{2}(\mathbf{b} + \mathbf{c}), ON = \frac{1}{2}(\mathbf{a} + \mathbf{c}), OP = \frac{1}{2}(\mathbf{a} + \mathbf{b}) \text{ [2]}$$

Question 5c i

We know that $OM \perp BC$ and $ON \perp AC$. Hence $OM \cdot BC = 0$ and $ON \cdot AC = 0$. [1]

$$\frac{1}{2}(\mathbf{b} + \mathbf{c}) \cdot (\mathbf{c} - \mathbf{b}) = 0$$

$$\mathbf{b} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{c} - \mathbf{c} \cdot \mathbf{b} = 0$$

$$|\mathbf{c}|^2 = |\mathbf{b}|^2$$

$$\therefore |\mathbf{c}| = |\mathbf{b}| \text{ since } |\mathbf{c}| > 0 \text{ and } |\mathbf{b}| > 0. \text{ [0.5]}$$

$$\frac{1}{2}(\mathbf{a} + \mathbf{c}) \cdot (\mathbf{c} - \mathbf{a}) = 0$$

$$\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{c} - \mathbf{c} \cdot \mathbf{a} = 0$$

$$|\mathbf{c}|^2 = |\mathbf{a}|^2$$

$$\therefore |\mathbf{c}| = |\mathbf{a}| \text{ since } |\mathbf{c}| > 0 \text{ and } |\mathbf{a}| > 0. [0.5]$$

$$\therefore |\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| [1]$$

Question 5c ii

$$\text{OP} \cdot \text{BA} = \frac{1}{2}(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \frac{1}{2}(\mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{b}) = \frac{1}{2}(|\mathbf{a}|^2 - |\mathbf{b}|^2) = \frac{1}{2}(|\mathbf{a}|^2 - |\mathbf{a}|^2) = 0$$

$$\therefore \text{OP} \perp \text{BA} \text{ since OP and BA} \neq \mathbf{0}. [1]$$

Question 5d

$$|\text{AC}|^2 = (\mathbf{c} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{a}) = \mathbf{c} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{a} = |\mathbf{c}|^2 + |\mathbf{a}|^2 - 2|\mathbf{a}||\mathbf{c}| \cos \alpha = d^2 + d^2 - 2d^2 \cos \alpha$$

$$= 2d^2(1 - \cos \alpha) [1]$$

$$\text{Similarly, } |\text{BC}|^2 = 2d^2(1 - \cos \beta) [1] \text{ and } |\text{BA}|^2 = 2d^2(1 - \cos \gamma) [1]$$

Hence,

$$|\text{AC}|^2 + |\text{BC}|^2 + |\text{BA}|^2 = 2d^2(1 - \cos \alpha + 1 - \cos \beta + 1 - \cos \gamma)$$

$$= 2d^2(3 - (\cos \alpha + \cos \beta + \cos \gamma)) [1]$$

Question 6a

$$x_0 = 1, y_0 = 1$$

$$x_1 = 1.1, y_1 = y(1.1) = y_0 + hf'(x_0) = 1 + 0.1 \left(\frac{2+1}{2} \right) = 1 + 0.1 \times 1.5 = 1.15 [1]$$

$$x_2 = 1.2, y_2 = y(1.2) = 1.15 + 0.1 \left(\frac{2.2+1}{2.1} \right) = 1.15 + 0.1 \times \frac{3.2}{2.1} = 1.302 [1]$$

Question 6b

$$\frac{dy}{dx} = \frac{2x+1}{x+1} = 2 - \frac{1}{x+1} \text{ using partial fractions or any other suitable method}$$

$$y = \int 2dx - \int \frac{1}{x+1} dx + c$$

$$y = 2x - \log_e |x+1| + c [1]$$

When $x = 1, y = 1$:

$$1 = 2 - \log_e 2 + c$$

$$c = \log_e 2 - 1$$

$$\therefore y = 2x - \log_e |x+1| + \log_e 2 - 1$$

$$y = 2x - 1 + \log_e \left| \frac{2}{x+1} \right|$$

$$y(1.2) = 2.4 - 1 + \log_e \left(\frac{2}{2.2} \right) = 1.4 + \log_e \left(\frac{1}{1.1} \right) [1]$$