

SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 1



2011 Trial Examination

SOLUTIONS

Question 1

The complex number $a + ib$ can be regarded as the rotation of the complex number $-\sqrt{3} - 3i$ anticlockwise by an angle of $\frac{\pi}{3}$ radians.....[M1]

$$\left. \begin{aligned} a + ib &= (-\sqrt{3} - 3i) \times \operatorname{cis}\left(\frac{\pi}{3}\right) \\ &= (-\sqrt{3} - 3i) \left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right) \end{aligned} \right\} \text{.....[M1]}$$

$$\left. \begin{aligned} &= (-\sqrt{3} - 3i) \left(\frac{1}{2} + i \frac{\sqrt{3}}{2} \right) = -\frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{2} + i \left(-\frac{3}{2} - \frac{3}{2} \right) \\ \therefore a + ib &= \sqrt{3} - 3i \quad \therefore a = \sqrt{3} \quad \text{and} \quad b = -3 \end{aligned} \right\} \text{.....[A1]}$$

Question 2

See the diagram

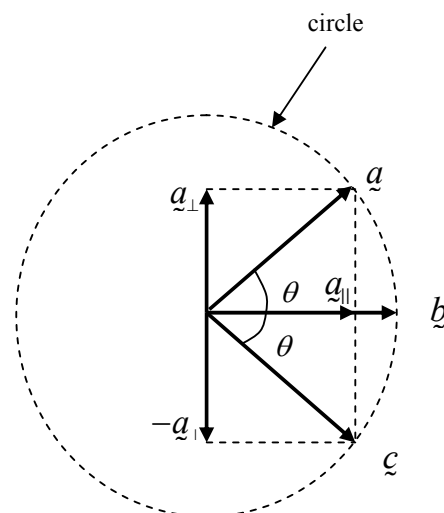
$$\underline{a} = \underline{i} - \underline{j} + \underline{k} \quad \text{and} \quad \underline{b} = \underline{i} + \underline{j} + \underline{k}$$

$$\underline{a}_{\parallel} = \frac{\underline{a} \cdot \underline{b}}{|\underline{b}|^2} \underline{b} = \frac{1-1+1}{1+1+1} (\underline{i} + \underline{j} + \underline{k}) = \frac{1}{3} \underline{i} + \frac{1}{3} \underline{j} + \frac{1}{3} \underline{k} \quad \dots\dots\dots [\text{M1}]$$

$$\underline{a}_{\perp} = \underline{a} - \underline{a}_{\parallel} \quad , \quad \underline{c} = \underline{a}_{\parallel} + (-\underline{a}_{\perp}) \quad \dots\dots\dots [\text{M1}]$$

$$\underline{c} = 2\underline{a}_{\parallel} - \underline{a} \quad \therefore \quad \underline{c} = 2\left(\frac{1}{3}\underline{i} + \frac{1}{3}\underline{j} + \frac{1}{3}\underline{k}\right) - (\underline{i} - \underline{j} + \underline{k}) \quad \dots\dots\dots [\text{A1}]$$

$$\underline{c} = -\frac{1}{3}\underline{i} + \frac{5}{3}\underline{j} - \frac{1}{3}\underline{k} \quad \dots\dots\dots [\text{A1}]$$

**Question 3****a.**

$$\sin^{-1} x + \frac{\pi}{6} = \sin^{-1} \left(\frac{\sqrt{3}}{2} x + \frac{1}{2} \sqrt{1-x^2} \right)$$

$$\left. \begin{array}{l} \text{We have to show that} \quad LHS = RHS \quad \text{for} \quad x = \frac{\sqrt{3}}{2} \\ LHS = \sin^{-1} \left(\frac{\sqrt{3}}{2} \right) + \frac{\pi}{6} = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2} \end{array} \right\} \dots\dots\dots [\text{A1}]$$

$$\left. \begin{array}{l} RHS = \sin^{-1} \left(\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \sqrt{1 - \left(\frac{\sqrt{3}}{2} \right)^2} \right) \\ = \sin^{-1} \left(\frac{3}{4} + \frac{1}{2} \times \frac{1}{2} \right) = \sin^{-1}(1) = \frac{\pi}{2} \quad \dots\dots\dots \therefore LHS = RHS \end{array} \right\} \dots\dots\dots [\text{A1}]$$

b.

$$\left. \begin{array}{l} \text{Apply sine to both sides:} \\ \sin\left(\sin^{-1}x + \frac{\pi}{6}\right) = \sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}x + \frac{1}{2}\sqrt{1-x^2}\right)\right) \\ \therefore \sin\left(\sin^{-1}x + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}x + \frac{1}{2}\sqrt{1-x^2} \end{array} \right\} \dots\dots\dots [\text{M1}]$$

$$\left. \begin{array}{l} \text{Let } LHS = \sin\left(\sin^{-1}x + \frac{\pi}{6}\right) \quad \text{and} \quad RHS = \frac{\sqrt{3}}{2}x + \frac{1}{2}\sqrt{1-x^2} \\ LHS = \sin\left(\sin^{-1}x + \frac{\pi}{6}\right) = \sin(\sin^{-1}x) \times \cos\left(\frac{\pi}{6}\right) + \cos(\sin^{-1}x) \times \sin\left(\frac{\pi}{6}\right) \end{array} \right\} \dots\dots\dots [\text{M1}]$$

$$\left. \begin{array}{l} \text{As } \sin^{-1}x \text{ is an angle in the first quadrant, } \cos(\sin^{-1}x) = \sqrt{1-x^2} \\ \therefore LHS = \frac{\sqrt{3}}{2}x + \frac{1}{2}\sqrt{1-x^2} \quad \therefore LHS = RHS \end{array} \right\} \dots\dots\dots [\text{A1}]$$

Question 4

$$\left. \begin{array}{l} 3x^2 - 2xy + 2y^3 = 7 \quad y = 1, x < 0 \\ 3x^2 - 2x + 2 = 7 \quad \therefore 3x^2 - 2x - 5 = 0 \quad \therefore (x+1)(3x-5) = 0 \\ \therefore x = -1 \text{ as } x < 0 \end{array} \right\} \dots\dots[\text{M1}]$$

$$\left. \begin{array}{l} \frac{d}{dx}(3x^2 - 2xy + 2y^3) = \frac{d}{dx}(7) \\ 6x - 2y - 2x \frac{dy}{dx} + 6y^2 \frac{dy}{dx} = 0 \\ -2x \frac{dy}{dx} + 6y^2 \frac{dy}{dx} = -6x + 2y \end{array} \right\} \dots\dots[\text{M1}]$$

$$\left. \begin{array}{l} \text{Find } \frac{dy}{dx} \text{ at } (-1, 1) \\ -2 \times (-1) \frac{dy}{dx} + 6 \times 1 \times \frac{dy}{dx} = 6 \times (-1) + 2 \times 1 \\ 8 \times \frac{dy}{dx} = 8 \quad \therefore \frac{dy}{dx} = 1 \end{array} \right\} \dots\dots[\text{A1}]$$

$$\text{Equation of the normal : } y - 1 = -(x + 1) \quad \therefore y = -x \quad \dots\dots[\text{A1}]$$

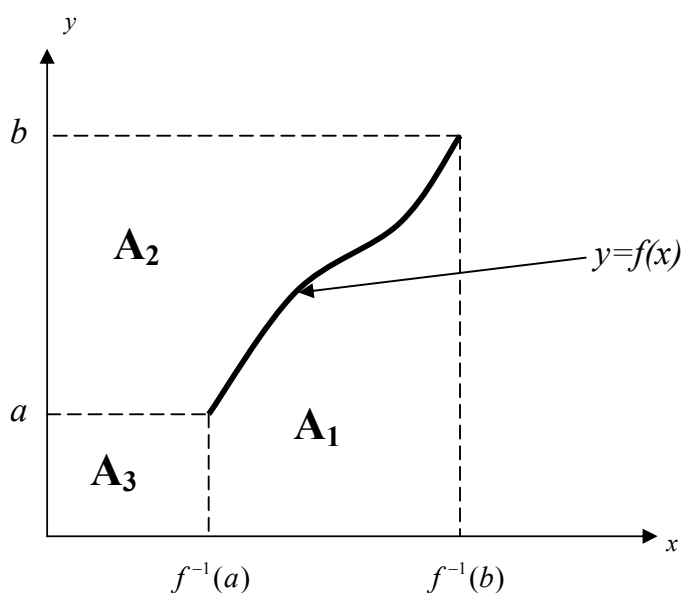
Question 5

See the diagram below

$$\left. \begin{array}{l} \text{Let } A \text{ be the area of the big rectangle , } A = b f^{-1}(b) \\ \text{As seen from the diagram, this area } A \text{ can be} \\ \text{written as the sum of three areas:} \\ A = A_1 + A_2 + A_3 \end{array} \right\} \dots\dots[M1]$$

$$\left. \begin{array}{l} A_1 = \int_{f^{-1}(a)}^{f^{-1}(b)} y dx \quad , \quad y = f(x) \quad \therefore \quad A_1 = \int_{f^{-1}(a)}^{f^{-1}(b)} f(x) dx \\ A_2 = \int_a^b x dy \quad , \quad x = f^{-1}(y) \quad \therefore \quad A_2 = \int_a^b f^{-1}(y) dy \end{array} \right\} \dots\dots[M1]$$

$$\left. \begin{array}{l} A_3 = \text{the area of the small rectangle , } A_3 = a f^{-1}(a) \\ b f^{-1}(b) = \int_{f^{-1}(a)}^{f^{-1}(b)} f(x) dx + \int_a^b f^{-1}(y) dy + a f^{-1}(a) \\ \int_{f^{-1}(a)}^{f^{-1}(b)} f(x) dx + \int_a^b f^{-1}(y) dy = b f^{-1}(b) - a f^{-1}(a) \\ \text{as required} \end{array} \right\} \dots\dots[A1]$$



Question 6

Refer to the graph below

$$\int_{-3}^{-1} \log_e |x| dx \text{ is equal to the area } A$$

But also, from symmetry,

$$A = \int_1^3 \log_e |x| dx = \int_1^3 \log_e (x) dx \quad \text{as } |x| = x$$

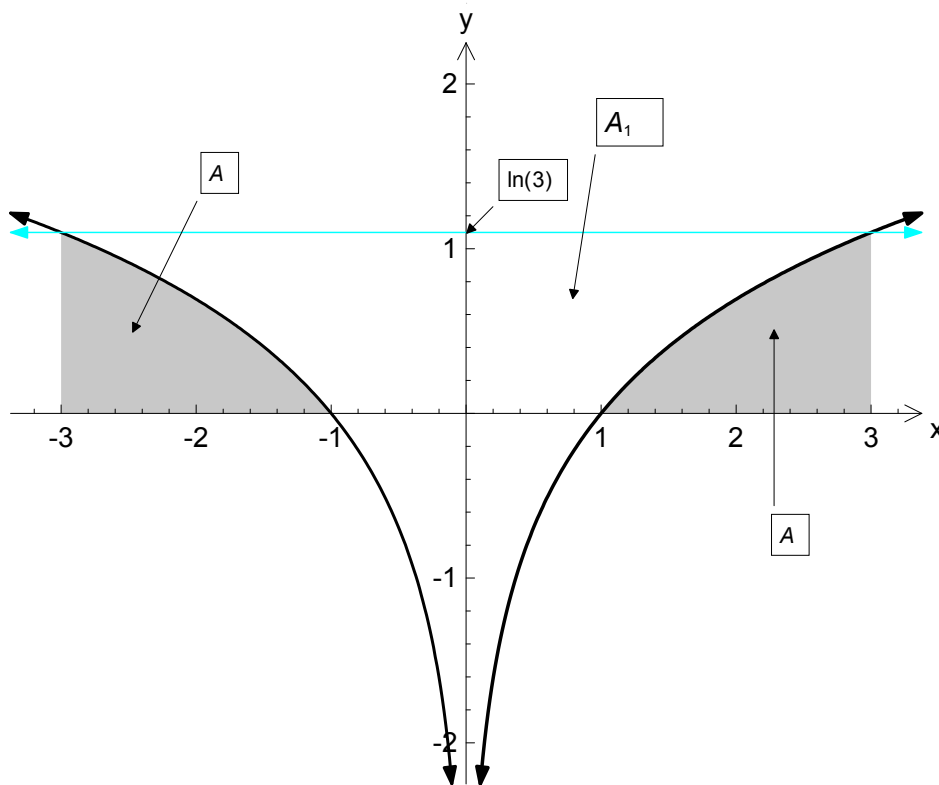
.....[M1]

$$A_1 = \int_0^{\log_e 3} x dy, \quad y = \log_e (x) \quad \therefore \quad x = e^y$$

$$\therefore A_1 = \int_0^{\log_e 3} e^y dy = [e^y]_0^{\log_e 3} = 2$$

.....[M1]

$$A = 3 \log_e 3 - 2 \quad \text{.....[A1]}$$



Question 7**a.**

$$\left. \begin{aligned} r_A(t) &= \int (-2\hat{i} + 3\hat{j}) dt \quad \therefore \quad r_A(t) = (-2\hat{i} + 3\hat{j})t + \underline{c}_1 \\ \text{At } t=0 \quad r_A(0) &= 4\hat{i} - 3\hat{j} \quad \therefore \quad \underline{c}_1 = 4\hat{i} - 3\hat{j} \\ \therefore \quad r_A(t) &= (-2\hat{i} + 3\hat{j})t + 4\hat{i} - 3\hat{j} \quad \therefore \quad r_A(t) = (-2t + 4)\hat{i} + (3t - 3)\hat{j} \end{aligned} \right\} \dots\dots[\text{M1}]$$

$$\left. \begin{aligned} r_B(t) &= \int \underline{v}_B dt \quad \therefore \quad r_B(t) = \underline{v}_B t + \underline{c}_2 \\ \text{At } t=0 \quad r_B(0) &= -\hat{i} + \hat{j} \quad \therefore \quad \underline{c}_2 = -\hat{i} + \hat{j} \quad \therefore \quad r_B(t) = \underline{v}_B t - \hat{i} + \hat{j} \end{aligned} \right\} \dots\dots[\text{M1}]$$

$$\left. \begin{aligned} \text{When collision occurs} \quad r_A(1) &= r_B(1) \\ (-2 \times 1 + 4)\hat{i} + (3 \times 1 - 3)\hat{j} &= \underline{v}_B \times 1 - \hat{i} + \hat{j} \\ 2\hat{i} = \underline{v}_B - \hat{i} + \hat{j} \quad \therefore \quad \underline{v}_B &= 3\hat{i} - \hat{j} \end{aligned} \right\} \dots\dots[\text{A1}]$$

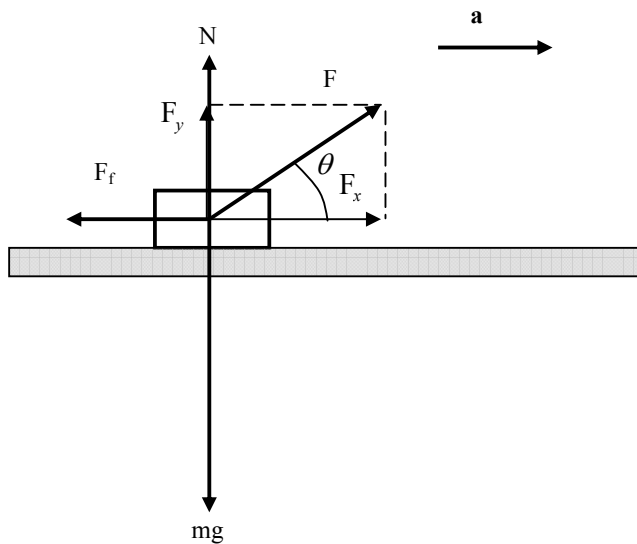
b.

$$\left. \begin{aligned} d &= |r_B(1) - r_A(0)| = |\underline{v}_B - \hat{i} + \hat{j} - (4\hat{i} - 3\hat{j})| \\ d &= |\underline{v}_B| = \sqrt{3^2 + 1} \quad \therefore \quad d = \sqrt{10} \end{aligned} \right\} \dots\dots[\text{A1}]$$

Question 8**a.**

See the diagram below

Draw in the weight, the
normal reaction force and
the friction force.....[A1]

**b.**

$$\left. \begin{aligned} F_x - F_f &= ma \\ N + F_y - mg &= 0 \\ F_f &= \mu N \end{aligned} \right\} \dots\dots[M1]$$

$$\Rightarrow \left. \begin{aligned} F \cos \theta - F_f &= ma \\ N + F \sin \theta - mg &= 0 \\ F_f &= \mu N \end{aligned} \right\} \dots\dots[M1]$$

$$\Rightarrow F \cos \theta - \mu(mg - F \sin \theta) = ma \dots\dots[M1]$$

$$\left. \begin{aligned} F(\cos \theta + \mu \sin \theta) - \mu mg &= ma \\ \therefore a &= \frac{F}{m}(\cos \theta + \mu \sin \theta) - \mu g \end{aligned} \right\} \dots\dots[A1]$$

Question 9**a.**

$$\left. \begin{array}{l} x = 2 \cos(nt + \varepsilon) \quad , \quad y = 3 \sin(nt + \varepsilon) \\ t = 0 \quad x = -2 \quad y = 0 \quad \therefore \quad -2 = 2 \cos \varepsilon \quad \text{and} \quad 0 = 3 \sin \varepsilon \\ \therefore \quad \varepsilon = \pi \quad \therefore \quad x = 2 \cos(nt + \pi) \quad \text{and} \quad y = 3 \sin(nt + \pi) \\ \therefore \quad x = -2 \cos(nt) \quad \text{and} \quad y = -3 \sin(nt) \end{array} \right\} \dots\dots[M1]$$

$$\left. \begin{array}{l} \underline{r}(t) = -2 \cos(nt) \underline{i} - 3 \sin(nt) \underline{j} \quad \therefore \quad \underline{v}(t) = 2n \sin(nt) \underline{i} - 3n \cos(nt) \underline{j} \\ \underline{v}(0) = 2n \sin(n \times 0) \underline{i} - 3n \cos(n \times 0) \underline{j} \quad \therefore \quad \underline{v}(0) = -3n \end{array} \right\} \dots\dots[M1]$$

$$speed = |\underline{v}(0)| \quad \therefore \quad 3n = 6 \quad \therefore \quad n = 2 \therefore \quad x = -2 \cos(2t) \quad y = -3 \sin(2t) \quad \dots\dots[A1]$$

b.

$$\underline{r}(t) = -2 \cos(2t) \underline{i} - 3 \sin(2t) \underline{j} \quad \therefore \quad \underline{v}(t) = 4 \sin(2t) \underline{i} - 6 \cos(2t) \underline{j} \quad \dots\dots[M1]$$

$$\left. \begin{array}{l} T = \pi \quad \therefore \quad \underline{v}\left(\frac{\pi}{4}\right) = 4 \sin\left(2 \times \frac{\pi}{4}\right) \underline{i} - 6 \cos\left(2 \times \frac{\pi}{4}\right) \underline{j} = 4 \underline{i} \\ \therefore \quad speed = 4ms^{-1} \end{array} \right\} \dots\dots[A1]$$

Question 10**a.**

$$\left. \begin{aligned} a &= -kv \quad \therefore \quad \frac{dv}{dt} = -kv \quad \therefore \quad \frac{dt}{dv} = -\frac{1}{kv} \\ \therefore \quad t &= -\frac{1}{k} \int \frac{dv}{v} \quad \therefore \quad t = -\frac{1}{k} \log_e v + c \\ \text{At } t=0 \quad v &= u \quad \therefore \quad 0 = -\frac{1}{k} \log_e u + c \quad \therefore \quad c = \frac{1}{k} \log_e u \end{aligned} \right\} \dots\dots\dots [\text{M1}]$$

$$\left. \begin{aligned} t &= -\frac{1}{k} \log_e v + \frac{1}{k} \log_e u \quad \therefore \quad t = \frac{1}{k} \log_e \left(\frac{u}{v} \right) \quad \therefore \quad \frac{u}{v} = e^{kt} \quad \therefore \quad v = ue^{-kt} \\ \frac{u}{e} &= ue^{-2k} \quad \therefore \quad e^{-2k} = e^{-1} \quad \therefore \quad 2k = 1 \quad \therefore \quad k = \frac{1}{2} \quad \text{and} \quad v(t) = ue^{-\frac{1}{2}t} \end{aligned} \right\} \dots\dots\dots [\text{A1}]$$

b.

$$d = \int_0^2 ue^{-\frac{1}{2}t} dt = -2u \left[e^{-\frac{1}{2}t} \right]_0^2 \dots\dots\dots [\text{M1}]$$

$$d = -2u(e^{-1} - 1) = 2u \left(1 - \frac{1}{e} \right) \dots\dots\dots [\text{A1}]$$