



Trial Examination 2011

VCE Specialist Mathematics Units 3 & 4

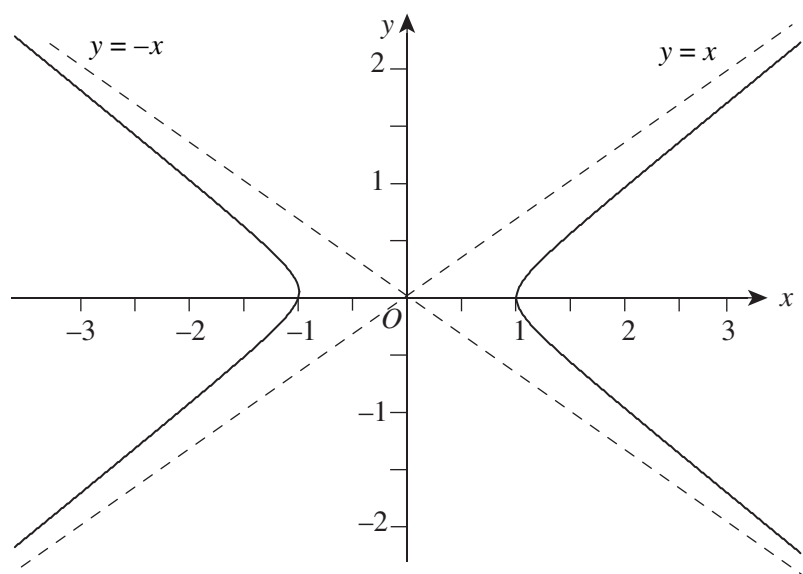
Written Examination 1

Suggested Solutions

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Question 1

a.

Two correct branches crossing the x -axis at $(\pm 1, 0)$.

A1

Asymptotes $y = \pm x$.

A1

b. Let the volume be V , where $V = \pi \int_1^{\sqrt{3}} y^2 dx$ and $y^2 = x^2 - 1$.

$$= \pi \left[\frac{x^3}{3} - x \right]_1^{\sqrt{3}}$$

A1

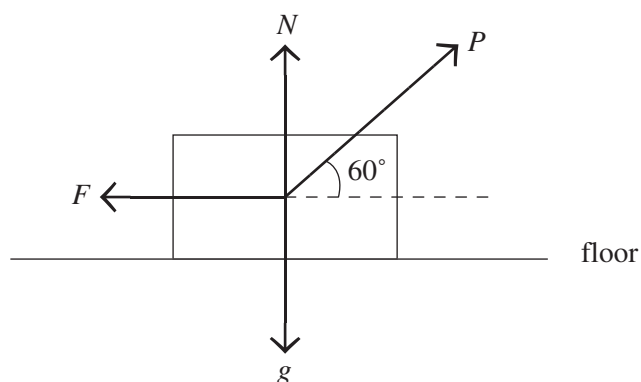
$$= \pi \left[(\sqrt{3} - \sqrt{3}) - \left(\frac{1}{3} - 1 \right) \right]$$

$$= \frac{2\pi}{3} \text{ (cubic units)}$$

A1

Question 2

a.



A1

b. The body is moving, hence $F = \frac{N}{\sqrt{2}}$.

Vertical: $N = g - P \sin(60^\circ)$, i.e. $N = g - \frac{\sqrt{3}P}{2}$. A1

Horizontal: $a = P \cos(60^\circ) - \frac{N}{\sqrt{2}}$, i.e. $a = \frac{P}{2} - \frac{N}{\sqrt{2}}$. A1

Substituting $N = g - \frac{\sqrt{3}P}{2}$ into $a = \frac{P}{2} - \frac{N}{\sqrt{2}}$ gives $a = \frac{P}{2} - \frac{1}{\sqrt{2}}\left(g - \frac{\sqrt{3}P}{2}\right)$.

$$a = \frac{P}{2} - \frac{g}{\sqrt{2}} + \frac{\sqrt{3}P}{2\sqrt{2}} \text{ (or equivalent)}$$

So $a = \frac{P - \sqrt{2}g}{2} + \frac{\sqrt{6}P}{4}$. A1

Note: $a = \frac{P}{2} - \frac{g}{\sqrt{2}} + \frac{\sqrt{3}P}{2\sqrt{2}}$ (or equivalent) is needed for the final A1.

Question 3

Let $u = \sin(x)$ and so $\frac{du}{dx} = \cos(x)$. When $x = 0$, $u = 0$ and when $x = \frac{\pi}{2}$, $u = 1$.

So $\int_0^{\frac{\pi}{2}} \frac{\cos(x)}{1 + \sin^2(x)} dx = \int_0^1 \frac{1}{1 + u^2} du$ A1

$$\begin{aligned} &= [\tan^{-1}(u)]_0^1 \\ &= \tan^{-1}(1) - \tan^{-1}(0) \\ &= \frac{\pi}{4} \end{aligned}$$

A1

Question 4

Using the scalar product, i.e. $\cos(\theta) = \frac{(\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (5\mathbf{i} - \mathbf{j} - \mathbf{k})}{\sqrt{3}\sqrt{27}}$ and so $\cos(\theta) = \frac{1}{3}$. M1

Using $\cos(\theta) = 2\cos^2\left(\frac{\theta}{2}\right) - 1$, we obtain $2\cos^2\left(\frac{\theta}{2}\right) - 1 = \frac{1}{3}$. M1

Rearranging $2\cos^2\left(\frac{\theta}{2}\right) - 1 = \frac{1}{3}$, we obtain $\cos^2\left(\frac{\theta}{2}\right) = \frac{2}{3}$. A1

As θ is an acute angle, we reject $\cos\left(\frac{\theta}{2}\right) = -\sqrt{\frac{2}{3}}$ and so $\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{2}{3}}$. A1

Question 5

Using implicit differentiation to differentiate $x^2 + xy = e^y$: M1

$$2x + y + x \frac{dy}{dx} = e^y \frac{dy}{dx} \text{ (or equivalent)} \quad \text{A1}$$

Let the gradient of the normal be m_N .

At $(-1, 0)$, $\frac{dy}{dx} = -1$ and so m_N is 1. A1

So the equation of the normal is $y = x + 1$. A1

Question 6

Attempting to use $a = \frac{d}{dx} \left(\frac{1}{2} v^2 \right)$. M1

$$\frac{d}{dx} \left(\frac{1}{2} v^2 \right) = \frac{3}{x^3} - \frac{1}{x^2}$$

$$\frac{1}{2} v^2 = \int \left(\frac{3}{x^3} - \frac{1}{x^2} \right) dx$$

$$= \frac{-3}{2x^2} + \frac{1}{x} + c \text{ where } c \text{ is an arbitrary constant} \quad \text{A1}$$

So $v^2 = \frac{-3}{x^2} + \frac{2}{x} + k$, where k is an arbitrary constant.

Given that $v = 0$ at $x = 1$, we find that $0 = -3 + 2 + k$, i.e. $k = 1$.

Hence, $v^2 = \frac{-3}{x^2} + \frac{2}{x} + 1$. M1

Writing as a single fraction, we obtain $v^2 = \frac{x^2 + 2x - 3}{x^2}$.

Taking the square root of both sides we obtain $v = \pm \sqrt{\frac{x^2 + 2x - 3}{x^2}}$.

As $a > 0$ when $x = 1$, v is initially positive. A1

Hence, $v = \frac{\sqrt{x^2 + 2x - 3}}{x}$ for $x \geq 1$. A1

Question 7

$$\text{a. } |\tilde{r}(t)| = \sqrt{\sin^2(2t) + 4\cos^2(t)} \quad \text{M1}$$

$$= \sqrt{4\sin^2(t)\cos^2(t) + 4\cos^2(t)} \quad (\text{using } \sin(2t) = 2\sin(t)\cos(t)) \quad \text{A1}$$

$$= \sqrt{4\cos^2(t)(1 + \sin^2(t))} \quad \text{A1}$$

$$= 2\sqrt{(1 - \sin^2(t))(1 + \sin^2(t))} \quad (\text{using } \cos^2(t) = 1 - \sin^2(t)) \quad \text{A1}$$

$$\text{Hence } |\tilde{r}(t)| = 2\sqrt{1 - \sin^4(t)}. \quad \text{A1}$$

Note: Only award the last A1 if the previous line of work is present.

$$\text{b. } \text{The minimum value of } \sin^4(t) \text{ is zero, and so } |\tilde{r}(t)|_{\max} = 2 \quad \text{A1}$$

$$|\tilde{r}(t)|_{\max} \text{ occurs at } t = n\pi, \text{ where } n = 0, 1, 2, 3, \dots \quad \text{A1}$$

Question 8

$$(z^4 - \text{cis}(\theta)) = 0 \text{ or } (z^4 - \text{cis}(-\theta)) = 0 \text{ where } 2\cos(\theta) = 1.$$

$$2\cos(\theta) = 1 \text{ and so } \theta = \frac{\pi}{3}. \quad \text{A1}$$

$$z^4 = \text{cis}\left(\frac{\pi}{3}\right) \text{ or } z^4 = \text{cis}\left(-\frac{\pi}{3}\right). \quad \text{A1}$$

$$z = \text{cis}\left(\frac{\pi}{12} + \frac{2k\pi}{4}\right) \text{ or } z = \text{cis}\left(-\frac{\pi}{12} + \frac{2k\pi}{4}\right), \text{ where } k \in \mathbb{Z}. \quad \text{M1}$$

$$\text{Hence, } z = \text{cis}\left(\pm\frac{\pi}{12}\right), \text{cis}\left(\pm\frac{5\pi}{12}\right), \text{cis}\left(\pm\frac{7\pi}{12}\right), \text{cis}\left(\pm\frac{11\pi}{12}\right). \quad \text{A1 A1}$$

$$\text{Note: Award A1 for } \text{cis}\left(\pm\frac{\pi}{12}\right) \text{ and } \text{cis}\left(\pm\frac{5\pi}{12}\right), \text{ and A1 for } \text{cis}\left(\pm\frac{7\pi}{12}\right) \text{ and } \text{cis}\left(\pm\frac{11\pi}{12}\right).$$

Question 9

a.
$$\frac{\tan\left(\tan^{-1}\left(\frac{1}{m}\right)\right) + \tan\left(\tan^{-1}\left(\frac{1}{n}\right)\right)}{1 - \tan\left(\tan^{-1}\left(\frac{1}{m}\right)\right)\tan\left(\tan^{-1}\left(\frac{1}{n}\right)\right)} = \tan\left(\frac{\pi}{4}\right) \quad (\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)})$$

So
$$\frac{\frac{1}{m} + \frac{1}{n}}{1 - \frac{1}{m} \times \frac{1}{n}} = 1 \quad \text{M1}$$

$$\frac{m+n}{mn-1} = 1 \quad (\text{multiplying numerator and denominator of the LHS by } mn) \quad \text{A1}$$

$$mn - m - n - 1 = 0$$

$$m(n-1) - n - 1 = 0$$

$$m(n-1) - n + 1 = 2 \quad (\text{adding 2 to both sides}) \quad \text{M1}$$

$$m(n-1) - 1(n-1) = 2$$

$$(m-1)(n-1) = 2 \quad \text{A1}$$

b. From $(m-1)(n-1) = 2$, we obtain $n = \frac{2}{m-1} + 1$, i.e. $n = \frac{m+1}{m-1}$. A1

Given that $m = k$, $\frac{1}{n} = \frac{k-1}{k+1}$, we obtain $\tan^{-1}\left(\frac{1}{k}\right) + \tan^{-1}\left(\frac{k-1}{k+1}\right) = \frac{\pi}{4}$. A1