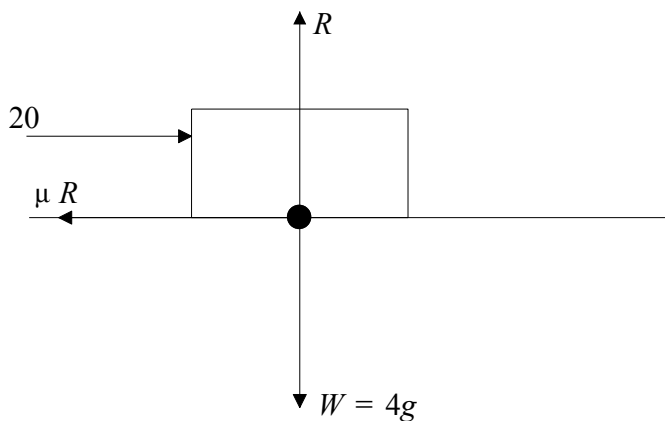


The Mathematical Association of Victoria
Trial Examination 2011
Specialist Maths Examination 1 - SOLUTIONS

Question 1**a.**Normal reaction force R .Weight force $W = 4g$.Friction force μR (since object is moving).

Pushing force of 20 Newton.



All forces labeled [A1]

b.

Net force in **vertical direction**: $\left. \begin{array}{l} F_{net} = 0 \\ F_{net} = R - 4g \end{array} \right\} \Rightarrow R = 4g . \quad \dots (1)$

Net force in **horizontal direction**: $\left. \begin{array}{l} F_{net} = ma = (4)(2) = 8 \\ F_{net} = 20 - \mu R \end{array} \right\} \Rightarrow 8 = 20 - \mu R \quad [M1]$
 $\Rightarrow 12 = \mu R . \quad \dots (2)$

Substitute equation (1) into equation (2):

$$12 = \mu 4g$$

$$\Rightarrow \mu = \frac{3}{g} . \quad [A1]$$

Total 3 marks

Question 2

Let $\vec{a} = -\vec{i} + \vec{j} - 2\vec{k}$ and $\vec{b} = 3\vec{i} - 2\vec{j} - 6\vec{k}$.

Parallel vector resolute:

$$\vec{u} = \left(\vec{a} \cdot \hat{\vec{b}} \right) \hat{\vec{b}}.$$

$$|\vec{b}| = \sqrt{3^2 + (-2)^2 + (-6)^2} = \sqrt{49} = 7.$$

$$\text{Therefore } \hat{\vec{b}} = \frac{\vec{b}}{|\vec{b}|} = \frac{3\vec{i} - 2\vec{j} - 6\vec{k}}{7}. \quad [\text{A1}]$$

$$\text{Therefore } \vec{u} = \left(\frac{(-\vec{i} + \vec{j} - 2\vec{k}) \cdot (3\vec{i} - 2\vec{j} - 6\vec{k})}{7} \right) \left(\frac{3\vec{i} - 2\vec{j} - 6\vec{k}}{7} \right) = \left(\frac{-3 - 2 + 12}{7} \right) \left(\frac{3\vec{i} - 2\vec{j} - 6\vec{k}}{7} \right)$$

$$= \frac{1}{7} (3\vec{i} - 2\vec{j} - 6\vec{k}). \quad [\text{M1}]$$

Perpendicular vector resolute:

$$\vec{a} = \vec{u} + \vec{v} \Rightarrow \vec{v} = \vec{a} - \vec{u}$$

$$\Rightarrow \vec{v} = -\vec{i} + \vec{j} - 2\vec{k} - \left(\frac{3}{7}\vec{i} - \frac{2}{7}\vec{j} - \frac{6}{7}\vec{k} \right)$$

$$= -\frac{10}{7}\vec{i} + \frac{9}{7}\vec{j} - \frac{8}{7}\vec{k}. \quad [\text{A1}]$$

Total 3 marks

Question 3

Substitute $y = \sqrt{3} - 2$ into $(x+1)^2 + \frac{(y+2)^2}{4} = 1$:

$$(x+1)^2 + \frac{3}{4} = 1$$

$$\Rightarrow x+1 = \pm \frac{1}{2}$$

$$\Rightarrow x = -\frac{1}{2}, -\frac{3}{2}.$$

But $x > -1$.

$$\text{Therefore } x = -\frac{1}{2}.$$

[A1]

$$\text{Implicit differentiation: } 2(x+1) + \frac{2(y+2)}{4} \times \frac{dy}{dx} = 0.$$

[M1]

Substitute $x = -\frac{1}{2}$ and $y = \sqrt{3} - 2$ into $2(x+1) + \frac{2(y+2)}{4} \times \frac{dy}{dx} = 0$:

$$1 + \frac{2\sqrt{3}}{4} \times \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{4}{2\sqrt{3}} = -\frac{2}{\sqrt{3}}.$$

$$m_{\text{normal}} = \frac{-1}{\frac{dy}{dx}} = \frac{\sqrt{3}}{2}.$$

[A1]**Total 3 marks**

Question 4

$$f(-i)=0 \Rightarrow z+i \text{ is a factor of } f(z)=z^3+2iz^2-2z-i.$$

Note: $z-i$ is NOT a second factor because the conjugate root theorem is NOT valid (the coefficients of the polynomial are not all real).

Polynomial long division:

$$\begin{array}{r} z^2 + iz - 1 \\ z+i \overline{) z^3 + 2iz^2 - 2z - i} \\ \underline{z^3 + iz^2} \\ iz^2 - 2z - i \\ \underline{iz^2 - z} \\ -z - i \\ \underline{-z - i} \\ 0 \end{array} \quad \text{[M2]}$$

Therefore the remaining roots of $f(z)=z^3+2iz^2-2z-i$ are solutions to $z^2+iz-1=0$.

Quadratic formula:

$$\begin{aligned} z &= \frac{-i \pm \sqrt{(i)^2 - 4(1)(-1)}}{2} \\ &= \frac{-i \pm \sqrt{3}}{2}. \end{aligned}$$

$$\text{Roots: } z = \frac{-i \pm \sqrt{3}}{2}, -i.$$

[A1]

Total 3 marks

Question 5**a.****Option 1:**

$$(z - \sqrt{2})(\bar{z} - \sqrt{2}) = 2$$

$$\Rightarrow (z - \sqrt{2})(\overline{z - \sqrt{2}}) = 2$$

$$\Rightarrow |z - \sqrt{2}|^2 = 2$$

$$\Rightarrow |z - \sqrt{2}| = \sqrt{2}.$$

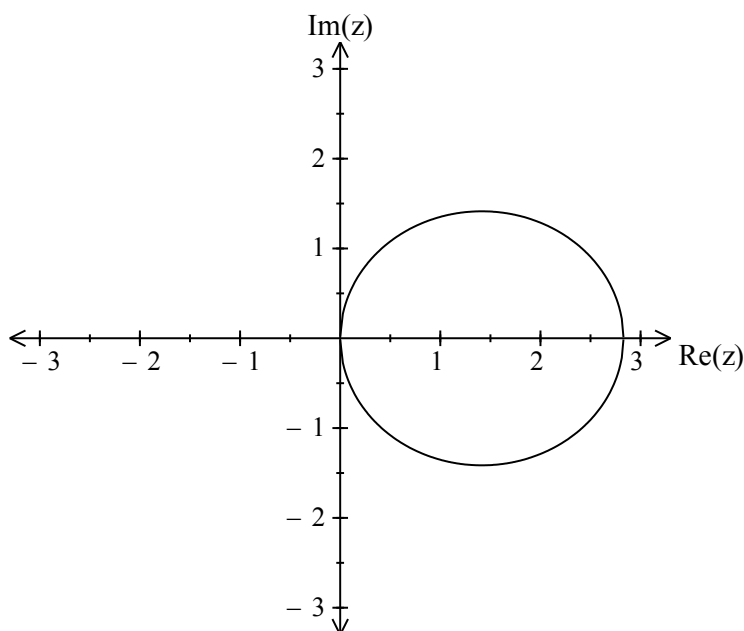
[M1]**Option 2:**Substitute $z = x + iy$:

$$(z - \sqrt{2})(\bar{z} - \sqrt{2}) = 2$$

$$\Rightarrow (x - \sqrt{2} + iy)(x - \sqrt{2} - iy) = 2$$

$$\Rightarrow x^2 - 2\sqrt{2}x + y^2 + 2 = 2$$

$$\Rightarrow (x - \sqrt{2})^2 + y^2 = 2.$$

[M1]Circle with radius $r = \sqrt{2}$ and centre at $(\sqrt{2}, 0)$.

Note: Shape must be consistent with the scale on imaginary and real axes.

Shape, centre and radius **[A1]**

b.**Option 1:**

$|z| = |z - 3\sqrt{2}|$ defines the perpendicular bisector of the line segment joining $z = 0$ and $z = 3\sqrt{2}$:

$$x = \frac{3\sqrt{2}}{2}. \quad [\text{A1}]$$

Option 2:

Substitute $z = x + iy$:

$$|z| = |z - 3\sqrt{2}| \Rightarrow |x + iy| = |x - 3\sqrt{2} + iy|$$

$$\Rightarrow \sqrt{x^2 + y^2} = \sqrt{(x - 3\sqrt{2})^2 + y^2}$$

$$\Rightarrow x^2 + y^2 = x^2 - 6\sqrt{2}x + 18 + y^2$$

$$\Rightarrow x = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2} \quad [\text{A1}]$$

Substitute $x = \frac{3\sqrt{2}}{2}$ into $(x - \sqrt{2})^2 + y^2 = 2$:

$$\frac{1}{2} + y^2 = 2$$

$$\Rightarrow y = \pm \frac{\sqrt{3}}{\sqrt{2}}.$$

[A1]

Therefore $z = \frac{3\sqrt{2}}{2} \pm i \frac{\sqrt{3}}{\sqrt{2}}.$

Polar form:

$$r = \sqrt{\left(\frac{3\sqrt{2}}{2}\right)^2 + \frac{3}{2}} = \sqrt{\frac{18}{4} + \frac{3}{2}} = \sqrt{6}.$$

$$\tan(\theta) = \frac{\pm \frac{\sqrt{3}}{\sqrt{2}}}{\frac{3\sqrt{2}}{2}} = \pm \frac{1}{\sqrt{3}}.$$

$$z = \sqrt{6}\text{cis}\left(\frac{\pi}{6}\right), \sqrt{6}\text{cis}\left(-\frac{\pi}{6}\right). \quad [\text{A1}]$$

Total 5 marks

Question 6**a.****Option 1:**

$$\begin{aligned}
 a &= v \frac{dv}{dx} = v \left(-\frac{1}{x^2} \right) \\
 &= \left(\frac{1}{x} + \frac{1}{2} \right) \left(-\frac{1}{x^2} \right) \\
 &= -\left(\frac{2+x}{2x^3} \right).
 \end{aligned}
 \tag{A1}$$

Option 2:

$$\begin{aligned}
 a &= \frac{1}{2} \frac{d}{dx} (v^2) = \frac{1}{2} \frac{d}{dx} \left(\frac{1}{x^2} + \frac{1}{x} + \frac{1}{4} \right) \\
 &= \frac{1}{2} \left(-\frac{2}{x^3} - \frac{1}{x^2} \right) \\
 &= -\left(\frac{2+x}{2x^3} \right).
 \end{aligned}
 \tag{A1}$$

b.

$$\begin{aligned}
 \frac{dx}{dt} &= \frac{2+x}{2x} \\
 \Rightarrow \frac{dt}{dx} &= \frac{2x}{2+x} \\
 &= 2 \left(1 - \frac{2}{x+2} \right) \\
 \Rightarrow t &= 2 \int \left(1 - \frac{2}{x+2} \right) dx \\
 &= 2x - 4 \log_e |x+2| + C.
 \end{aligned}
 \tag{M1}$$

Substitute $t=0$ and $x=2$: $C = 4 \ln(4) - 4$.

Therefore:

$$\begin{aligned}
 t &= 2x - 4 + 4 \log_e(4) - 4 \log_e |x+2| \\
 &= 2x - 4 + 4 \log_e \left| \frac{4}{x+2} \right|.
 \end{aligned}
 \tag{A1}$$

Total 4 marks

Question 7

Let $y = \arccos\left(\frac{1}{2\sqrt{x}}\right)$.

Chain rule: Let $u = \frac{1}{2\sqrt{x}} \Rightarrow y = \arccos(u)$.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{-1}{\sqrt{1-u^2}} \times \left(-\frac{1}{4x^{3/2}}\right) \quad [\text{M1}]$$

$$= \frac{-1}{\sqrt{1-\frac{1}{4x}}} \times \left(-\frac{1}{4x\sqrt{x}}\right)$$

$$= \frac{1}{\sqrt{\frac{4x-1}{4x}}} \times \left(\frac{1}{4x\sqrt{x}}\right) \quad [\text{M1}]$$

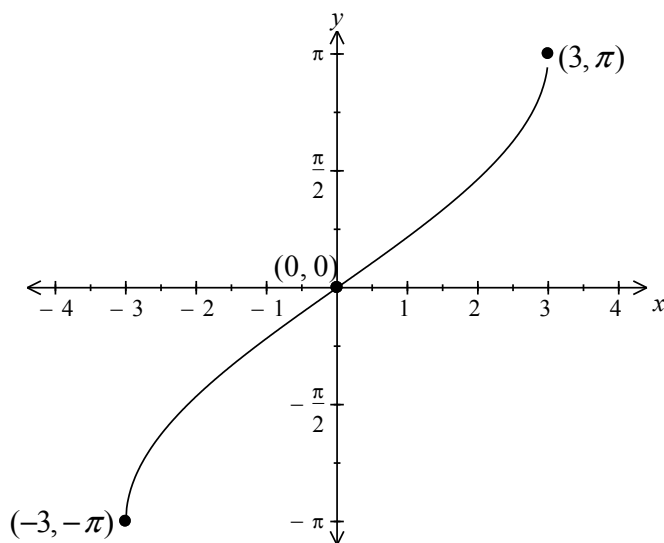
$$= \frac{2\sqrt{x}}{\sqrt{4x-1}} \times \left(\frac{1}{4x\sqrt{x}}\right)$$

$$= \frac{1}{2x\sqrt{4x-1}}. \quad [\text{A1}]$$

Total 3 marks

Question 8**a.**

The graph of $y = \arcsin(x)$ is dilated from the x -axis by a factor of 2 and dilated from the y -axis by a factor of 3:



Endpoints [A1]
Shape and inflection point [A1]

b.

$$x = \frac{3}{2} \Rightarrow y = 2 \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{3}.$$

$$x = 3 \Rightarrow y = 2 \arcsin(1) = \pi.$$

$$V = \pi \int_{\pi/3}^{\pi} x^2 dy. \quad [\text{M1}]$$

$$y = 2 \arcsin\left(\frac{x}{3}\right) \Rightarrow 3 \sin\left(\frac{y}{2}\right) = x.$$

Therefore:

$$V = 9\pi \int_{\pi/3}^{\pi} \sin^2\left(\frac{y}{2}\right) dy \quad [\text{M1}]$$

$$= \frac{9\pi}{2} \int_{\pi/3}^{\pi} (1 - \cos(y)) dy \quad [\text{M1}]$$

$$= \frac{9\pi}{2} [y - \sin(y)]_{\pi/3}^{\pi} = \frac{9\pi}{2} \left[\pi - \frac{\pi}{3} + \frac{\sqrt{3}}{2} \right] = \frac{9\pi}{2} \left[\frac{4\pi + 3\sqrt{3}}{6} \right]$$

$$= \frac{(27\sqrt{3} + 36\pi)\pi}{12} = \frac{(9\sqrt{3} + 12\pi)\pi}{4}. \quad [\text{A1}]$$

Total 6 marks

Question 9**a.**

$$\frac{x^2}{(x+1)^2} = \frac{(x^2 + 2x + 1) - (2x + 1)}{x^2 + 2x + 1}$$

$$= 1 - \frac{2x + 1}{x^2 + 2x + 1} \quad [\text{M1}]$$

$$\frac{2x + 1}{x^2 + 2x + 1} = \frac{2x + 1}{(x + 1)^2} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2} = \frac{A(x + 1) + B}{(x + 1)^2}.$$

$$2x + 1 \equiv A(x + 1) + B = Ax + (A + B)$$

$$\Rightarrow A = 2, \quad B = -1.$$

$$\text{Therefore } 1 - \frac{2x + 1}{x^2 + 2x + 1} = 1 - \frac{2}{x + 1} + \frac{1}{(x + 1)^2}. \quad [\text{A1}]$$

b.

$$\frac{dx}{dt} = \left(\frac{x + 1}{x} \right)^2 = \frac{(x + 1)^2}{x^2}$$

$$\Rightarrow \frac{dt}{dx} = \frac{x^2}{(x + 1)^2} = 1 - \frac{2}{x + 1} + \frac{1}{(x + 1)^2} \quad [\text{M1}]$$

$$\Rightarrow t = \int \left(1 - \frac{2}{x + 1} + \frac{1}{(x + 1)^2} \right) dx$$

$$= x - 2 \log_e |x + 1| - \frac{1}{x + 1} + C. \quad [\text{M1}]$$

Substitute $x = 0$ and $t = 0$: $C = 1$.

Therefore:

$$t = x - 2 \log_e |x + 1| - \frac{1}{x + 1} + 1. \quad [\text{A1}]$$

Total 5 marks

Question 10**a.**

$$x = \sin(t) \quad \dots (1)$$

$$y = -\cos(2t)$$

$$= 2\sin^2(t) - 1 \quad \dots (2)$$

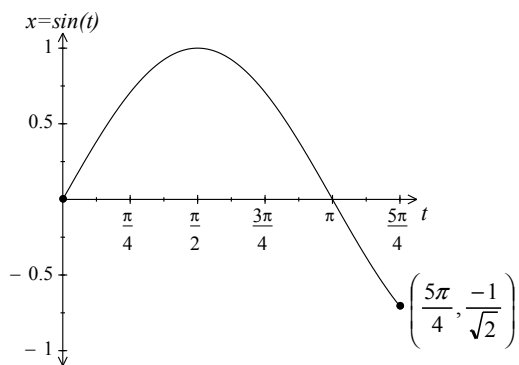
[M1]

Substitute equation (1) into equation (2):

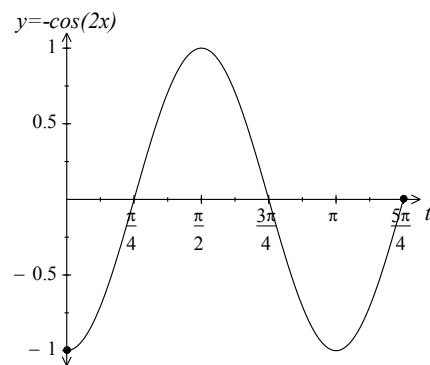
$$y = 2x^2 - 1.$$

[A1]

Domain:



Range:



$$\text{Domain: } -\frac{1}{\sqrt{2}} \leq x \leq 1.$$

$$\text{Range: } -1 \leq y \leq 1.$$

Domain and range **[A1]****b.**

$$\dot{\mathbf{r}} = \cos(t) \mathbf{i} + 2\sin(2t) \mathbf{j}.$$

[M1]

$$\text{At } t = \frac{\pi}{4}: \dot{\mathbf{r}} = \frac{1}{\sqrt{2}} \mathbf{i} + 2 \mathbf{j}.$$

[A1]**Total 5 marks****END OF SOLUTIONS**