

Year 2011
VCE
Specialist Mathematics
Solutions
Trial Examination 2



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SECTION 1

ANSWERS

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E
12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E
21	A	B	C	D	E
22	A	B	C	D	E

SECTION 1

Question 1

Answer C

The equation of the hyperbola is

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1 \text{ since its domain is}$$

R and does not cross the x -axis.

The two asymptotes intersect at the centre (h, k) of the hyperbola, so that

$$2x + 5 = 1 - 2x \Rightarrow 4x = -4$$

$$\Rightarrow x = -1 \text{ and } y = 3 \Rightarrow h = -1 \text{ and } k = 3$$

The distance from the centre to the

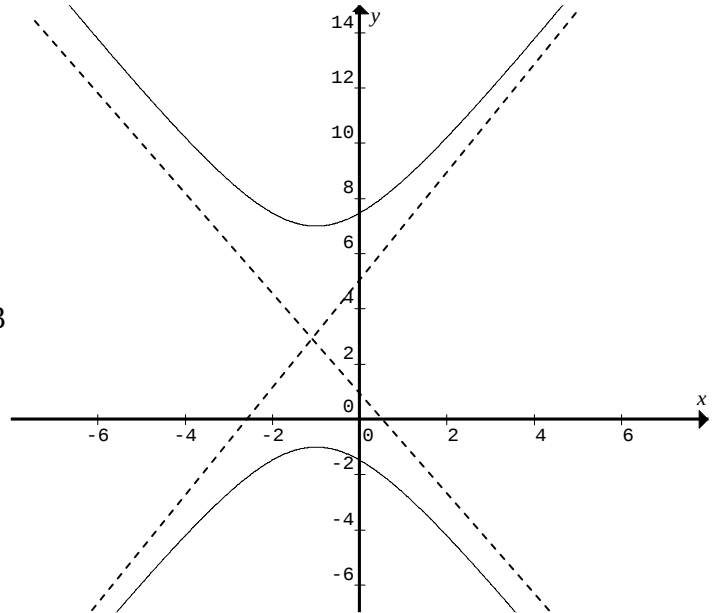
vertices $(-1, 7)$ and $(-1, -1)$ is 4 units,

so that $b = 4$, now the gradient of the

asymptotes is $\frac{b}{a} = \pm 2$ so that $a = 2$.

The equation of the hyperbola is

$$\frac{(y-3)^2}{16} - \frac{(x+1)^2}{4} = 1$$



Question 2

Answer C

Let the roots be α and $\bar{\alpha}$, since the roots occur in conjugate pairs.

$$\alpha = \text{cis}(\theta) = \cos(\theta) + i \sin(\theta) \text{ and } \bar{\alpha} = \text{cis}(-\theta) = \cos(-\theta) + i \sin(-\theta) = \cos(\theta) - i \sin(\theta)$$

$$\alpha + \bar{\alpha} = 2 \cos(\theta) \text{ and}$$

$$\alpha \cdot \bar{\alpha} = \cos^2(\theta) - i^2 \sin^2(\theta) = \cos^2(\theta) + \sin^2(\theta) = 1$$

$$\text{The quadratic is } z^2 - (\alpha + \bar{\alpha})z + \alpha\bar{\alpha} = 0 \text{ or } z^2 - 2z \cos(\theta) + 1 = 0$$

Question 3

Answer E

The graph is that of a reciprocal inverted parabola, with asymptotes at $x = a$ and $x = b$,

note that $a < 0$ while $b > 0$. Its rule is

$$f(x) = \frac{-1}{(x-a)(x-b)} = \frac{-1}{x^2 - (a+b)x + ab} = \frac{1}{(a+b)x - ab - x^2}$$

Question 4**Answer D**

Let $z = x + yi$ so that $\bar{z} = x - yi$ and $c = a + bi$, $\bar{c} = a - bi$

$$(z - c)(\bar{z} - \bar{c}) = r^2$$

$$z\bar{z} - \bar{z}c - z\bar{c} + c\bar{c} = r^2$$

$$(x + yi)(x - yi) - (x - yi)(a + bi) - (x + yi)(a - bi) + (a + bi)(a - bi) = r^2$$

$$x^2 + y^2 - (xa - iya + bxi + by) - (xa + iya - bxi + yb) + a^2 + b^2 = r^2$$

$$x^2 - 2xa + a^2 + y^2 - 2by + b^2 = r^2$$

$$(x - a)^2 + (y - b)^2 = r^2$$

this is a circle with centre (a, b) radius r .

Question 5**Answer A**

If $z = a + 1 + ai$ $\bar{z} = a + 1 - ai$, then

$$\frac{2a}{1 - \bar{z}} = \frac{2a}{1 - (a + 1 - ai)} = \frac{2a}{-a + ai} = \frac{2a}{a(-1 + i)} = \frac{2}{-1 + i} \times \frac{-1 - i}{-1 - i}$$

$$\frac{2a}{1 - \bar{z}} = \frac{-2(1 + i)}{1 - i^2} = -1 - i = \sqrt{2} \operatorname{cis}\left(-\frac{3\pi}{4}\right)$$

Question 6**Answer A**

$$y = \frac{x^n + a}{x} = x^{n-1} + ax^{-1} \Rightarrow \frac{dy}{dx} = (n-1)x^{n-2} - ax^{-2} \text{ for turning points}$$

$$\frac{dy}{dx} = 0 \Rightarrow \frac{(n-1)x^n}{x^2} = \frac{a}{x^2} \text{ or } x^n = \frac{a}{n-1}, \text{ since there are two turning points, the only possibility listed is for } n = 2 \text{ and } a > 0.$$

Question 7**Answer D**

$$u = 4 \operatorname{cis}(\theta), v = r \operatorname{cis}\left(-\frac{2\pi}{3}\right) \text{ and } uv = 12i$$

$$uv = 4r \operatorname{cis}\left(\theta - \frac{2\pi}{3}\right) = 12i = 12 \operatorname{cis}\left(\frac{\pi}{2}\right)$$

$$4r = 12 \Rightarrow r = 3$$

$$\theta - \frac{2\pi}{3} = \frac{\pi}{2} \Rightarrow \theta = \frac{2\pi}{3} + \frac{\pi}{2} = \frac{4\pi + 3\pi}{6} = \frac{7\pi}{6}$$

$$\text{or } \theta = \frac{7\pi}{6} - 2\pi = -\frac{5\pi}{6}$$

Question 8**Answer E**All of options **A. B. C.** and **D.** are correct, **E.** is false

$$\operatorname{Arg}\left(\frac{1}{u}\right) = -\alpha$$

Question 9**Answer B**Since the area is below the x-axis, the area A is equal to

$$A = -\int_{-1}^1 (x^2 - 1)\sqrt{2x+3} dx$$

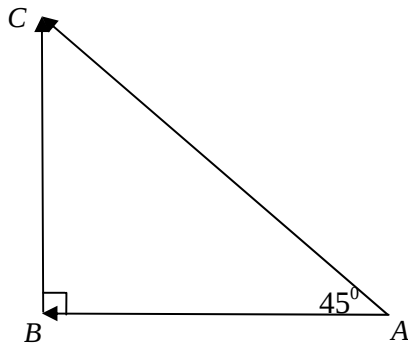
$$\text{let } u = 2x+3 \quad \frac{du}{dx} = 2 \quad \Rightarrow \quad dx = \frac{1}{2} du$$

terminals $x = -1 \quad u = 1$ and when $x = 1 \quad u = 5$

$$x = \frac{1}{2}(u-3) \quad \text{so that}$$

$$\begin{aligned} x^2 - 1 &= \frac{1}{4}(u-3)^2 - 1 = \frac{1}{4}(u^2 - 6u + 9) - 1 \\ &= \frac{1}{4}(u^2 - 6u + 5) = \frac{1}{4}(u-5)(u-1) \end{aligned}$$

$$A = -\frac{1}{8} \int_1^5 (u-5)(u-1)\sqrt{u} du = \frac{1}{8} \int_1^5 (5-u)(u-1)\sqrt{u} du$$

Question 10**Answer B**Since it is an isosceles triangle, $|\overrightarrow{AB}| = |\overrightarrow{BC}|$ and

$$\angle CAB = 45^\circ$$

$$\frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}| |\overrightarrow{AC}|} = \cos(45^\circ) = \frac{1}{\sqrt{2}}$$

$$|\overrightarrow{AB}| |\overrightarrow{AC}| = \sqrt{2} \overrightarrow{AB} \cdot \overrightarrow{AC}$$

D. States it is a right-angled triangle at B , but not necessarily isosceles.**E.** States it is an isosceles triangle, but not necessarily right-angled.**Question 11****Answer D**

Using parametric graphing, shows that when $n = m$ and $a = b$ the path is a circle, when $n = m$ and $a \neq b$ the path is an ellipse, and when $n = 2m$ the path is part of a parabola.

Question 12**Answer C**

$$\text{Let } \underline{a} = 2\hat{i} - 2\hat{j} + \hat{k} \quad |\underline{a}| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$$

$$-\hat{a} = -\frac{\underline{a}}{|\underline{a}|} = -\frac{1}{3}(2\hat{i} - 2\hat{j} + \hat{k}) = \frac{1}{3}(-2\hat{i} + 2\hat{j} - \hat{k})$$

Question 13**Answer E**

$$y = \log_e(x-1) \Rightarrow x-1 = e^y \quad \text{so that } x = 1 + e^y, \text{ rotating about the } y\text{-axis,}$$

$$V_y = \pi \int_a^b (x_2^2 - x_1^2) dy \quad x_2 = 1 + e^y \quad \text{and} \quad x_1 = 1 \quad a = 0 \quad \text{and} \quad b = 1$$

$$V = \pi \int_0^1 ((e^y + 1)^2 - 1) dy = \pi \int_0^1 ((e^{2y} + 2e^y + 1) - 1) dy$$

$$V = \pi \int_0^1 (e^{2y} + 2e^y) dy = \pi \int_0^1 (e^{2x} + 2e^x) dx \quad \text{by dummy variable property}$$

Question 14**Answer E**

$$\frac{dx}{dt} = \cos\left(\frac{1}{t}\right)$$

$$x = \int_1^t \cos\left(\frac{1}{u}\right) du + C \quad \text{now to find } C, \quad x = 3 \text{ when } t = 1,$$

$$3 = \int_1^1 \cos\left(\frac{1}{u}\right) du + C \Rightarrow C = 3$$

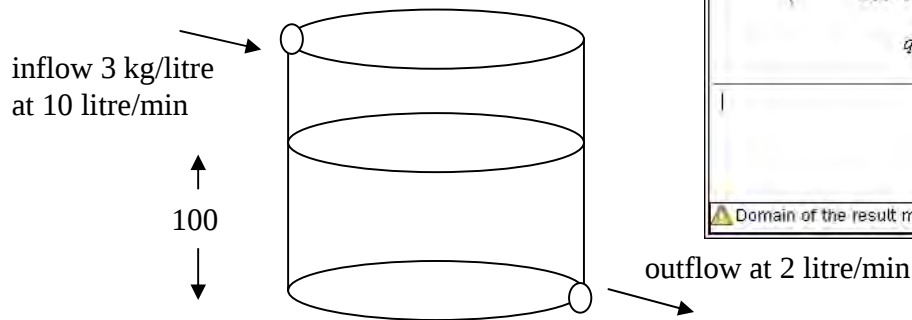
$$x = \int_1^t \cos\left(\frac{1}{u}\right) du + 3 \quad \text{now when } t = 2 \quad x = \int_1^2 \cos\left(\frac{1}{u}\right) du + 3$$

Question 15**Answer C**

The stone takes on the initial upwards speed of the balloon, but its acceleration is just due to gravity. Taking upwards as positive and downwards as negative,

$$s = -250 \quad u = 3 \quad a = -9.8 \quad t = ? \quad \text{using } s = ut + \frac{1}{2}at^2$$

$$-250 = 3t - 4.9t^2 \quad \text{solving } \Rightarrow t = 7.46$$

Question 16**Answer B**

deSolve($q' = 30 - \frac{2q}{100+t}, t, q$)

$q = \frac{Ct}{(t+100)^2} + 10(t+100)$

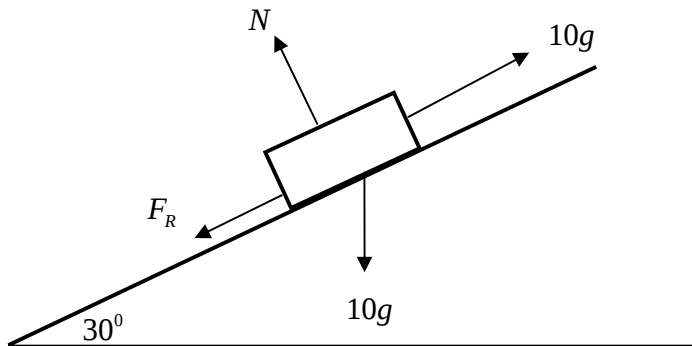
Domain of the result might be larger than the do...

The differential equation for Q , the amount of salt in kilograms in the tank at a time t minutes, is given by

$$\frac{dQ}{dt} = \text{inflow} - \text{outflow} = 10 \times 3 - \frac{2Q}{V} \quad \text{and the volume } V = V(t) = 100 + (3-2)t$$

$$\frac{dQ}{dt} = 30 - \frac{2Q}{100+t}, \quad Q(0) = 2, \text{ solving using CAS, the general solution is}$$

$$Q = \frac{C}{(100+t)^2} + 10(100+t) = C(100+t)^{-2} + 10(100+t) \quad \text{so that } n = -2$$

Question 17**Answer A**

Note that all forces are in newtons.

resolving perpendicular to the plane

$$N - 10g \cos(30^\circ) = 0 \Rightarrow N = 10g \cos(30^\circ) = 5\sqrt{3}g$$

resolving up and parallel to the plane

$$10g - 10g \sin(30^\circ) - F_R = 0 \Rightarrow F_R = 10g - 10g \sin(30^\circ) = 5g$$

$$F_R \leq \mu N$$

$$5g \leq \mu 5\sqrt{3}g \Rightarrow \mu \geq \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Question 18**Answer B**

By Newton's Law $R = ma$ using $a = \frac{dv}{dt}$ with $R = f(t)$

$$f(t) = m \frac{dv}{dt}$$

$$m \int_{v_1}^{v_2} 1 \cdot dv = m [v]_{v_1}^{v_2} = mv_2 - mv_1 = \int_{t_1}^{t_2} f(t) dt$$

Question 19**Answer D**

By Newton's law, the equation of motion is given by

$$ma = F - kv^3, \text{ using } a = v \frac{dv}{dx}$$

$$mv \frac{dv}{dx} = F - kv^3 \text{ integrating from } v = 0 \text{ to } v = V$$

the distance s , travelled from rest is given by

$$s = \int_0^V \frac{mv}{F - kv^3} dv$$

Question 20**Answer A**

$$a = \frac{d\left(\frac{1}{2}v^2\right)}{dx} = \frac{3bx - 2b^2}{x^3}$$

$$\frac{1}{2}v^2 = \int \frac{3bx - 2b^2}{x^3} dx = \int (3bx^{-2} - 2b^2x^{-3}) dx$$

$$\frac{1}{2}v^2 = -3bx^{-1} + b^2x^{-2} + c$$

Now when $v = 0$ $x = b$

$$0 = -3 + 1 + c \Rightarrow c = 2$$

$$\frac{1}{2}v^2 = \frac{-3b}{x} + \frac{b^2}{x^2} + 2 = \frac{2x^2 - 3bx + b^2}{x^2}$$

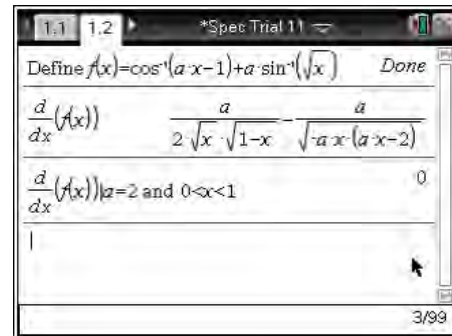
$$\frac{1}{2}v^2 = \frac{(2x - b)(x - b)}{x^2} \text{ so that when } v = 0 \text{ } x = b \text{ and } x = \frac{b}{2}$$

Question 21**Answer D**

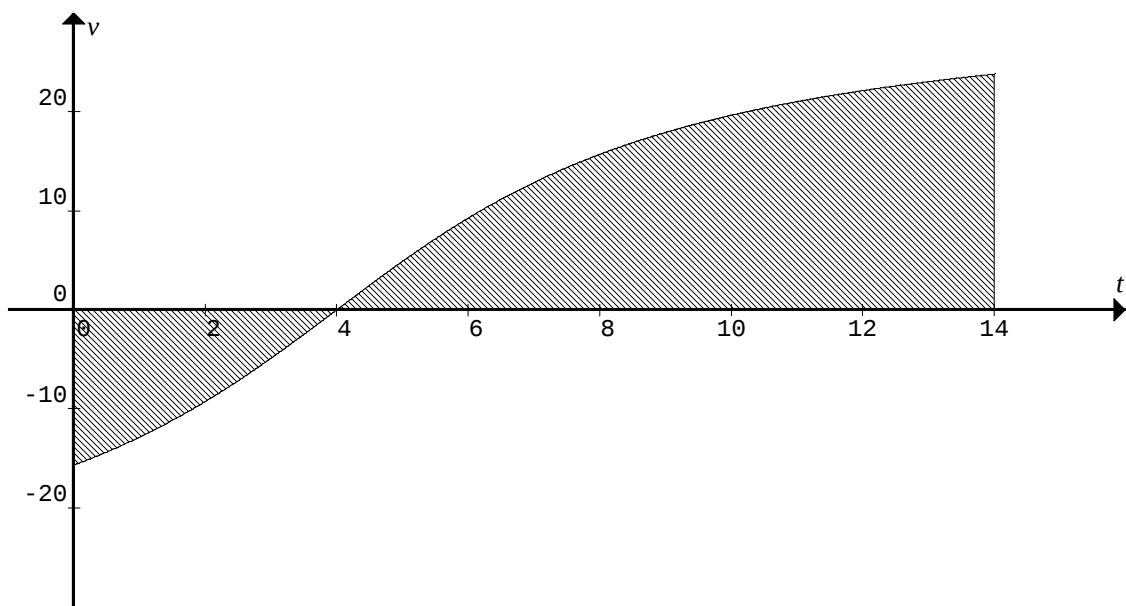
$$f(x) = \cos^{-1}(ax-1) + a \sin^{-1}(\sqrt{x})$$

$$f'(x) = \frac{a}{2\sqrt{x}\sqrt{1-x}} - \frac{a}{\sqrt{-ax(ax-2)}}$$

if $a=2$ for $x \in (0,1)$ so $b=1$ then $f'(x)=0$

**Question 22****Answer C**

$$v(t) = 20 \tan^{-1}\left(\frac{t}{4} - 1\right)$$



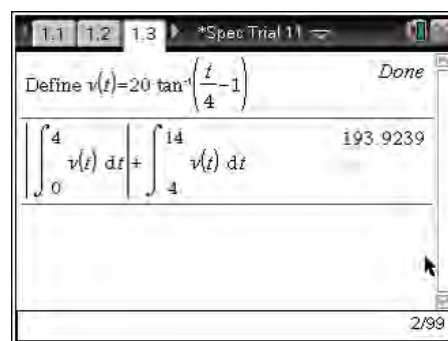
distance travelled d is given by

$$d = \left| \int_0^4 v(t) dt \right| + \int_4^{14} v(t) dt$$

$$d = \left| \int_0^4 20 \tan^{-1}\left(\frac{t}{4} - 1\right) dt \right| + \int_4^{14} 20 \tan^{-1}\left(\frac{t}{4} - 1\right) dt$$

$$d = |-35.11| + 158.82 = 35.11 + 158.82 = 193.9$$

$$d = 194 \text{ m}$$

**END OF SECTION 1 SUGGESTED ANSWERS**

SECTION 2

Question 1

a. $u = a + bi \Rightarrow |u|^2 = a^2 + b^2$ and $\bar{u} = a - bi$
 $u\bar{u} = (a + bi)(a - bi) = a^2 - b^2i^2 = a^2 + b^2$ A1

b. The cubic is $(z - 1)(z - u)(z - \bar{u}) = 0$
 $(z - 1)(z^2 - (u + \bar{u})z + u\bar{u}) = 0$
 $(z - 1)(z^2 - 2az + a^2 + b^2) = 0$ M1
 $z^3 - (2a + 1)z^2 + (a^2 + b^2 + 2a)z - (a^2 + b^2) = 0$
 $p = -2a - 1$, $q = a^2 + b^2 + 2a$ and $s = -(a^2 + b^2)$ A2

c. $iu = i(a + bi) = ai + bi^2$
 $iu = -b + ai$ A1

d. $|u|^2 = a^2 + b^2$ and $|iu|^2 = (-b)^2 + a^2 = a^2 + b^2$
 so that $|u|^2 + |iu|^2 = 2(a^2 + b^2)$ M1
 now $|u - iu| = |(a + b) + (a - b)i|$
 $|u - iu|^2 = (a + b)^2 + (a - b)^2 = a^2 + 2ab + b^2 + a^2 - 2ab + b^2$
 $|u - iu|^2 = 2(a^2 + b^2)$ shown A1

Since iu is a rotation 90° anticlockwise from u , and both $|u|^2 = |iu|^2 = a^2 + b^2$. In the triangle, which passes through O , u and iu it is a right angled isosceles triangle. By Pythagoras's Theorem, the hypotenuse is $\sqrt{2}$ times the two equal side lengths, and $|u - iu| = \sqrt{2(a^2 + b^2)}$ A1
 and represents the distance between the complex numbers u and iu .

e. From d. since the angle at the origin is a right-angle, the line joining the complex numbers u and iu . is a diameter of the circle, the radius r is therefore

$$r = \frac{1}{2}|u - iu| = \frac{1}{2}\sqrt{2(a^2 + b^2)} = \sqrt{\frac{a^2 + b^2}{2}} \quad \text{A1}$$

The centre is the mid-point of the line joining the complex numbers u and iu .

$$c = \frac{1}{2}(u + iu) = \frac{1}{2}(a - b) + \frac{1}{2}(a + b)i \quad \text{A1}$$

- f. The shaded area equals the area of the circle minus the area of the triangle

$$\text{Area} = \pi r^2 - \frac{1}{2}|u||iu|$$

M1

$$\text{Area} = \pi \left(\frac{a^2 + b^2}{2} \right) - \frac{1}{2}(a^2 + b^2)$$

$$\text{Area} = \frac{1}{2}(a^2 + b^2)(\pi - 1)$$

A1

Question 2

- a. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ differentiating implicitly, gives

$$\frac{2x}{a^2} - \frac{2y}{b^2} \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{b^2 x}{a^2 y} \text{ at } P(a \sec(\theta), b \tan(\theta))$$

A1

$$\frac{dy}{dx} = \frac{b^2 a \sec(\theta)}{a^2 b \tan(\theta)} = \frac{b}{a \sin(\theta)}, \text{ the equation of the tangent at } P \text{ is}$$

$$y - b \tan(\theta) = \frac{b}{a \sin(\theta)}(x - a \sec(\theta))$$

$$y = \frac{bx}{a \sin(\theta)} + \frac{b \sin(\theta)}{\cos(\theta)} - \frac{b}{\sin(\theta) \cos(\theta)}$$

$$y = \frac{bx}{a \sin(\theta)} - \frac{b(1 - \sin^2(\theta))}{\sin(\theta) \cos(\theta)} = \frac{bx}{a \sin(\theta)} - \frac{b \cos^2(\theta)}{\sin(\theta) \cos(\theta)}$$

A1

$$y = \frac{bx}{a \sin(\theta)} - \frac{b \cos(\theta)}{\sin(\theta)}$$

- b. This crosses the asymptote $y = \frac{bx}{a}$ at A, when

$$\frac{bx}{a} = \frac{bx}{a \sin(\theta)} - \frac{b \cos(\theta)}{\sin(\theta)}$$

A1

$$\frac{b \cos(\theta)}{\sin(\theta)} = \frac{bx}{a \sin(\theta)} - \frac{bx}{a} = \frac{bx}{a} \left(\frac{1}{\sin(\theta)} - 1 \right) = \frac{bx(1 - \sin(\theta))}{a \sin(\theta)}$$

$$x_A = \frac{a \cos(\theta)}{1 - \sin(\theta)} \Rightarrow y_A = \frac{b \cos(\theta)}{1 - \sin(\theta)}$$

A1

$$A \left(\frac{a \cos(\theta)}{1 - \sin(\theta)}, \frac{b \cos(\theta)}{1 - \sin(\theta)} \right)$$

This crosses the asymptote $y = -\frac{bx}{a}$ at B , when

$$-\frac{bx}{a} = \frac{bx}{a \sin(\theta)} - \frac{b \cos(\theta)}{\sin(\theta)} \quad \text{A1}$$

$$\frac{b \cos(\theta)}{\sin(\theta)} = \frac{bx}{a \sin(\theta)} + \frac{bx}{a} = \frac{bx}{a} \left(\frac{1}{\sin(\theta)} + 1 \right) = \frac{bx(1 + \sin(\theta))}{a \sin(\theta)}$$

$$x_B = \frac{a \cos(\theta)}{1 + \sin(\theta)} \Rightarrow y_B = \frac{-b \cos(\theta)}{1 + \sin(\theta)} \quad \text{A1}$$

$$B \left(\frac{a \cos(\theta)}{1 + \sin(\theta)}, \frac{-b \cos(\theta)}{1 + \sin(\theta)} \right)$$

c. Since O is the origin,

$$\overrightarrow{OA} = \left(\frac{a \cos(\theta)}{1 - \sin(\theta)} \right) \hat{i} + \left(\frac{b \cos(\theta)}{1 - \sin(\theta)} \right) \hat{j}$$

$$\begin{aligned} |\overrightarrow{OA}| &= \sqrt{\left(\frac{a \cos(\theta)}{1 - \sin(\theta)} \right)^2 + \left(\frac{b \cos(\theta)}{1 - \sin(\theta)} \right)^2} \\ &= \sqrt{\frac{a^2 \cos^2(\theta) + b^2 \cos^2(\theta)}{(1 - \sin(\theta))^2}} = \sqrt{a^2 + b^2} \left(\frac{\cos(\theta)}{1 - \sin(\theta)} \right) \end{aligned} \quad \text{A1}$$

$$\overrightarrow{OB} = \left(\frac{a \cos(\theta)}{1 + \sin(\theta)} \right) \hat{i} + \left(\frac{-b \cos(\theta)}{1 + \sin(\theta)} \right) \hat{j}$$

$$\begin{aligned} |\overrightarrow{OB}| &= \sqrt{\left(\frac{a \cos(\theta)}{1 + \sin(\theta)} \right)^2 + \left(\frac{-b \cos(\theta)}{1 + \sin(\theta)} \right)^2} \\ &= \sqrt{\frac{a^2 \cos^2(\theta) + b^2 \cos^2(\theta)}{(1 + \sin(\theta))^2}} = \sqrt{a^2 + b^2} \left(\frac{\cos(\theta)}{1 + \sin(\theta)} \right) \end{aligned} \quad \text{A1}$$

$$\overrightarrow{OA} \cdot \overrightarrow{OB} = \left(\frac{a \cos(\theta)}{1 - \sin(\theta)} \right) \left(\frac{a \cos(\theta)}{1 + \sin(\theta)} \right) + \left(\frac{b \cos(\theta)}{1 - \sin(\theta)} \right) \left(\frac{-b \cos(\theta)}{1 + \sin(\theta)} \right)$$

$$\overrightarrow{OA} \cdot \overrightarrow{OB} = \frac{(a^2 - b^2) \cos^2(\theta)}{1 - \sin^2(\theta)} = a^2 - b^2$$

$$\text{Now } \cos(\alpha) = \frac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{|\overrightarrow{OA}| |\overrightarrow{OB}|} \quad \text{M1}$$

$$\begin{aligned} &= \frac{a^2 - b^2}{\sqrt{a^2 + b^2} \left(\frac{\cos(\theta)}{1 - \sin(\theta)} \right) \sqrt{a^2 + b^2} \left(\frac{\cos(\theta)}{1 + \sin(\theta)} \right)} \\ &= \frac{a^2 - b^2}{(a^2 + b^2) \frac{\cos^2(\theta)}{1 - \sin^2(\theta)}} = \frac{a^2 - b^2}{a^2 + b^2} \quad \text{M1} \end{aligned}$$

$$\text{so that } \alpha = \angle AOB = \cos^{-1} \left(\frac{a^2 - b^2}{a^2 + b^2} \right)$$

d. Area of the triangle $\triangle AOB = \frac{1}{2} |\overrightarrow{OA}| |\overrightarrow{OB}| \sin(\alpha)$, where

$$\sin(\alpha) = \frac{\sqrt{(a^2 + b^2)^2 - (a^2 - b^2)^2}}{(a^2 + b^2)}$$

$$\sin(\alpha) = \frac{\sqrt{(a^4 + 2a^2b^2 + b^4) - (a^4 - 2a^2b^2 + b^4)}}{(a^2 + b^2)}$$

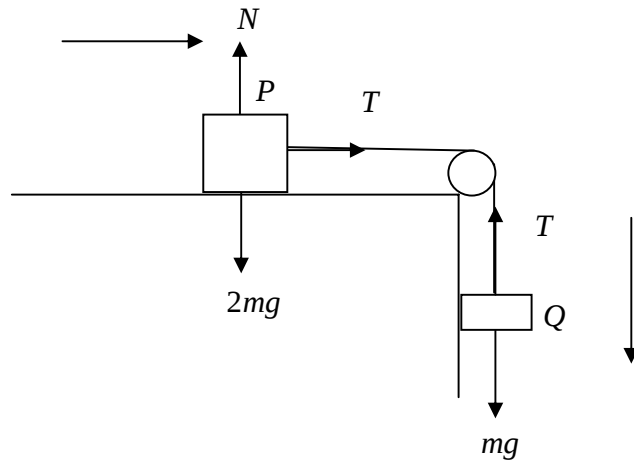
$$\sin(\alpha) = \frac{\sqrt{4a^2b^2}}{(a^2 + b^2)} = \frac{2ab}{(a^2 + b^2)} \quad \text{since } a > b > 0 \quad \text{A1}$$

$$\text{Area } \triangle AOB = \frac{1}{2} \left(\sqrt{a^2 + b^2} \left(\frac{\cos(\theta)}{1 - \sin(\theta)} \right) \right) \times \left(\sqrt{a^2 + b^2} \left(\frac{\cos(\theta)}{1 + \sin(\theta)} \right) \right) \times \frac{2ab}{(a^2 + b^2)}$$

$$\text{Area } \triangle AOB = ab \quad \text{A1}$$

Question 3

a. smooth surface



resolving downwards around Q, (1) $mg - T = ma$

resolving around P, (2) $N - 2mg = 0$

(3) $T = 2ma$

A1

From (3) into (1) $mg - 2ma = ma \Rightarrow 3ma = mg \Rightarrow a = \frac{g}{3}$

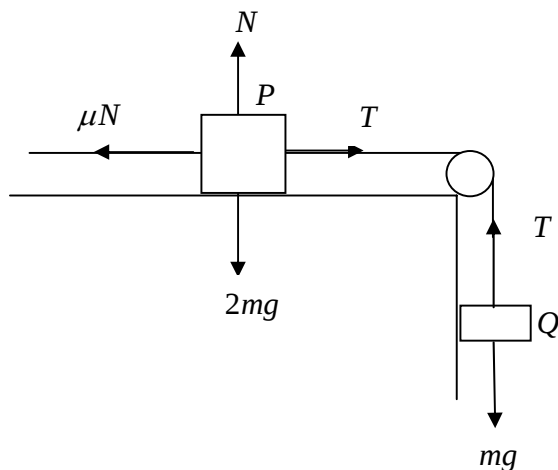
A1

$a = \frac{g}{3}$ $u = 0$ $t = 0.5$ $s = ?$, using $s = ut + \frac{1}{2}at^2$

$s = 0 + \frac{1}{2} \times \frac{g}{3} \times 0.5^2 = 0.41 \text{ m}$

A1

b. rough surface



T is the tension in the string in newtons, N the normal reaction and μN the frictional force. correct forces and labelled

A1

- i. resolving downwards around Q, (1) $mg - T = 0$
 since the acceleration of the system is zero.
 $T = mg$ newtons A1
- ii. consider when the acceleration is non-zero
 resolving downwards around Q, (1) $mg - T = ma$
 resolving around P, (2) $N - 2mg = 0$ A1
 (3) $T - \mu N = 2ma$
 from (2) $N = 2mg$ substituting into (3) gives
 $T - 2\mu mg = 2ma$ but from (1) $mg - T = ma$ adding eliminating T gives M1
 $3ma = mg - 2\mu mg = mg(1 - 2\mu)$
 $a = \frac{g}{3}(1 - 2\mu)$
 for $a \geq 0$, it follows that the maximum value of M1
 μ , is $\mu = \frac{1}{2}$, and since $\mu > 0$,
 $0 < \mu \leq \frac{1}{2}$ A1

Question 4

$$\overrightarrow{OA} = 2\hat{i} + \hat{j} + 4\hat{k}, \quad \overrightarrow{OB} = \alpha\hat{i} - 2\hat{j} - \hat{k} \quad \text{and} \quad \overrightarrow{OC} = -5\hat{i} + 8\hat{j} + 11\hat{k}$$

- a. for perpendicular $\overrightarrow{OA} \cdot \overrightarrow{OB} = 0$
 $2\alpha - 2 - 4 = 0$
 $2\alpha = 6$
 $\alpha = 3$ A1
- b. for linearly dependant $\overrightarrow{OC} = x\overrightarrow{OA} + y\overrightarrow{OB}$
 $\hat{c} = -5\hat{i} + 8\hat{j} + 11\hat{k} = x(2\hat{i} + \hat{j} + 4\hat{k}) + y(\alpha\hat{i} - 2\hat{j} - \hat{k})$ M1
 $\hat{i} \quad (1) \quad -5 = 2x + y\alpha \quad (2) \quad 8 = x - 2y$
 $\hat{j} \quad (2) \quad 8 = x - 2y \quad -2x(3) \quad -22 = -8x + 2y$ M1
 $\hat{k} \quad (3) \quad 11 = 4x - y \quad \text{adding} \quad -7x = -14$
 so that $x = 2$, $y = -3$ and $\alpha = 3$ A1

c. $|\vec{OA}| = \sqrt{2^2 + 1^2 + 4^2} = \sqrt{21}$

$$|\vec{OB}| = \sqrt{\alpha^2 + (-2)^2 + (-1)^2} = \sqrt{5 + \alpha^2} \quad \text{A1}$$

$$|\vec{OA}| = |\vec{OB}| \Rightarrow \sqrt{21} = \sqrt{5 + \alpha^2}$$

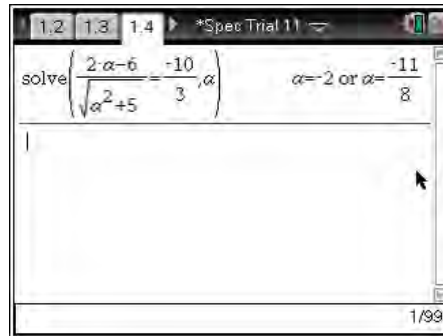
$$5 + \alpha^2 = 21$$

$$\alpha^2 = 16$$

$$\alpha = \pm 4 \text{ both answers are acceptable} \quad \text{A1}$$

d. the scalar resolute of $\vec{OA}$ parallel to $\vec{OB}$ is $\frac{\vec{OA} \cdot \vec{OB}}{|\vec{OB}|} = -\frac{10}{3}$

$$\frac{\vec{OA} \cdot \vec{OB}}{|\vec{OB}|} = \frac{2\alpha - 6}{\sqrt{\alpha^2 + 5}} = -\frac{10}{3}$$



A1

solving on a CAS gives

$$\alpha = -2 \text{ or } \alpha = -\frac{11}{8} \text{ both answers are acceptable.} \quad \text{A1}$$

e. Let θ be the angle the vector $\vec{OB}$ makes with the x -axis, then

$$\theta = \cos^{-1}\left(-\frac{2}{3}\right) \Rightarrow \cos(\theta) = -\frac{2}{3} = \frac{\vec{OB} \cdot \vec{i}}{|\vec{OB}|}$$

$$-\frac{2}{3} = \frac{\alpha}{\sqrt{\alpha^2 + 5}}, \text{ and } \alpha < 0$$

M1

$$-2\sqrt{\alpha^2 + 5} = 3\alpha$$

$$4(\alpha^2 + 5) = 9\alpha^2$$

$$20 = 5\alpha^2$$

$$\alpha^2 = 4$$

$$\text{solving gives } \alpha = -2 \text{ as the only solution} \quad \text{A1}$$

Question 5

a. $\dot{x} = \frac{dx}{dt} = V \cos(\alpha) - kx$

$$\frac{dt}{dx} = \frac{1}{V \cos(\alpha) - kx}$$

$$t = \int \frac{1}{V \cos(\alpha) - kx} dx \quad \text{M1}$$

$$t = -\frac{1}{k} \log_e(V \cos(\alpha) - kx) + c$$

Now when $t = 0$ $x = 0$

$$0 = -\frac{1}{k} \log_e(V \cos(\alpha)) + c \Rightarrow c = \frac{1}{k} \log_e(V \cos(\alpha))$$

$$t = \frac{1}{k} \log_e(V \cos(\alpha)) - \frac{1}{k} \log_e(V \cos(\alpha) - kx) \quad \text{M1}$$

$$t = \frac{1}{k} \log_e \left(\frac{V \cos(\alpha)}{V \cos(\alpha) - kx} \right)$$

$$e^{kt} = \frac{V \cos(\alpha)}{V \cos(\alpha) - kx}$$

$$\frac{V \cos(\alpha) - kx}{V \cos(\alpha)} = e^{-kt} \quad \text{M1}$$

$$V \cos(\alpha) - kx = V \cos(\alpha) e^{-kt}$$

$$kx = V \cos(\alpha) - V \cos(\alpha) e^{-kt} = V \cos(\alpha) (1 - e^{-kt})$$

$$x = x(t) = \frac{V \cos(\alpha)}{k} (1 - e^{-kt}) \quad \text{A1}$$

b. $\dot{y} = V \sin(\alpha) - gt$ since $y(0) = 0$

$$y = Vt \sin(\alpha) - \frac{1}{2} gt^2 = t \left(V \sin(\alpha) - \frac{gt}{2} \right) \quad \text{M1}$$

$$\text{when } y = 0 \quad t = T = \frac{2V \sin(\alpha)}{g}$$

c. $x = x(t) = \frac{V \cos(\alpha)}{k} - \frac{V \cos(\alpha)}{k} e^{-kt}$ $y(t) = Vt \sin(\alpha) - \frac{1}{2}gt^2$
 $\dot{x}(t) = -k \times \frac{-V \cos(\alpha)}{k} e^{-kt} = V \cos(\alpha) e^{-kt}$ $\dot{y}(t) = V \sin(\alpha) - gt$ M1
 when it hits the ground $T = \frac{2V \sin(\alpha)}{g}$

$\dot{x}(T) = V \cos(\alpha) e^{-\frac{2Vk \sin(\alpha)}{g}}$ A1
 $\dot{y}(T) = V \sin(\alpha) - g \times \frac{2V \sin(\alpha)}{g} = -V \sin(\alpha)$

the angle at which it hits the ground is

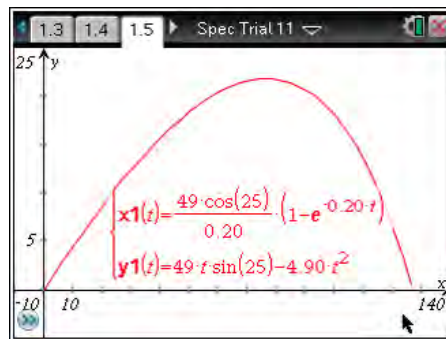
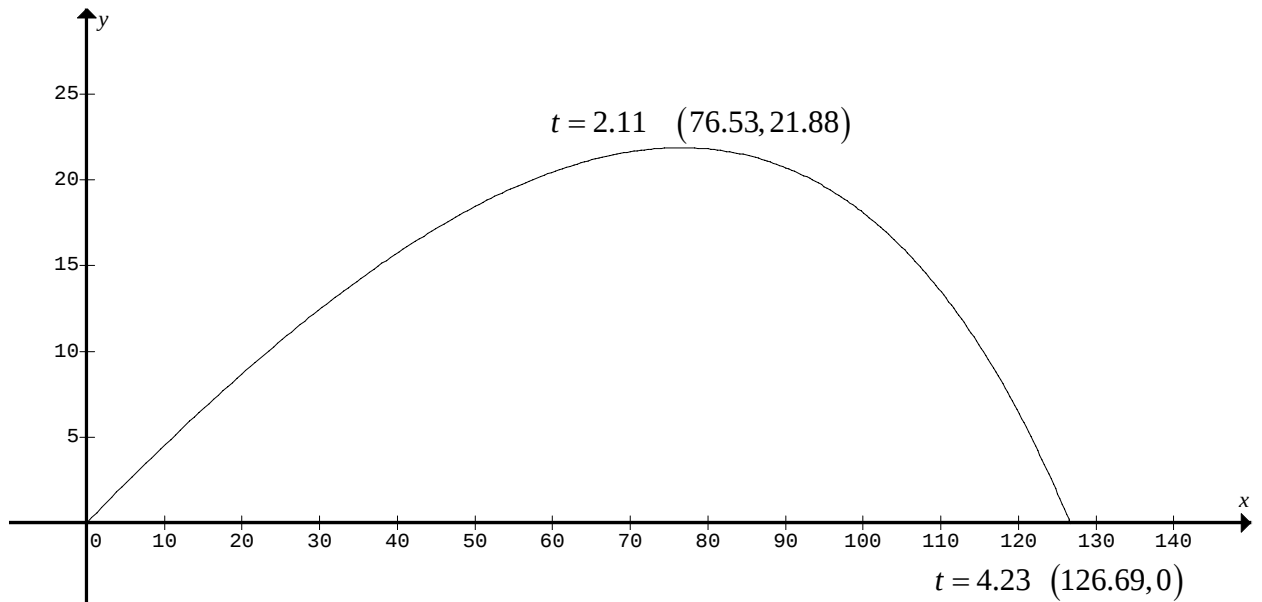
$\tan^{-1} \left(\left| \frac{\dot{y}(T)}{\dot{x}(T)} \right| \right) = \tan^{-1} \left(\frac{V \sin(\alpha)}{V \cos(\alpha) e^{-\frac{2Vk \sin(\alpha)}{g}}} \right)$
 $= \tan^{-1} \left(\tan(\alpha) e^{\frac{2Vk \sin(\alpha)}{g}} \right)$ A1

d. since $\ddot{\mathbf{r}}(t) = -k\dot{\mathbf{x}}(t)\hat{i} - g\hat{j}$ the golf ball is subjected to gravity,
 in the vertical y direction, but an air resistance force, proportional to the
 velocity in the horizontal x direction of motion. A1

e. $V = 49$ $\alpha = 25^\circ$ $k = 0.2$
 $T = \frac{2V \sin(\alpha)}{g} = \frac{2 \times 49 \times \sin(25^\circ)}{9.8}$
 $T = 4.23$ sec
 $R = x(T) = \frac{V \cos(\alpha)}{k} (1 - e^{-kT}) = \frac{49 \times \cos(25^\circ)}{0.2} (1 - e^{-0.2 \times 4.23})$
 $R = 126.69$ m A1

f. at maximum height
 $\dot{y} = 0 \Rightarrow t = \frac{1}{2}T = 2.11$ sec A1
 $x\left(\frac{1}{2}T\right) = x(2.11) = x(t) = \frac{49 \cos 25^\circ}{0.2} (1 - e^{-0.2 \times 2.11}) = 76.53$ m A1
 $y_{\max} = H = \frac{V^2 \sin^2(\alpha)}{2g} = \frac{49^2 \sin^2(25^\circ)}{2 \times 9.8} = 21.88$ m A1

- g. correct graph, shape, critical points,
 the maximum height at maximum (76.53, 21.88) when $t = 2.11$
 and the range, hits the ground at (126.69, 0) when $t = 4.23$ A1
 correct shape of the graph (not parabolic) A1



$t_f = \frac{2 \cdot 49 \sin(25)}{9.8}$	4.2262
$x1(t_f)$	126.6867
$ht_f = \frac{t_f^2}{2}$	2.1131
$x1(ht_f)$	76.5327
$y1(ht_f)$	21.8793
	5.09

END OF SECTION 2 SUGGESTED ANSWERS