

Year 2011
VCE
Specialist Mathematics
Solutions
Trial Examination 1



KILBAHA MULTIMEDIA PUBLISHING
PO BOX 2227
KEW VIC 3101
AUSTRALIA

TEL: (03) 9018 5376
FAX: (03) 9817 4334
kilbaha@gmail.com
<http://kilbaha.com.au>

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Question 1

$$z = \sqrt{5} + bi$$

$$|z| = \sqrt{(\sqrt{5})^2 + b^2} = \sqrt{5 + b^2} = 3 \quad b^2 + 5 = 9 \quad b^2 = 4$$

$$\Rightarrow b = \pm 2$$

$$\text{since } b = \text{Im}(z) < 0 \text{ then } b = -2$$

A1

$$\text{Arg}(z) = \theta = \tan^{-1}\left(-\frac{2}{\sqrt{5}}\right), \text{ so that } \tan(\theta) = -\frac{2}{\sqrt{5}}$$

$$\tan(2\theta) = \frac{2 \tan(\theta)}{1 - \tan^2(\theta)} = \frac{-\frac{4}{\sqrt{5}}}{1 - \left(-\frac{2}{\sqrt{5}}\right)^2} = \frac{-\frac{4}{\sqrt{5}}}{1 - \frac{4}{5}} = \frac{-\frac{4}{\sqrt{5}}}{\frac{1}{5}} \quad \text{M1}$$

$$\tan(2\theta) = -\frac{4}{\sqrt{5}} \times \frac{5}{1} \times \frac{\sqrt{5}}{\sqrt{5}} = -4\sqrt{5}$$

$$k = -4$$

A1

Question 2

$$\text{a.} \quad P(z) = z^3 + (3i - 3\sqrt{3})z^2 + 5z - 15\sqrt{3} + 15i$$

$$\begin{aligned} P(3\sqrt{3} - 3i) &= (3\sqrt{3} - 3i)^3 + (3i - 3\sqrt{3})(3\sqrt{3} - 3i)^2 + 5(3\sqrt{3} - 3i) - 15\sqrt{3} + 15i \\ &= (3\sqrt{3} - 3i)^3 - (3\sqrt{3} - 3i)^3 + 15\sqrt{3} - 15i - 15\sqrt{3} + 15i \\ &= 0 \text{ shown} \end{aligned} \quad \text{M1}$$

$$\text{b.} \quad \text{Hence } (z - 3\sqrt{3} + 3i) \text{ is a factor}$$

$$z^3 + (3i - 3\sqrt{3})z^2 + 5z - 15\sqrt{3} + 15i = 0$$

$$(z - 3\sqrt{3} + 3i)(z^2 + 5) = 0 \quad \text{A1}$$

$$(z - 3\sqrt{3} + 3i)(z^2 - 5i^2) = 0$$

$$(z - 3\sqrt{3} + 3i)(z + \sqrt{5}i)(z - \sqrt{5}i) = 0$$

$$z = 3\sqrt{3} - 3i \text{ and } \pm \sqrt{5}i \quad \text{A1}$$

Question 3

$$y = \frac{2x^3 - 4}{x} = 2x^2 - \frac{4}{x}$$

$y = 2x^2$ and $x = 0$ are asymptotes

A1

crosses x -axis $y = 0 \Rightarrow 2x^3 = 4 \quad x = \sqrt[3]{2} \quad (\sqrt[3]{2}, 0)$

A1

Note that $1 < \sqrt[3]{2} < 2$

$$\frac{dy}{dx} = 4x + \frac{4}{x^2} = 0 \text{ for turning points}$$

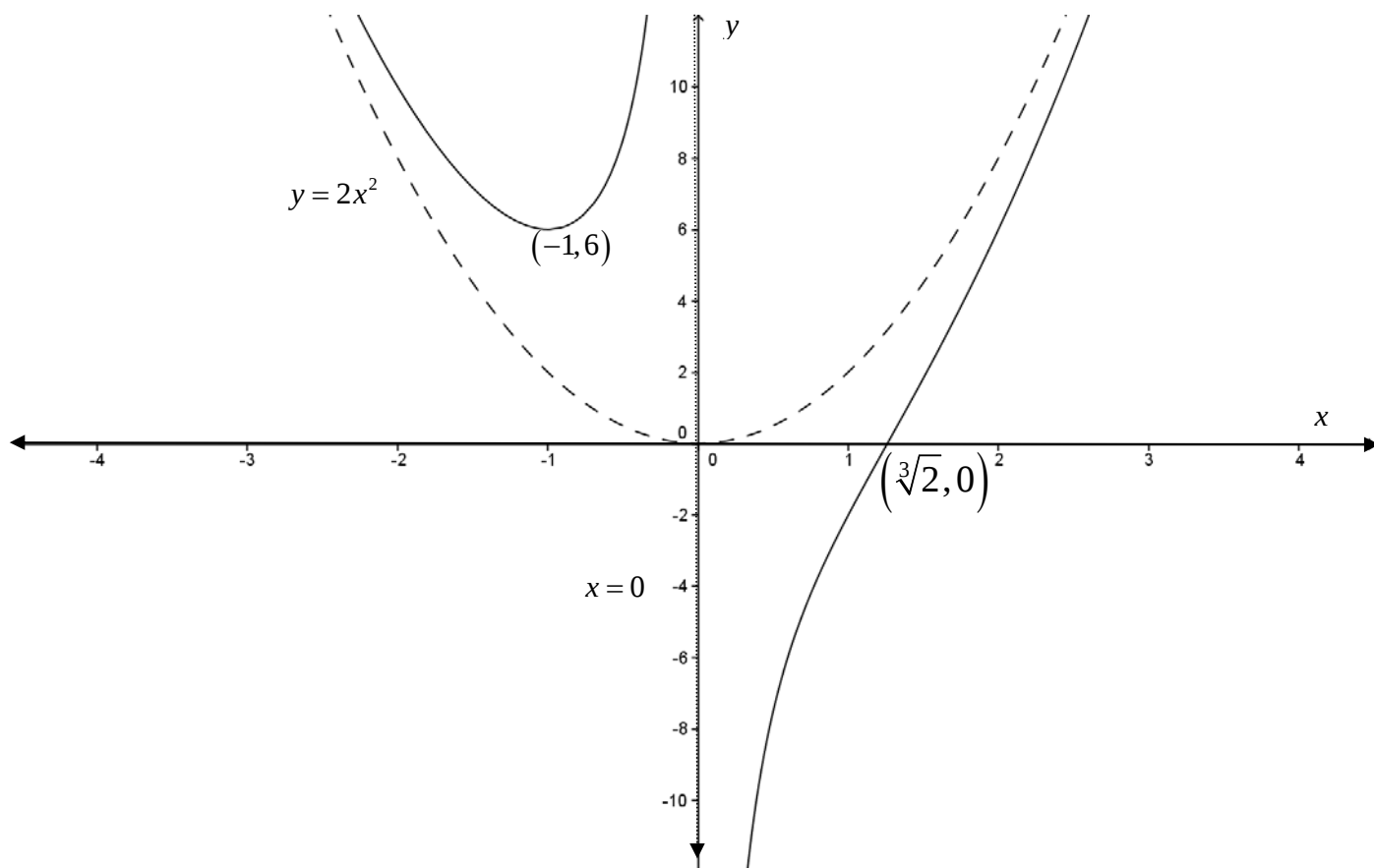
$$\Rightarrow x^3 = -1 \quad x = -1 \text{ and } y = 2 + 4 = 6$$

$(-1, 6)$ is a minimum turning point

A1

correct graph, shape asymptotes, turning points

A1



Question 4

a. $F = ma$ $F = 4x^3 + 12x$ newtons $m = 2 \text{ kg}$

$$2a = 4x^3 + 12x \quad \text{since } a = a(x) \quad \text{use } a = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$$

$$a = \frac{d\left(\frac{1}{2}v^2\right)}{dx} = 2x^3 + 6x \quad \text{M1}$$

$$\frac{1}{2}v^2 = \int (2x^3 + 6x) dx$$

$$\frac{1}{2}v^2 = \frac{1}{2}x^4 + 3x^2 + c$$

Now when $x=1$ $v=4$

$$8 = \frac{1}{2} + 3 + c \Rightarrow c = \frac{9}{2} \quad \text{A1}$$

$$v^2 = x^4 + 6x^2 + 9 = (x^2 + 3)^2 \quad \text{since } v > 0$$

$$v = x^2 + 3 \quad \text{shown}$$

b. $v = \frac{dx}{dt} = x^2 + 3$

$$\Rightarrow \frac{dt}{dx} = \frac{1}{x^2 + 3} \quad \text{M1}$$

$$t = \int \frac{1}{x^2 + 3} dx = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c$$

Now when $t=0$ $x=1$

$$0 = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + c \Rightarrow c = -\frac{\pi}{6\sqrt{3}} \quad \text{A1}$$

$$t = \frac{1}{\sqrt{3}} \left(\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \frac{\pi}{6} \right) \quad \text{now when } x = \sqrt{3} \quad \text{M1}$$

$$t = \frac{\sqrt{3}}{3} \left(\tan^{-1}(1) - \frac{\pi}{6} \right) = \frac{\sqrt{3}}{3} \left(\frac{\pi}{4} - \frac{\pi}{6} \right)$$

$$t = \frac{\sqrt{3}\pi}{36} \text{ sec} \quad \text{A1}$$

Question 5

a.
$$I = \int_a^b \frac{f(x)}{f(x) + f(a+b-x)} dx$$

$$u = a+b-x \Rightarrow \frac{du}{dx} = -1 \text{ and } x = a+b-u$$

M1

terminals when $x=b$ $u=a$ and when $x=a$ $u=b$

$$I = - \int_b^a \frac{f(a+b-u)}{f(a+b-u) + f(u)} du = \int_a^b \frac{f(a+b-u)}{f(a+b-u) + f(u)} du$$

but u is a dummy variable, so
$$I = \int_a^b \frac{f(a+b-x)}{f(a+b-x) + f(x)} dx$$

$$I = \int_a^b \frac{f(x)}{f(x) + f(a+b-x)} dx \quad \text{adding}$$

M1

$$2I = \int_a^b \frac{f(a+b-x) + f(x)}{f(a+b-x) + f(x)} dx = \int_a^b 1 dx = [x]_a^b = b-a$$

$$I = \frac{1}{2}(b-a)$$

b. $f(x) = \sqrt{x} \quad a=1 \quad b=4$

$$A = \int_1^4 \frac{\sqrt{x}}{\sqrt{5-x} + \sqrt{x}} dx \quad \text{from a.}$$

$$A = \frac{1}{2}(4-1) = \frac{3}{2} = 1.5$$

A1

Question 6

$$V = \pi \int_a^b x^2 dy$$

$$y = \frac{\sqrt{9x^2 - 1}}{2x} \Rightarrow 2xy = \sqrt{9x^2 - 1}$$

M1

$$4x^2y^2 = 9x^2 - 1 \Rightarrow 1 = x^2(9 - 4y^2) \quad \text{so} \quad x^2 = \frac{1}{9 - 4y^2}$$

$$V = \pi \int_0^{\sqrt{2}} x^2 dy = \pi \int_0^{\sqrt{2}} \frac{1}{9 - 4y^2} dy \quad \text{partial fractions}$$

$$\frac{1}{9 - 4y^2} = \frac{A}{3 - 2y} + \frac{B}{3 + 2y} = \frac{A(3 + 2y) + B(3 - 2y)}{(3 - 2y)(3 + 2y)} = \frac{3(A + B) + 2y(A - B)}{9 - 4y^2}$$

M1

$$(1) \quad A - B = 0 \Rightarrow A = B$$

$$(2) \quad 3(A + B) = 1 \quad A = B = \frac{1}{6}$$

$$V = \frac{\pi}{6} \int_0^{\sqrt{2}} \left(\frac{1}{3 - 2y} + \frac{1}{3 + 2y} \right) dy$$

A1

$$V = \frac{\pi}{6} \left[-\frac{1}{2} \log_e |3 - 2y| + \frac{1}{2} \log_e |3 + 2y| \right]_0^{\sqrt{2}}$$

$$V = \frac{\pi}{12} \left[\log_e \left| \frac{3 + 2y}{3 - 2y} \right| \right]_0^{\sqrt{2}}$$

$$V = \frac{\pi}{12} \log_e \left(\frac{3 + 2\sqrt{2}}{3 - 2\sqrt{2}} \right) \text{ units}^3$$

A1

Question 7

a. $\frac{dy}{dx} = 2x - y$ $y_0 = y(0) = 0$ $h = \frac{1}{4}$ $x_0 = 0$ $f(x, y) = 2x - y$

$$y_1 = y_0 + hf(x_0, y_0) \qquad y_1 = 0 + \frac{1}{4}(2 \times 0 - 0) = 0$$

$$y_2 = y_1 + hf(x_1, y_1) \qquad x_1 = x_0 + h = \frac{1}{4} \qquad y_2 = 0 + \frac{1}{4}\left(2 \times \frac{1}{4} - 0\right) = \frac{1}{8}$$

$$y_3 = y_2 + hf(x_2, y_2) \qquad x_2 = x_1 + h = \frac{1}{2}$$

$$y_3 = \frac{1}{8} + \frac{1}{4}\left(2 \times \frac{1}{2} - \frac{1}{8}\right) = \frac{1}{8} + \frac{1}{4} \times \frac{7}{8} = \frac{11}{32}$$

M1

A1

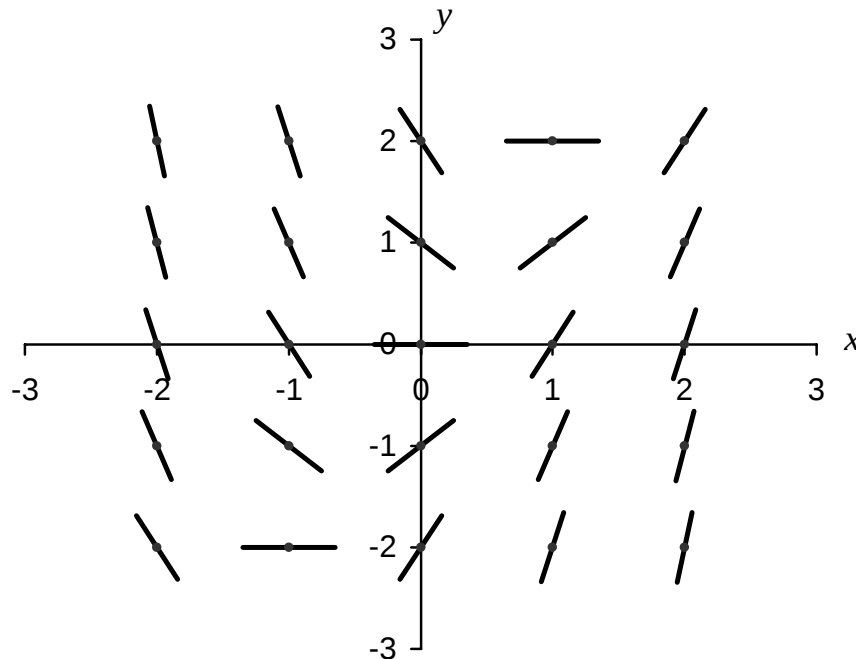
b. correct slopes in the table

A1

correct slopes in the graph below

A1

	$x = -2$	$x = -1$	$x = 0$	$x = 1$	$x = 2$
$y = -2$	-2	0	2	4	6
$y = -1$	-3	-1	1	3	5
$y = 0$	-4	-2	0	2	4
$y = 1$	-5	-3	-1	1	3
$y = 2$	-6	-4	-2	0	2



Question 8

a. $y = x \sin^{-1}\left(\frac{2x}{3}\right)$ using product rule

$$\frac{dy}{dx} = \sin^{-1}\left(\frac{2x}{3}\right) \frac{d}{dx}(x) + x \frac{d}{dx}\left(\sin^{-1}\left(\frac{2x}{3}\right)\right) \quad \text{M1}$$

$$\frac{dy}{dx} = \sin^{-1}\left(\frac{2x}{3}\right) + \frac{2x}{\sqrt{9-4x^2}} \quad \text{A1}$$

b. $\frac{dy}{dx} = \arcsin\left(\frac{2x}{3}\right)$

$$y = \int \sin^{-1}\left(\frac{2x}{3}\right) dx$$

$$\text{since } \frac{d}{dx}\left[x \sin^{-1}\left(\frac{2x}{3}\right)\right] = \sin^{-1}\left(\frac{2x}{3}\right) + \frac{2x}{\sqrt{9-4x^2}}$$

$$y = x \sin^{-1}\left(\frac{2x}{3}\right) - \int \frac{2x}{\sqrt{9-4x^2}} dx$$

$$\text{let } u = 9 - 4x^2 \quad \frac{du}{dx} = -8x$$

$$y = x \sin^{-1}\left(\frac{2x}{3}\right) + \frac{1}{4} \int u^{-\frac{1}{2}} du \quad \text{M1}$$

$$y = x \sin^{-1}\left(\frac{2x}{3}\right) + \frac{1}{2} u^{\frac{1}{2}} + c \quad \text{A1}$$

$$y = x \sin^{-1}\left(\frac{2x}{3}\right) + \frac{1}{2} \sqrt{9-4x^2} + c$$

$$\text{Now when } x = \frac{3}{2} \quad y = 0$$

$$0 = \frac{3}{2} \sin^{-1}(1) + 0 + c \Rightarrow c = -\frac{3\pi}{4}$$

$$y = x \sin^{-1}\left(\frac{2x}{3}\right) + \frac{1}{2} \sqrt{9-4x^2} - \frac{3\pi}{4} \quad \text{A1}$$

Question 9

a. $\vec{r}(t) = (t - \sin(t))\vec{i} + (1 + \cos(t))\vec{j}$ for $0 \leq t \leq 2\pi$

$$\dot{\vec{r}}(t) = (1 - \cos(t))\vec{i} - \sin(t)\vec{j}$$

$$|\dot{\vec{r}}(t)| = \sqrt{(1 - \cos(t))^2 + (-\sin(t))^2} \quad \text{A1}$$

$$|\dot{\vec{r}}(t)| = \sqrt{1 - 2\cos(t) + \cos^2(t) + \sin^2(t)}$$

$$|\dot{\vec{r}}(t)| = \sqrt{2 - 2\cos(t)}$$

$$|\dot{\vec{r}}(t)| = \sqrt{2(1 - \cos(t))} \quad \text{using half-angle formulae} \quad \text{M1}$$

$$|\dot{\vec{r}}(t)| = \sqrt{2\left(2\sin^2\left(\frac{t}{2}\right)\right)} = \left|2\sin\left(\frac{t}{2}\right)\right| \quad \text{since } 0 \leq \frac{t}{2} \leq \pi$$

$$|\dot{\vec{r}}(t)| = 2\sin\left(\frac{t}{2}\right)$$

b. $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$$\frac{dy}{dt} = \dot{y} = -\sin(t) \quad \frac{dx}{dt} = \dot{x} = 1 - \cos(t)$$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{-\sin(t)}{1 - \cos(t)} \quad \text{M1}$$

$$\left. \frac{dy}{dx} \right|_{t=\frac{\pi}{3}} = \frac{-\sin\left(\frac{\pi}{3}\right)}{1 - \cos\left(\frac{\pi}{3}\right)} = \frac{-\frac{\sqrt{3}}{2}}{1 - \frac{1}{2}} = -\sqrt{3} \quad \text{A1}$$

Question 10

a. $9x^2 + 36x + 4y^2 - 8y + 4 = 0$

Using implicit differentiation

$$18x + 36 + 8y \frac{dy}{dx} - 8 \frac{dy}{dx} = 0 \quad \text{M1}$$

$$\frac{dy}{dx}(8 - 8y) = 18x + 36 = 18(x + 2)$$

$$\frac{dy}{dx} = \frac{9(x + 2)}{4(1 - y)}$$

$$\frac{dy}{dx} = 0 \Rightarrow x = -2$$

$$36 - 72 + 4y^2 - 8y + 4 = 0$$

$$4y^2 - 8y - 32 = 0$$

$$y^2 - 2y - 8 = (y - 4)(y + 2) = 0$$

the points are $(-2, 4)$ and $(-2, -2)$

A1

b. $9x^2 + 36x + 4y^2 - 8y + 4 = 0$

$$9(x^2 + 4x) + 4(y^2 - 2y) = -4$$

$$9(x^2 + 4x + 4) + 4(y^2 - 2y + 1) = -4 + 36 + 4$$

$$9(x + 2)^2 + 4(y - 1)^2 = 36$$

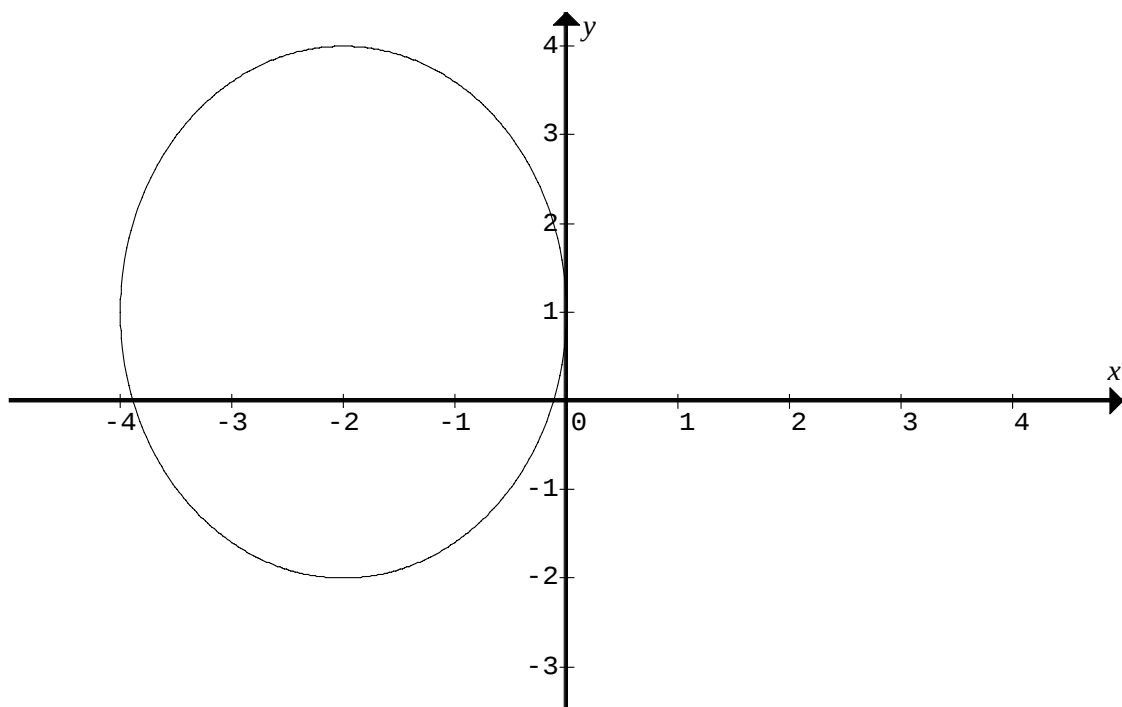
$$\frac{(x + 2)^2}{4} + \frac{(y - 1)^2}{9} = 1$$

ellipse centre $(-2, 1)$, domain $[-4, 0]$ range $[-2, 4]$

A1

graph, correct shape, scale

A1



END OF SUGGESTED SOLUTIONS