

2010 VCAA Specialist Math Exam 2 Solutions

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Note: Some steps can be done by CAS

SECTION 1

1	2	3	4	5	6	7	8	9	10	11
D	B	E	D	C	B	C	E	A	C	E

12	13	14	15	16	17	18	19	20	21	22
C	C	D	D	A	A	B	A	E	B	E

$$Q1 \quad y = \frac{2x^3 + x^2 - 1}{x^2 - x - 2} = 2x + 3 + \frac{7x + 5}{(x-2)(x+1)}$$

Straight line asymptotes are: $y = 2x + 3$, $x = 2$ and $x = -1$ D

Q2 Circle $x^2 - 6x + y^2 + 4y = b$, centre $(a, -2)$, radius 5

$$x^2 - 6x + 9 + y^2 + 4y + 4 = b + 9 + 4$$

$$(x-3)^2 + (y+2)^2 = b + 13$$

$$\therefore a = 3 \text{ and } b + 13 = 25, b = 12$$

B

$$Q3 \quad -1 \leq ax \leq 1, -\frac{1}{a} \leq x \leq \frac{1}{a}$$

E

$$Q4 \quad \frac{(y+3)^2}{9} - \frac{(x-6)^2}{4} = 1, \frac{(x-6)^2}{2^2} - \frac{(y+3)^2}{3^2} = -1$$

$$\text{Asymptotes: } y + 3 = \pm \frac{3}{2}(x - 6)$$

$$\therefore y = \frac{3}{2}x - 12 \text{ and } y = -\frac{3}{2}x + 6$$

D

$$Q5 \quad \frac{1}{z} = \frac{1}{z} \times \frac{z}{z} = \frac{z}{|z|^2}, \therefore \frac{1}{z} \text{ has the same argument as } z \text{ but a}$$

smaller modulus, given $|z| > 1$.

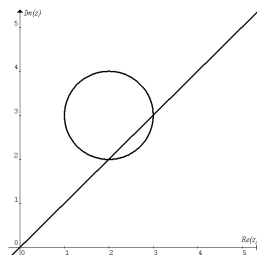
C

Q6 $P(z) = 0$ has real coefficients, $\therefore$ the conjugate roots theorem can be applied to it. Given $z = 3i$ as one of the roots, $\therefore z = -3i$ is also a root.

B

Q7

C



Q8 $z^3 = -27i = (3i)^3$, $\therefore$ one of the three roots of $z^3 = -27i$ is $3i$. The three roots are evenly spaced around the circle of radius 3. The points which represent the roots are P_4 , P_8 and P_{12} .

E

$$Q9 \quad x \sin(x) \sec(2x) = \frac{x \sin(x)}{\cos(2x)} = 0, x \in [0, 2\pi]$$

$$\therefore x = 0, \sin(x) = 0 \text{ and } \cos(2x) \neq 0$$

$$\therefore x = 0, \pi \text{ or } 2\pi$$

A

Q10 The diagonals of a rhombus are perpendicular.

$$\therefore (\tilde{a} + \tilde{b}) \cdot (\tilde{a} - \tilde{b}) = 0$$

C

$$Q11 \quad \tilde{F}_1 + \tilde{F}_2 + \tilde{F}_3 = \frac{1}{2}\tilde{i} + \frac{\sqrt{3}}{2}\tilde{j}, \therefore |\tilde{F}_1 + \tilde{F}_2 + \tilde{F}_3| = |\tilde{F}_3| = 1$$

E

$$Q12 \quad \cos \theta = \frac{(3\tilde{i} + 6\tilde{j} - 2\tilde{k}) \cdot (2\tilde{i} - 2\tilde{j} + \tilde{k})}{|3\tilde{i} + 6\tilde{j} - 2\tilde{k}| |2\tilde{i} - 2\tilde{j} + \tilde{k}|} = -\frac{8}{21}$$

$$\therefore \theta = \cos^{-1}\left(-\frac{8}{21}\right) \approx 112.4^\circ$$

C

$$Q13 \quad \tilde{r}(t) = (\sqrt{t-2})\tilde{i} + (2t)\tilde{j}, t \geq 2$$

$$y = 2t \text{ and } x = \sqrt{t-2}, \therefore x \geq 0 \text{ and } t = x^2 + 2$$

$$\therefore y = 2(x^2 + 2) = 2x^2 + 4, x \geq 0$$

C

Q14 $f(0) = 0$, either B or D.

$f'(0) = 0$ for both also.

Only D gives $f''(x) = 2e^x \sin(x)$

D

$$Q15 \quad \text{Let } u = \tan(2x), \therefore \frac{1}{2} \frac{du}{dx} = \sec^2(2x)$$

$$\text{When } x = 0, u = 0; \text{ when } x = \frac{\pi}{24}, u = 2 - \sqrt{3}$$

$$\therefore \int_0^{\frac{\pi}{24}} \tan(2x) \sec^2(2x) dx = \frac{1}{2} \int_0^{2-\sqrt{3}} (u) du$$

D

$$Q16 \quad \text{Gradient of the tangent} = \frac{dy}{dx},$$

$$\text{gradient of the normal} = -\frac{1}{\frac{dy}{dx}} = -2 \times \frac{y-1}{x-1}$$

$$\therefore \frac{dy}{dx} = -\frac{x-1}{2(y-1)}, \therefore \frac{dy}{dx} + \frac{x-1}{2(y-1)} = 0$$

A

Q17 The gradient of the little line segment at (1,0) is 2, i.e.

$$\frac{dy}{dx} = -2. \text{ Only } \frac{dy}{dx} = \frac{y-2x}{2y+x} \text{ satisfies this condition.}$$

A

$$Q18 \quad \frac{dx}{dt} = -\frac{10}{10-t}$$

$$t = 0, x = 5, \quad \frac{dx}{dt} = -1$$

$$t = 0.5, x = 5 + 0.5(-1) = 4.5, \quad \frac{dx}{dt} = -\frac{10}{10-0.5} = -1.05263$$

$$t = 1.0, x = 4.5 + 0.5(-1.05263) \approx 3.97$$

B

Q19 Position = area of the trapezium – the area of the triangle

$$= \frac{1}{2}(5+15)3 - \frac{1}{2} \times 10 \times 2.5 = 17.5 \text{ m}$$

A

Q20 Given $v = 2\sqrt{1-x^2}$, $v^2 = 4(1-x^2)$,

$$a = \frac{1}{2} \times \frac{d}{dx}(v^2) = -4x$$

E

Q21 $u = 0$, $v = 12$, $s = 16$, $a = ?$

$$v^2 = u^2 + 2as$$
, $a = 4.5$, $F = ma = 2 \times 4.5 = 9.0 \text{ N}$

B

Q22 $v = \frac{dx}{dt} = 25 + x^2$, $\frac{dt}{dx} = \frac{1}{5^2 + x^2}$, $t = \int_5^x \frac{1}{5^2 + x^2} dx + 0$,

$$\therefore t = \frac{1}{5} \int_5^x \frac{5}{5^2 + x^2} dx = \frac{1}{5} \left[\tan^{-1}\left(\frac{x}{5}\right) \right]_5^x$$

$$= \frac{1}{5} \left(\tan^{-1}\left(\frac{x}{5}\right) - \frac{\pi}{4} \right) = \frac{1}{5} \tan^{-1}\left(\frac{x}{5}\right) - \frac{\pi}{20}$$

E

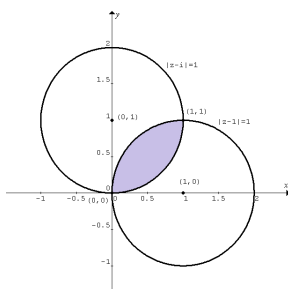
SECTION 2

Q1a $|z-i|=1$, $|x+yi-i|=1$, $|x+yi-i|^2=1$, $|x+(y-1)i|^2=1$,
 $\therefore x^2 + (y-1)^2 = 1$

Q1b The 2 circles are reflections of each other in the line $y = x$,
 $\therefore$ the intersections are on $y = x$.

Solve $x^2 + (y-1)^2 = 1$ and $y = x$, $\therefore x^2 + (x-1)^2 = 1$,
 $2x^2 - 2x = 0$, $\therefore x = 0$ and $y = 0$ OR $x = 1$ and $y = 1$
 The points of intersection are $(0,0)$ and $(1,1)$.

Q1c and ei



Q1di $y = x$

Q1dii $z = a\bar{z}$, $x + yi = a(x - yi)$,
 $\therefore 1 + i = a(1 - i)$ since $(1,1)$ is on it.
 $\therefore a = \frac{1+i}{1-i} = i$

Q1eii Area of the shaded region $= 2 \left(\frac{1}{4} \times \pi \times 1^2 - \frac{1}{2} \right) = \frac{\pi}{2} - 1$

Q2a Given $\tilde{a}(t) = -9.8\tilde{k}$, $\tilde{v}(0) = 35\tilde{i} + 5\tilde{j} + 24.5\tilde{k}$, $\tilde{r}(0) = \tilde{0}$.
 $\tilde{v}(t) = -9.8t\tilde{k} + \tilde{c}$, $\therefore \tilde{c} = 35\tilde{i} + 5\tilde{j} + 24.5\tilde{k}$
 $\therefore \tilde{v}(t) = -9.8t\tilde{k} + 35\tilde{i} + 5\tilde{j} + 24.5\tilde{k} = 35\tilde{i} + 5\tilde{j} + (24.5 - 9.8t)\tilde{k}$
 $\therefore \tilde{r}(t) = 35t\tilde{i} + 5t\tilde{j} + (24.5t - 4.9t^2)\tilde{k} + \tilde{d}$, $\tilde{d} = \tilde{0}$ since $\tilde{r}(0) = \tilde{0}$.
 $\therefore \tilde{r}(t) = 35t\tilde{i} + 5t\tilde{j} + (24.5t - 4.9t^2)\tilde{k}$

Q2b The k -component of $\tilde{r}(t)$ is zero at the start and finish.
 $\therefore$ let $24.5t - 4.9t^2 = 0$, $t = 0$ or 5 . $\therefore \Delta t = 5 - 0 = 5$ seconds

Q2c At $t = \frac{5}{2} = 2.5$,
 maximum height $= 24.5 \times 2.5 - 4.9(2.5)^2 = 30.625 \text{ m}$

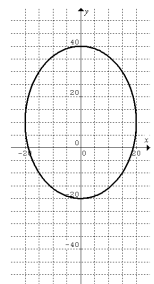
Q2d $\tilde{v}(t) = 35\tilde{i} + 5\tilde{j} + (24.5 - 9.8t)\tilde{k}$,
 $\therefore \tilde{v}(5) = 35\tilde{i} + 5\tilde{j} - 24.5\tilde{k}$ is the velocity when the ball hits the ground.
 Speed $= |\tilde{v}(5)| = \sqrt{35^2 + 5^2 + (-24.5)^2} \approx 43 \text{ ms}^{-1}$

Q2e $\tilde{r}_{\text{hole}} = 200\tilde{i}$, $\tilde{r}_{\text{ball}}(5) = 175\tilde{i} + 25\tilde{j}$
 Distance from the hole to the ball $= |\tilde{r}_{\text{ball}}(5) - \tilde{r}_{\text{hole}}|$
 $= |-25\tilde{i} + 25\tilde{j}| = \sqrt{(-25)^2 + 25^2} \approx 35 \text{ m}$

Q3a x -intercepts:

Let $y = 0$, $\frac{x^2}{400} + \frac{1}{9} = 1$, $x^2 = \frac{3200}{9}$,
 $x = \pm \frac{40\sqrt{2}}{3}$.

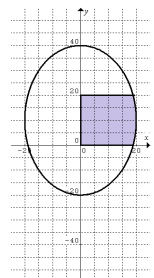
x -intercepts $\left(-\frac{40\sqrt{2}}{3}, 0 \right)$, $\left(\frac{40\sqrt{2}}{3}, 0 \right)$



Q3bi

$$V = \int_0^{20} \pi x^2 dy = \int_0^{20} \pi \left(400 - \frac{4(y-10)^2}{9} \right) dy$$

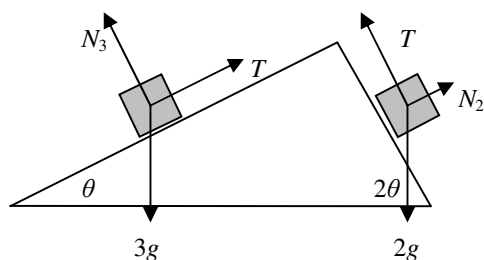
Q3bii $V \approx 24202 \text{ cm}^3$



Q3c Given $\frac{dV}{dt} = 500$ and $\frac{dV}{dh} = \frac{25\pi}{36}(800 + 20h - h^2)$.
 $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$, $\therefore 500 = \frac{25\pi}{36}(800 + 20h - h^2) \times \frac{dh}{dt}$.

When $h = 15$, $\frac{dh}{dt} = \frac{500}{\frac{25\pi}{36}(800 + 20(15) - 15^2)} \approx 0.26 \text{ cm min}^{-1}$

Q4



Q4a $T - 3g \sin \theta = 3a$ (1)

Q4b $2g \sin(2\theta) - T = 2a$ (2)

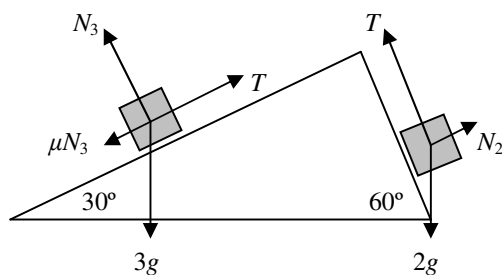
Q4c (1) + (2): $2g \sin(2\theta) - 3g \sin \theta = 5a$

$4g \sin \theta \cos \theta - 3g \sin \theta = 5a$, $\therefore a = \frac{g \sin \theta}{5} (4 \cos \theta - 3)$

Q4d Let $a = 0$ for equilibrium. $\therefore \frac{g \sin \theta}{5} (4 \cos \theta - 3) = 0$

Since $0 < \theta < \frac{\pi}{2}$, $\therefore 4 \cos \theta - 3 = 0$, $\theta = \cos^{-1}\left(\frac{3}{4}\right) \approx 41.4^\circ$

Q4e



$N_3 = 3g \cos 30^\circ = \frac{3\sqrt{3}g}{2}$, $\mu N_3 = 0.05 \times \frac{3\sqrt{3}g}{2} = \frac{3\sqrt{3}g}{40}$

3-kg mass: $T - 3g \sin 30^\circ - \mu N_3 = 3a$,

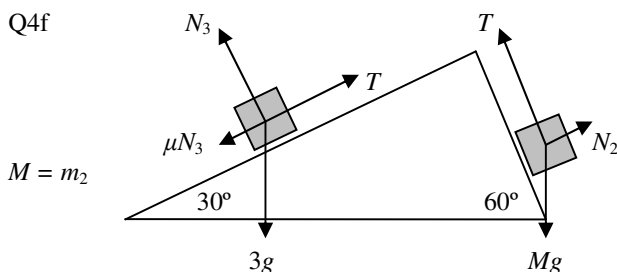
$\therefore T - \frac{3g}{2} - \frac{3\sqrt{3}g}{40} = 3a$ (1)

2-kg mass: $2g \sin 60^\circ - T = 2a$

$\therefore \sqrt{3}g - T = 2a$ (2)

Solve (1) and (2): $a \approx 0.2 \text{ m s}^{-2}$

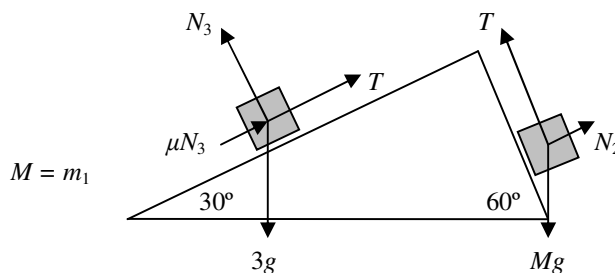
Q4f



3-kg mass: $T - 3g \sin 30^\circ - \mu N_3 = 0$, $T - 15.97306 = 0$... (1)

M mass: $Mg \sin 60^\circ - T = 0$, $8.487M - T = 0$ (2)

Solve (1) and (2): $M \approx 1.88$, i.e. $m_2 \approx 1.88 \text{ kg}$



3-kg mass: $3g \sin 30^\circ - \mu N_3 - T = 0$, $13.427 - T = 0$ (1)

M mass: $T - Mg \sin 60^\circ = 0$, $T - 8.487M = 0$ (2)

Solve (1) and (2): $M \approx 1.58$, i.e. $m_1 \approx 1.58 \text{ kg}$

Q5a At time t min, volume = $10t + 10$ litres, mass = x grams,

$\therefore \text{concentration} = \frac{x}{10(t+1)}$ grams per litre

Q5b Rate of inflow of chemical = $20 \times e^{-0.2t}$ grams per min,

rate of outflow of chemical = $\frac{x}{10(t+1)} \times 10 = \frac{x}{t+1}$ grams per min

Rate of change of the amount of chemical

= rate of inflow - rate of outflow

$\therefore \frac{dx}{dt} = 20e^{-0.2t} - \frac{x}{t+1}$,

$\therefore \frac{dx}{dt} + \frac{x}{t+1} = 20e^{-0.2t}$.

Q5ci $x(t) = \frac{600}{t+1} - \frac{100e^{-0.2t}(t+6)}{t+1}$

$\frac{dx}{dt} = \frac{20e^{-0.2t}(t+6)}{t+1} + \frac{500e^{-0.2t} - 600}{(t+1)^2}$

Q5cii Differential equation: $\frac{dx}{dt} + \frac{x}{t+1} = 20e^{-0.2t}$

$LHS = \frac{20e^{-0.2t}(t+6)}{t+1} + \frac{500e^{-0.2t} - 600}{(t+1)^2} + \frac{600}{t+1} - \frac{100e^{-0.2t}(t+6)}{t+1}$

$= \frac{20e^{-0.2t}(t+6)(t+1)}{(t+1)^2} + \frac{500e^{-0.2t} - 600}{(t+1)^2} + \frac{600 - 100e^{-0.2t}(t+6)}{(t+1)^2}$

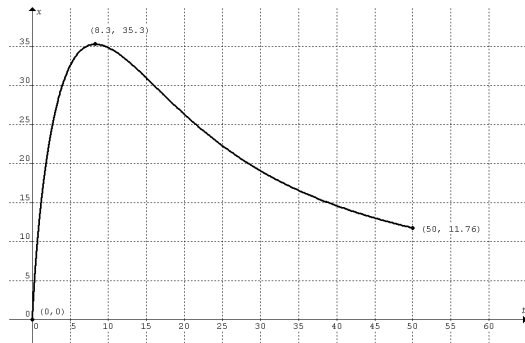
$= \frac{20e^{-0.2t}(t+6)(t+1)}{(t+1)^2} + \frac{500e^{-0.2t}}{(t+1)^2} - \frac{100e^{-0.2t}(t+6)}{(t+1)^2}$

$= \frac{20e^{-0.2t}}{(t+1)^2} ((t+6)(t+1) + 25 - 5(t+6))$

$= \frac{20e^{-0.2t}}{(t+1)^2} (t+1)^2 = 20e^{-0.2t} = RHS$

Initial condition: $x(0) = \frac{600}{0+1} - \frac{100e^0(0+6)}{0+1} = 0$

Q5d



Turning point: (8.3, 0.353)

$$\text{Q5e Rate of outflow} = \frac{x}{t+1} = \frac{600 - 100e^{-0.2t}(t+6)}{(t+1)^2}$$

$$\text{Amount of outflow} = \int_0^{10} \frac{600 - 100e^{-0.2t}(t+6)}{(t+1)^2} dt \approx 51.6 \text{ grams}$$

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