

2011 Specialist Maths Trial Exam 2 Solutions

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Section 1

1	2	3	4	5	6	7	8	9	10	11
D	B	C	C	B	A	E	E	A	A	D

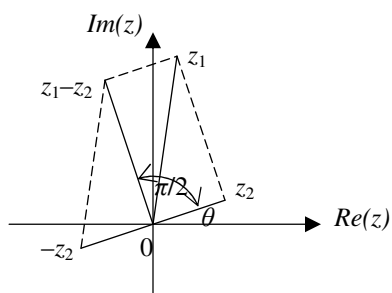
12	13	14	15	16	17	18	19	20	21	22
C	A	B	D	A	C	B	E	A	D	A

Q1 $z_1^2 + 3z_2^2 = 0$, $z_1 = \pm i\sqrt{3}z_2$

$$|z_1 - z_2| = |\pm i\sqrt{3}z_2 - z_2| = |(\pm i\sqrt{3} - 1)z_2| = |\pm i\sqrt{3} - 1||z_2| = 2|z_2|$$

D

Q2

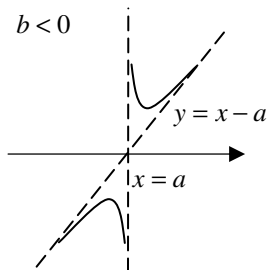


Q3 $P(z) = z^3 + 3iz^2 - 3z - i = (z+i)^3$

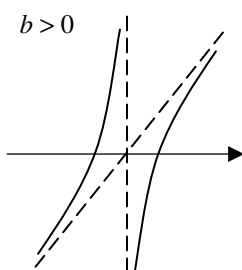
Q4 Let $z = x + iy$, $\text{Im}(z) = |z - i|$, $(\text{Im}(z))^2 = |z - i|^2$

$$y^2 = x^2 + (y-1)^2, y = \frac{1}{2}x^2 + \frac{1}{2}$$

Q5 $b < 0$



$b > 0$



B

Q6 For $y = \frac{2}{a+bx+4ax^2}$ to have only one asymptote, the

discriminant of $a+bx+4ax^2$ must be a negative value.

$$\therefore \Delta = b^2 - 16a^2 < 0, \therefore -4a < b < 4a$$

Since $a > 1$, $-1 \leq b \leq 4$ satisfies the requirement $-4a < b < 4a$.

A

Q7 The equation of the hyperbola is $\frac{(x+1)^2}{1^2} - \frac{(y-2)^2}{b^2} = 1$.

The gradients of the asymptotes are $\pm b = \pm 2$ (best approximation determined from the scaled graph).

Equations of the asymptotes: $y - 2 = \pm 2(x + 1)$,

i.e. $y = -2x$, $y = 2x + 4$

E

Q8 $\tan^{-1}(a) = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

$$\sin^{-1}(b) = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

$$\therefore \sec(\tan^{-1}(a) + \sin^{-1}(b)) = \sec\left(-\frac{2\pi}{3}\right) = \frac{1}{\cos\left(-\frac{2\pi}{3}\right)} = -2 \quad \text{E}$$

Q9 The values of a and b do not change the range of f . The

range of $\cos^{-1}(x)$ is $[0, \pi]$, $\therefore$ the range of $\cos^{-1}\left(\frac{x}{a} + b\right) + c$ is

$[c, \pi + c]$. A

Q10 $\tan^{-1}(x-a+1) = \tan^{-1}(x-a) + \frac{\pi}{4}$

$$\tan^{-1}(x-a+1) - \tan^{-1}(x-a) = \frac{\pi}{4}$$

$$\tan(\tan^{-1}(x-a+1) - \tan^{-1}(x-a)) = \tan\left(\frac{\pi}{4}\right)$$

$$\frac{(x-a+1) - (x-a)}{1 + (x-a+1)(x-a)} = 1, \therefore 1 = 1 + (x-a+1)(x-a)$$

$$\therefore (x-a+1)(x-a) = 0, \therefore x = a-1 \text{ or } x = a$$

B

C

Q11 $\vec{OP} = \tilde{i} - \tilde{j}$, $\vec{OQ} = -\tilde{j} + \tilde{k}$, $\vec{PQ} = \vec{OQ} - \vec{OP} = -\tilde{i} + \tilde{k}$

$$\therefore |\vec{PQ}| = |\vec{OQ}| = |\vec{OP}|$$

$\therefore \triangle OPQ$ is equilateral.

$$\therefore \angle OPQ = 60^\circ$$

C

D

Q12 $2\tilde{i} + p\tilde{j} + 3\tilde{k}$, $-\tilde{i} + 3\tilde{j} + q\tilde{k}$ and $\tilde{i} - \tilde{j} + \tilde{k}$ are linearly dependent.

$$2\tilde{i} + p\tilde{j} + 3\tilde{k} + m(-\tilde{i} + 3\tilde{j} + q\tilde{k}) + n(\tilde{i} - \tilde{j} + \tilde{k}) = 0, m, n \neq 0$$

$$\therefore 2 - m + n = 0 \quad \dots\dots\dots (1)$$

$$p + 3m - n = 0 \quad \dots\dots\dots (2)$$

$$3 + qm + n = 0 \quad \dots\dots\dots (3)$$

$$(1) + (2): m = \frac{-p-2}{2}$$

$$(2) + (3): m = \frac{-p-3}{q+3}$$

$$\therefore \frac{-p-2}{2} = \frac{-p-3}{q+3}$$

$$\therefore q = \frac{-p}{p+2}$$

B

C

Q13 $\angle ACB = \angle ADB$

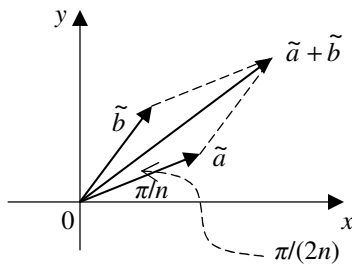
$$\cos \angle ACB = \cos \angle ADB$$

$$\frac{\vec{AC} \cdot \vec{BC}}{|\vec{AC}| |\vec{BC}|} = \frac{\vec{AD} \cdot \vec{BD}}{|\vec{AD}| |\vec{BD}|}$$

A

Q14 $\tilde{a} = \cos\left(\frac{\pi}{n}\right)\tilde{i} + \sin\left(\frac{\pi}{n}\right)\tilde{j}$, $\tilde{b} = \cos\left(\frac{2\pi}{n}\right)\tilde{i} + \sin\left(\frac{2\pi}{n}\right)\tilde{j}$

$\tilde{a}$ and $\tilde{b}$ have the same magnitude.



Angle between vectors $\tilde{a} + \tilde{b}$ and $\tilde{i}$ is $\frac{\pi}{n} + \frac{\pi}{2n} = \frac{3\pi}{2n}$.

Q15 $v(t) = 2\sin^{-1}\left(\frac{t}{10} - 1\right) + \pi$, $0 \leq t \leq 10$

Average velocity = $\frac{\text{displacement}}{\text{time}}$

$$= \frac{\int_0^{10} \left(2\sin^{-1}\left(\frac{t}{10} - 1\right) + \pi\right) dt}{10} = 2$$

Q16 $\int_0^1 \frac{1-2x-x^2}{\sqrt{1-x^2}} dx = \int_0^1 \frac{1-x^2}{\sqrt{1-x^2}} dx + \int_0^1 \frac{-2x}{\sqrt{1-x^2}} dx$

$$= \int_0^1 \sqrt{1-x^2} dx + \int_0^1 \frac{-2x}{\sqrt{1-x^2}} dx$$

$$= \frac{\pi}{4} - 2$$

Q17 $\tilde{r} = 2\cos^{-1}(t)\tilde{i} - 2\cos^{-1}(t)\tilde{j} + \cos^{-1}(t)\tilde{k}$, $0 \leq t \leq 1$

$$\tilde{r} = (2\tilde{i} - 2\tilde{j} + \tilde{k})\cos^{-1}(t)$$

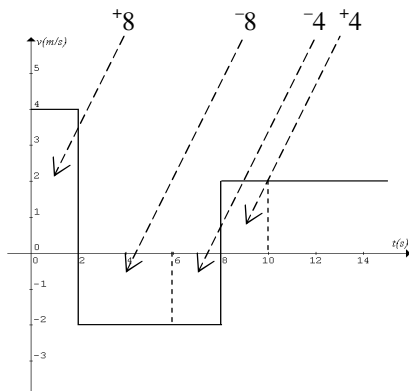
$$\tilde{v} = (2\tilde{i} - 2\tilde{j} + \tilde{k}) \frac{-1}{\sqrt{1-t^2}}$$

$$\tilde{a} = (2\tilde{i} - 2\tilde{j} + \tilde{k}) \frac{-t}{(1-t^2)^{\frac{3}{2}}}$$

Q18 $a = +2$, $u = -10$, $s = +16 - +5 = +11$, $t = ?$

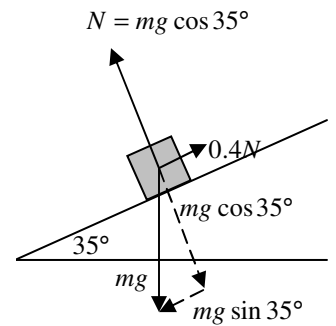
$$s = ut + \frac{1}{2}at^2, t = 11$$

Q19



Q20

The diagram shows the particle sliding down the inclined plane. If the particle slides up the plane the force due to friction points in the opposite direction. Since $mg \sin 35^\circ > 0.4mg \cos 35^\circ$ $\therefore$ there is always a resultant force down the plane. $\therefore$ the particle does not move at constant velocity.



A

B

Q21 $v^2 = x - 2$, $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2}$

Since $v^2 = x - 2 \geq 0$, $\therefore x \geq 2$, i.e. the particle moves along the positive x -axis.

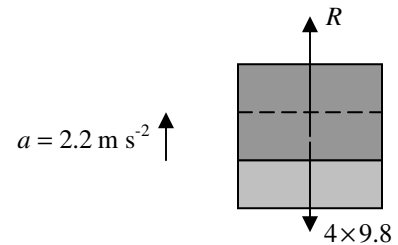
Since $v^2 = x - 2$, $\therefore v = \pm\sqrt{x-2}$. When $v = -\sqrt{x-2}$, the particle moves towards the origin.

D

Q22 $+R + 4 \times -9.8 = 4 \times +2.2$, $\therefore R = +48 \text{ N}$

A

D

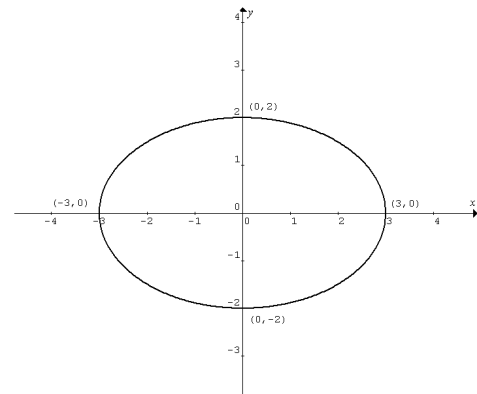


A

Section 2

Q1a $4x^2 + 9y^2 = 36$, $\therefore \frac{x^2}{9} + \frac{y^2}{4} = 1$

C



B

E

Q1b $4x^2 + 9y^2 = 36$

Implicit differentiation: $8x + 18y \frac{dy}{dx} = 0$, $\frac{dy}{dx} = -\frac{4x}{9y}$

At $y = 1$, $4x^2 = 27$, $\therefore x = \pm \frac{3\sqrt{3}}{2}$

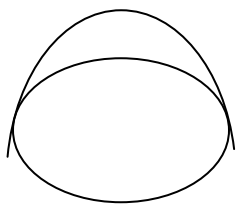
$$\therefore \frac{dy}{dx} = \pm \frac{2\sqrt{3}}{3}$$

Q1c $x^2 + 3y = c$ is an inverted parabola.

It touches the ellipse at $(0, -2)$ if $c = -6$.

It touches the ellipse at $(0, 2)$ if $c = 6$.

There are two other possible points:



Solve simultaneously, $x^2 + 3y = c$, $4x^2 + 9y^2 = 36$

$$\therefore 4(c - 3y) + 9y^2 = 36,$$

$$\therefore 9y^2 - 12y + (4c - 36) = 0$$

Same y -coordinate at the contact points.

To have only one y value, let the discriminant be 0.

$$\therefore (-12)^2 - 4 \times 9 \times (4c - 36) = 0$$

$$\therefore c = 10$$

$$\therefore 9y^2 - 12y + 4 = 0, \quad y = \frac{2}{3}, \quad x = \pm 2\sqrt{2}$$

Two other possible points are $\left(2\sqrt{2}, \frac{2}{3}\right)$ and $\left(-2\sqrt{2}, \frac{2}{3}\right)$ if $c = 10$.

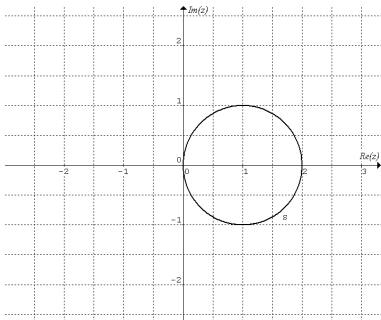
Q1di At $P(0, -2)$, $x^2 + 3y = -6$ is the minimum value.

Q1dii At $P\left(-2\sqrt{2}, \frac{2}{3}\right)$ or $P\left(2\sqrt{2}, \frac{2}{3}\right)$, $x^2 + 3y = 10$ is the maximum value.

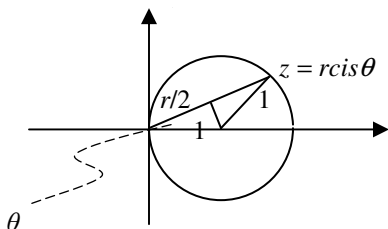
Q2ai Let $z = x + iy$

$$|z - 1| = 1, \quad |(x - 1) + yi| = 1, \quad |(x - 1) + yi|^2 = 1, \quad (x - 1)^2 + y^2 = 1$$

Q2aii



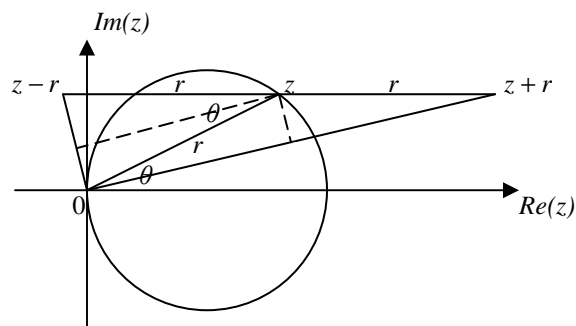
Q2bi



$$\frac{r}{2} = 1 \cos \theta, \therefore r = 2 \cos \theta$$

$$\begin{aligned} \text{Q2bii } \frac{1}{z} &= \frac{1}{r \text{cis } \theta} = \frac{\text{cis}(-\theta)}{r} = \frac{\cos(-\theta) + i \sin(-\theta)}{2 \cos \theta} \\ &= \frac{\cos \theta - i \sin \theta}{2 \cos \theta} = \frac{1}{2} - i \frac{\tan \theta}{2} \end{aligned}$$

Q2biii



$$\text{Arg}\left(\frac{z-r}{z+r}\right) = \text{Arg}(z-r) - \text{Arg}(z+r)$$

= angle formed by $z-r$, 0 and $z+r$

= $\frac{\pi}{2}$ because 0 is on the circumference of the circle of radius r centred at z .

Refer to the diagram: $|z-r| = 2 \times r \sin \frac{\theta}{2}$, $|z+r| = 2 \times r \cos \frac{\theta}{2}$

$$\therefore \frac{|z-r|}{|z+r|} = \frac{|z-r|}{|z+r|} = \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} = \tan \frac{\theta}{2}$$

$$\text{Q2ci } \frac{z-r}{z+r} = \left| \frac{z-r}{z+r} \right| \text{cis}\left(\text{Arg}\left(\frac{z-r}{z+r}\right)\right) = \left(\tan \frac{\theta}{2}\right) \text{cis} \frac{\pi}{2} = i \tan \frac{\theta}{2}$$

$$\therefore \frac{z_1 - r_1}{z_1 + r_1} = i \tan \frac{\theta_1}{2}, \quad \frac{z_2 - r_2}{z_2 + r_2} = i \tan \frac{\theta_2}{2}, \quad \frac{z_3 - r_3}{z_3 + r_3} = i \tan \frac{\theta_3}{2}$$

$\therefore$ all three complex numbers are purely imaginary (has no real part), so they are on the imaginary axis and $\therefore$ collinear.

$$\text{Q2cii } \frac{1}{z} = \frac{1}{2} - i \frac{\tan \theta}{2}$$

$$\therefore \frac{1}{z_1} = \frac{1}{2} - i \frac{\tan \theta_1}{2}, \quad \frac{1}{z_2} = \frac{1}{2} - i \frac{\tan \theta_2}{2}, \quad \frac{1}{z_3} = \frac{1}{2} - i \frac{\tan \theta_3}{2}$$

All three complex numbers have the same real part of $\frac{1}{2}$, $\therefore$ they line up vertically on the line $\text{Re}(z) = \frac{1}{2}$.

$$\text{Q3a } \tilde{r}(t) = \sqrt{\frac{1+(t-1)^2}{2}} \tilde{i} - (t-1) \tilde{j} + \frac{\sqrt{2}|t-2|}{4} \tilde{k}, \quad t \geq 0$$

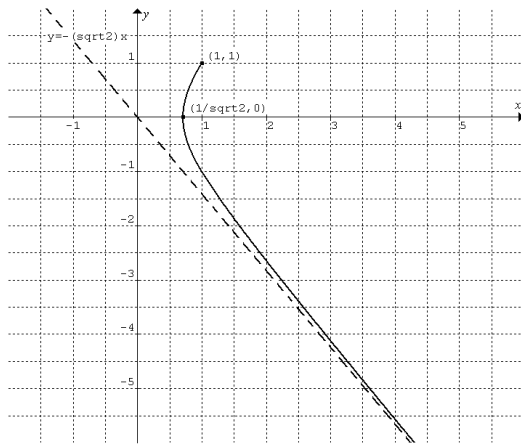
$$x = \sqrt{\frac{1+(t-1)^2}{2}}, \quad y = -(t-1)$$

$$x^2 = \frac{1+(t-1)^2}{2}, \quad y^2 = (t-1)^2$$

$$\therefore x^2 = \frac{1+y^2}{2}, \quad 2x^2 - y^2 = 1$$

$$\therefore \left(\frac{x^2}{\frac{1}{2}}\right) - y^2 = 1$$

Q3b When $t = 0$, the shadow is at $(1,1)$.



$$\text{Q3c } |\tilde{r}|^2 = \frac{1+(t-1)^2}{2} + (t-1)^2 + \frac{(t-2)^2}{8}$$

$$|\tilde{r}|^2 = \frac{1}{2} + \frac{3(t-1)^2}{2} + \frac{(t-2)^2}{8}$$

$$\text{Let } \frac{d}{dt}|\tilde{r}|^2 = 0 \therefore 3(t-1) + \frac{t-2}{4} = 0$$

$\therefore t = \frac{14}{13}$, the time when the aeroplane was closest to the controller.

$$\text{Q3di } \tilde{r}(t) = \sqrt{\frac{1+(t-1)^2}{2}} \tilde{i} - (t-1) \tilde{j} + \frac{\sqrt{2}|t-2|}{4} \tilde{k}$$

$$\therefore \tilde{v} = \frac{d}{dt} \tilde{r} = \frac{t-1}{2\sqrt{\frac{1+(t-1)^2}{2}}} \tilde{i} - \tilde{j} - \frac{\sqrt{2}}{4} \tilde{k} \text{ for } t < 2$$

$$\text{When } t = 0, \tilde{v} = -\frac{1}{2} \tilde{i} - \tilde{j} - \frac{\sqrt{2}}{4} \tilde{k}$$

$$\text{and speed } |\tilde{v}| = \sqrt{\left(-\frac{1}{2}\right)^2 + (-1)^2 + \left(-\frac{\sqrt{2}}{4}\right)^2} = \sqrt{\frac{11}{8}} = \frac{\sqrt{22}}{4}$$

$$\text{Q3dii When } t = 0, \tilde{v} = -\frac{1}{2} \tilde{i} - \tilde{j} - \frac{\sqrt{2}}{4} \tilde{k}$$

$$\hat{v} = \frac{\tilde{v}}{|\tilde{v}|} = \sqrt{\frac{8}{11}} \left(-\frac{1}{2} \tilde{i} - \tilde{j} - \frac{\sqrt{2}}{4} \tilde{k} \right)$$

Let θ be the angle between $\hat{v}$ and $\tilde{k}$.

$$\hat{v} \cdot \tilde{k} = -\sqrt{\frac{8}{11}} \times \frac{\sqrt{2}}{4} = -\frac{1}{\sqrt{11}} = \cos \theta, \therefore \theta \approx 108^\circ$$

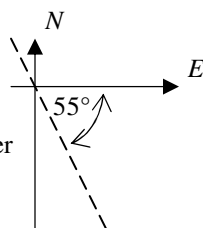
$\therefore$ angle between flight path and ground $= 108 - 90 = 18^\circ$

$$\text{Q3e Asymptote: } y = -\sqrt{2}x$$

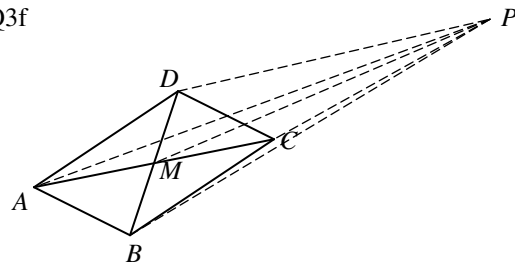
$$\therefore \tan \phi = -\sqrt{2}$$

$$\therefore \phi \approx -55^\circ$$

True bearing of destination from controller $= 90 + 55 = 145^\circ T$



Q3f



P is at any position above the rectangle $ABCD$.

$$\overrightarrow{AC} = \overrightarrow{PC} - \overrightarrow{PA}, \overrightarrow{BD} = \overrightarrow{PD} - \overrightarrow{PB}$$

$$|\overrightarrow{AC}| = |\overrightarrow{BD}|, \text{ diagonals of rectangle } ABCD$$

$$\therefore |\overrightarrow{PC} - \overrightarrow{PA}|^2 = |\overrightarrow{PD} - \overrightarrow{PB}|^2$$

$$\therefore |\overrightarrow{PC}|^2 + |\overrightarrow{PA}|^2 - 2 \times \overrightarrow{PC} \cdot \overrightarrow{PA} = |\overrightarrow{PD}|^2 + |\overrightarrow{PB}|^2 - 2 \times \overrightarrow{PD} \cdot \overrightarrow{PB} \dots (1)$$

M is the midpoint of the diagonals.

$$\therefore \overrightarrow{PM} = \frac{1}{2}(\overrightarrow{PC} + \overrightarrow{PA}) = \frac{1}{2}(\overrightarrow{PD} + \overrightarrow{PB})$$

$$\therefore |\overrightarrow{PM}|^2 = \frac{1}{4}|\overrightarrow{PC} + \overrightarrow{PA}|^2 = \frac{1}{4}|\overrightarrow{PD} + \overrightarrow{PB}|^2$$

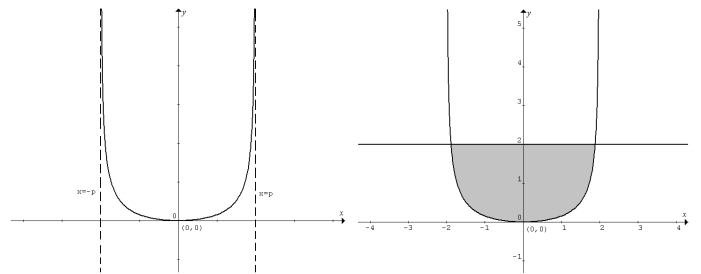
$$\therefore |\overrightarrow{PC}|^2 + |\overrightarrow{PA}|^2 + 2 \times \overrightarrow{PC} \cdot \overrightarrow{PA} = |\overrightarrow{PD}|^2 + |\overrightarrow{PB}|^2 + 2 \times \overrightarrow{PD} \cdot \overrightarrow{PB} \dots (2)$$

$$\frac{(1)+(2)}{2}: |\overrightarrow{PA}|^2 + |\overrightarrow{PC}|^2 = |\overrightarrow{PB}|^2 + |\overrightarrow{PD}|^2$$

$$\text{Q4a } f(x) = \frac{p}{\sqrt{p^2 - x^2}} - 1, p \in \mathbb{R}^+$$

Asymptotes: $p^2 - x^2 = 0, x = \pm p$

y-intercept: $x = 0, y = 0$



$$\text{Q4b } p = 2, \therefore f(x) = \frac{2}{\sqrt{4 - x^2}} - 1$$

$$\text{At } y = 2, \frac{2}{\sqrt{4 - x^2}} - 1 = 2, x = \pm \frac{4\sqrt{2}}{3}$$

$$\text{Area} = 2 \left[\frac{4\sqrt{2}}{3} \times 2 - \int_0^{\frac{4\sqrt{2}}{3}} \left(\frac{2}{\sqrt{4 - x^2}} - 1 \right) dx \right]$$

$$= 2 \left[\frac{8\sqrt{2}}{3} - \left[2 \sin^{-1} \left(\frac{x}{2} \right) - x \right]_0^{\frac{4\sqrt{2}}{3}} \right]$$

$$= 2 \left[\frac{8\sqrt{2}}{3} - 2 \sin^{-1} \left(\frac{2\sqrt{2}}{3} \right) + \frac{4\sqrt{2}}{3} \right] = 8\sqrt{2} - 4 \sin^{-1} \left(\frac{2\sqrt{2}}{3} \right)$$

$$\text{Q4ci } y = \frac{2}{\sqrt{4-x^2}} - 1, \therefore x^2 = 4 \left(1 - \frac{1}{(y+1)^2} \right)$$

$$V = \int_0^2 \pi x^2 dy = \int_0^2 4\pi \left(1 - \frac{1}{(y+1)^2} \right) dy$$

$$\text{Q4cii } V = 4\pi \left[y + \frac{1}{y+1} \right]_0^2 = 4\pi \left(2 + \frac{1}{3} - 1 \right) = \frac{16\pi}{3}$$

$$\text{Q4d Required time} = \frac{\frac{16\pi}{3}}{\frac{\pi}{3}} = 16 \text{ seconds}$$

$$\text{Q4e } \frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

When the depth of water is h , volume of water V

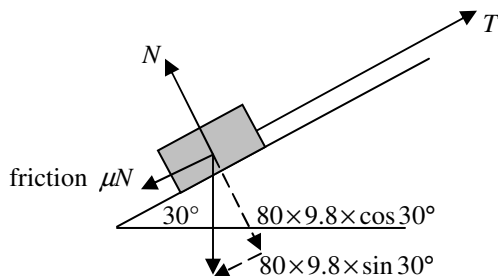
$$= \int_0^h 4\pi \left(1 - \frac{1}{(y+1)^2} \right) dy = 4\pi \left(h + \frac{1}{h+1} - 1 \right)$$

$$\therefore \frac{dV}{dh} = 4\pi \left(1 - \frac{1}{(h+1)^2} \right)$$

$$\text{When } h = 1, \frac{\pi}{3} = 3\pi \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{1}{9} \text{ cm per second}$$

Q5a

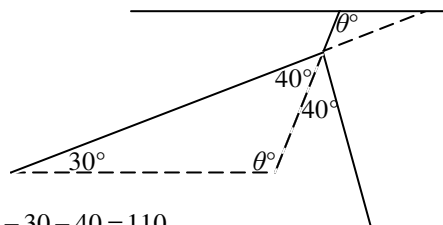


Normal reaction force of the inclined plane on the crate N
 $= 80 \times 9.8 \times \cos 30^\circ$

Force of friction $= \mu N = 0.25 \times 80 \times 9.8 \times \cos 30^\circ \approx 170 \text{ N}$

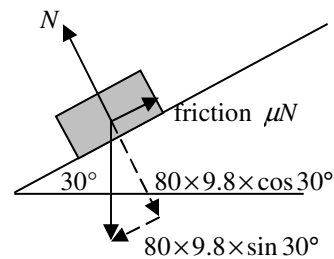
Q5b Applied force $= T = 170 + 80 \times 9.8 \times \sin 30^\circ \approx 562 \text{ N}$

Q5ci



$$\text{Q5cii } T_{\text{chain}} = 2 \times 562 \cos 40^\circ \approx 861 \text{ N}$$

Q5d



Resultant force $= 80 \times 9.8 \times \sin 30^\circ - 170 \approx 222 \text{ N}$

$$a = \frac{F}{m} = \frac{222}{80} \approx 2.8 \text{ m s}^{-2}$$

Q5ei $u = -0.2$, $a = +2.8$, $t = 0.25$, v ?

$$v = u + at, v = -0.2 + 2.8 \times 0.25 = +0.5$$

$$\therefore \text{speed} = 0.5 \text{ m s}^{-1}$$

$$\text{Q5eii } |\text{momentum}| = mv = 80 \times 0.5 = 40 \text{ kg m s}^{-1}$$

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