INSIGHT

YEAR 12 *Trial Exam Paper*

2011

SPECIALIST MATHEMATICS
UNIT 3

Written examination 1

Worked solutions

This book presents:

- Worked solutions, giving you a series of points to show you how to work through the questions
- Mark allocations
- Tips on how to approach the questions

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Question 1

Consider the function defined by $e^{x+y} = y + x^2 + e - 1$.

a. Show that $y = 1$ when $x = 0$.

Worked solution

$e^{x+y} = y + x^2 + e - 1$
Substitute (0, 1), giving
LHS = $e^{0+1} = e$
RHS = $1 + 0^2 + e - 1 = e$
$\therefore$ LHS = RHS

1 mark

Mark allocation

- 1 mark for substituting (0, 1) into the equation and showing that the left-hand side is equal to the right-hand side.

b. Find the gradient of the tangent to the function given in part **a** at $x = 0$.

Worked solution

$$e^{x+y} = y + x^2 + e - 1$$
$$e^x.e^y = y + x^2 + e - 1$$
$$\frac{d}{dx}\ e^x.e^y\ = \frac{d}{dx}\ y + x^2 + e - 1)$$
$$\frac{d}{dx}(e^x).e^y + e^x.\frac{d}{dy}\ e^y\ .\frac{dy}{dx} = \frac{d}{dx}(y) + \frac{d}{dx}(x^2 + e - 1)$$
$$e^x.e^y + e^x.e^y.\frac{dy}{dx} = \frac{dy}{dx} + 2x$$
$$e^{x+y} + e^x.e^y.\frac{dy}{dx} = \frac{dy}{dx} + 2x$$
$$e^{x+y}.\frac{dy}{dx} - \frac{dy}{dx} = 2x - e^{x+y}$$
$$\frac{dy}{dx}(e^{x+y} - 1) = 2x - e^{x+y}$$
$$\frac{dy}{dx} = \frac{2x - e^{x+y}}{e^{x+y} - 1}$$

Substitute (0, 1), giving
$$\frac{dy}{dx} = \frac{-e}{e-1} = \frac{e}{1-e}$$
which is the gradient of the tangent at $x = 0$.

3 marks

Question 1 – continued

Mark allocation

- 1 mark for implicitly differentiating the relation correctly.
- 1 mark for finding the correct gradient function.
- 1 mark for the correct answer.

Tip

- *The relation could not be expressed explicitly as a function of* x. *Therefore, implicit differentiation must be used for this problem.*

Total 1 + 3 = 4 marks

Question 2

The position of a particle at any time t seconds is $\underset{\sim}{r}(t) = \cos\ t\ \underset{\sim}{i} + \sin\ 2t\ \underset{\sim}{j},\ t \geq 0$

a. Show that the relation which describes the position of the particle is $y^2 = 4x^2(1 - x^2)$.

Worked solution

$$\underset{\sim}{r}(t) = \cos\ t\ \underset{\sim}{i} + \sin\ 2t\ \underset{\sim}{j},\ t \geq 0$$

$x = \cos(t)$

$y = \sin(2t) = 2\sin(t)\cos(t)$

$y^2 = 4\sin^2(t)\cos^2(t)$

$y^2 = 4(1 - \cos^2(t))\cos^2(t)$

$y^2 = 4x^2(1 - x^2)$ as required.

2 marks

Mark allocation

- 1 mark for finding the correct parametric equations.
- 1 mark for using the correct trigonometric identity.

Question 2 – continued

b. Show that the angle, θ, between the direction of motion of the particle at $t = \dfrac{\pi}{2}$ and $t = \dfrac{3\pi}{2}$ is $\theta = \cos^{-1} 0.6$.

Worked solution

$\underset{\sim}{r}(t) = \cos(t)\,\underset{\sim}{i} + \sin(2t)\,\underset{\sim}{j},\ t \geq 0$

$\underset{\sim}{v}(t) = -\sin(t)\,\underset{\sim}{i} + 2\cos(2t)\,\underset{\sim}{j},\ t \geq 0$

$\underset{\sim}{v}\left(\dfrac{\pi}{2}\right) = -\sin\left(\dfrac{\pi}{2}\right)\underset{\sim}{i} + 2\cos(\pi)\,\underset{\sim}{j}$

$\underset{\sim}{v}\left(\dfrac{\pi}{2}\right) = -\underset{\sim}{i} - 2\,\underset{\sim}{j}$

$\underset{\sim}{v}\left(\dfrac{3\pi}{2}\right) = -\sin\left(\dfrac{3\pi}{2}\right)\underset{\sim}{i} + 2\cos(3\pi)\,\underset{\sim}{j}$

$\underset{\sim}{v}\left(\dfrac{3\pi}{2}\right) = \underset{\sim}{i} - 2\,\underset{\sim}{j}$

$\underset{\sim}{v}\left(\dfrac{\pi}{2}\right).\underset{\sim}{v}\left(\dfrac{3\pi}{2}\right) = \left|\underset{\sim}{v}\left(\dfrac{\pi}{2}\right)\right|.\left|\underset{\sim}{v}\left(\dfrac{3\pi}{2}\right)\right|\cos\theta$, where θ is the angle between $\underset{\sim}{v}\left(\dfrac{\pi}{2}\right)$ and $\underset{\sim}{v}\left(\dfrac{3\pi}{2}\right)$.

$\Rightarrow (-\underset{\sim}{i} - 2\,\underset{\sim}{j}).(\underset{\sim}{i} - 2\,\underset{\sim}{j}) = \sqrt{5}.\sqrt{5}.\cos\theta$

$-1 + 4 = 5\cos\theta$

$\Rightarrow \cos\theta = \dfrac{3}{5}$

$\Rightarrow \theta = \cos^{-1} 0.6$

3 marks

Mark allocation

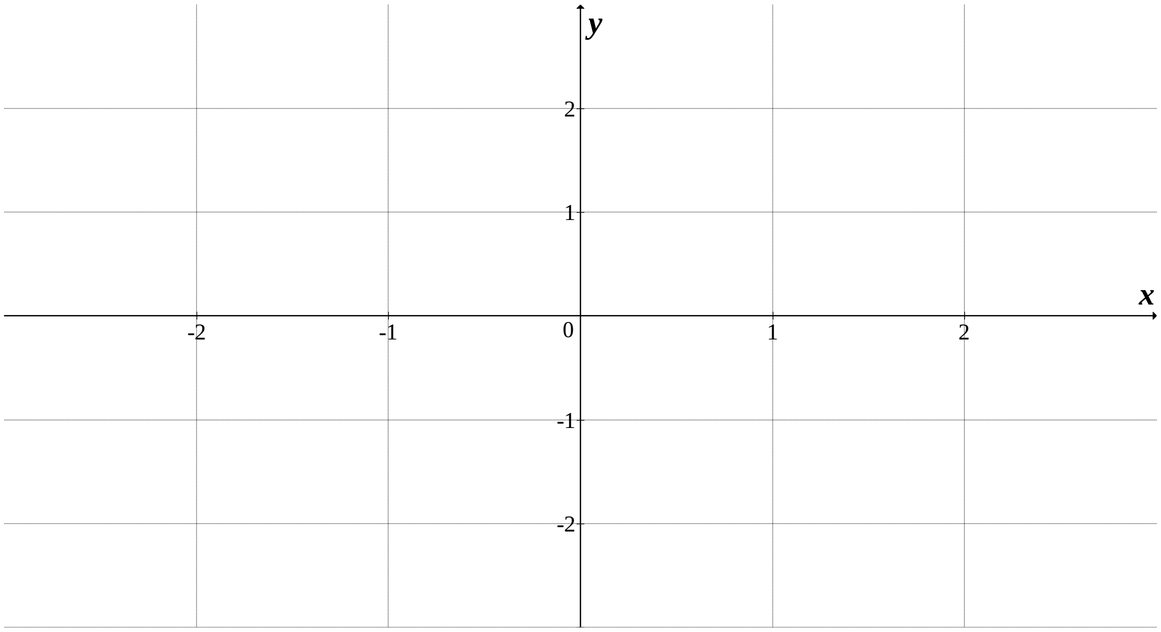
- 1 mark for calculating the velocity vector correctly.
- 1 mark for the finding the correct vectors for $\underset{\sim}{v}\left(\dfrac{\pi}{2}\right)$ and $\underset{\sim}{v}\left(\dfrac{3\pi}{2}\right)$.
- 1 mark for correctly finding $\cos\theta$ using the dot product.

Total 2 + 3 = 5 marks

End of Question 2

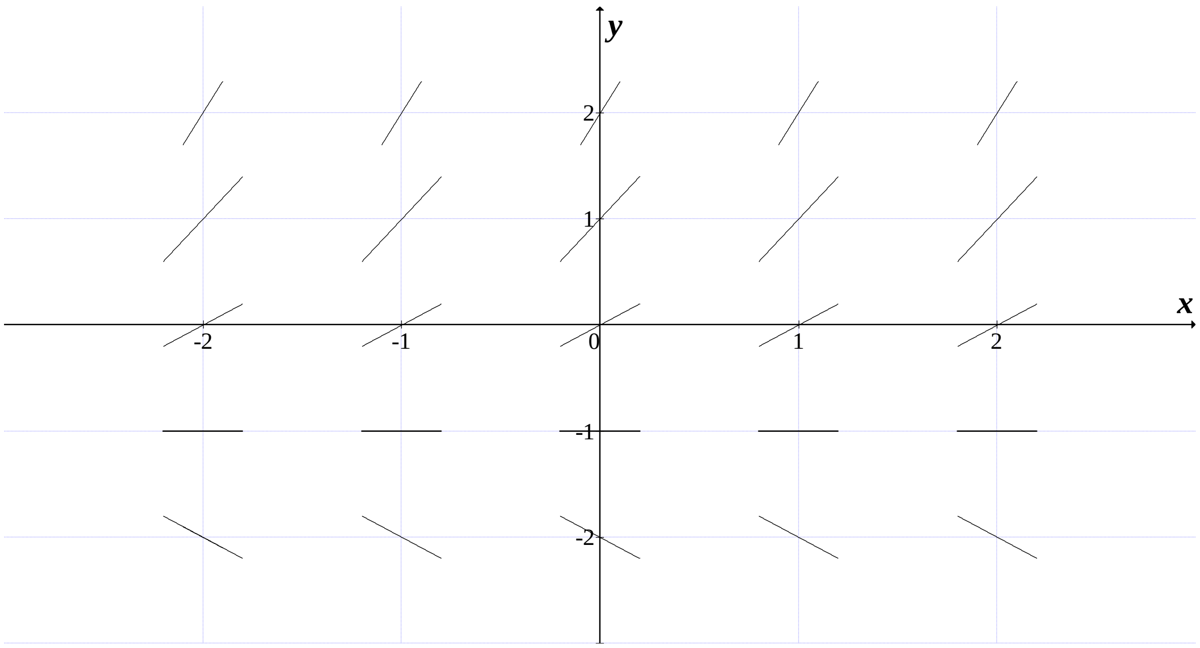
Question 3

a. On the set of axes below, sketch the slope field of the differential equation $\dfrac{dy}{dx} = y + 1$ for $y = -2, -1, 0, 1, 2$ at the x values $x = -2, -1, 0, 1, 2$.



Worked solution

y	-2	-1	0	1	2
y'	-1	0	1	2	3



1 mark

Mark allocation

- 1 mark for correct direction of the tangent slopes on the slope field.

Question 3 – continued

b. Solve the differential equation given in part **a**, for y in terms of x, if it is known that $y = 0$ when $x = 1$.

Worked solution

$$\frac{dy}{dx} = y+1$$
$$\frac{dx}{dy} = \frac{1}{y+1}$$
$$x = \int \frac{1}{y+1}.dy$$
$$x = \log_e|y+1| + c$$

Substituting (1, 0) gives

$$1 = \log_e 1 + c$$

So $c = 1$

$$x = \log_e|y+1| + 1$$
$$x-1 = \log_e|y+1|$$
$$|y+1| = e^{x-1}$$
$$y+1 = \pm e^{x-1}$$
$$y = \pm e^{x-1} - 1$$
$$y = e^{x-1} - 1 \text{ or } y = -e^{x-1} - 1$$

Since $y = 0$ when $x = 1$

$$\Rightarrow y = e^{x-1} - 1$$

3 marks

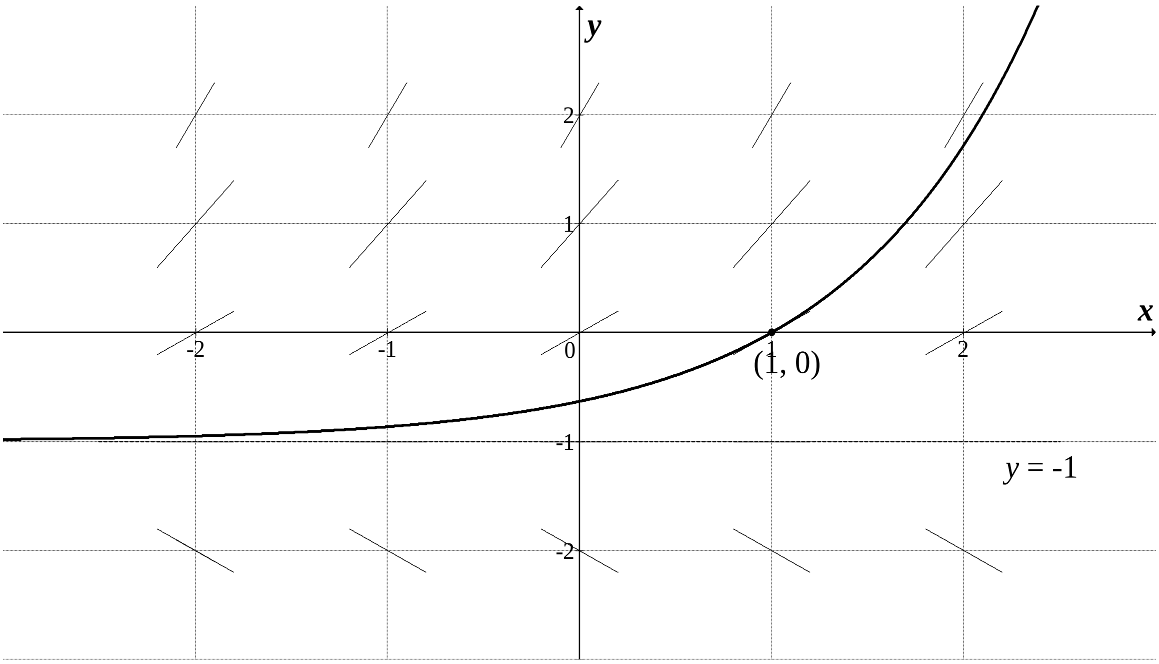
Mark allocation

- 1 mark for finding the correct antiderivative.
- 1 mark for the correct evaluation of the constant.
- 1 mark for the correct answer.

Question 3 – continued

c. Sketch the graph of the solution found in part **b** on the slope field found in part **a**.

Worked solution



1 mark

Mark allocation

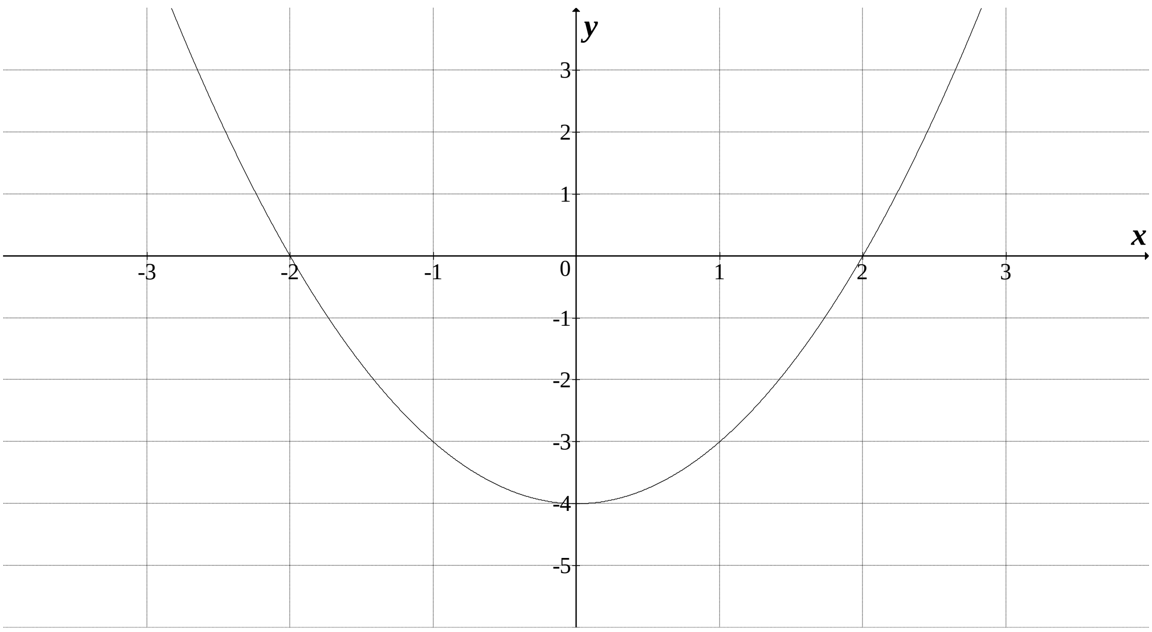
- 1 mark for the correct graph with the correct *x*-intercept and horizontal asymptote.

Total 1 + 3 + 1 = 5 marks

Question 4

The graph of $f(x) = x^2 - 4$ is shown below.

On the same axes sketch the graph of $g(x) = \dfrac{1}{f(x)}$. Clearly label any asymptotes and axes intercepts.



Question 4 – continued

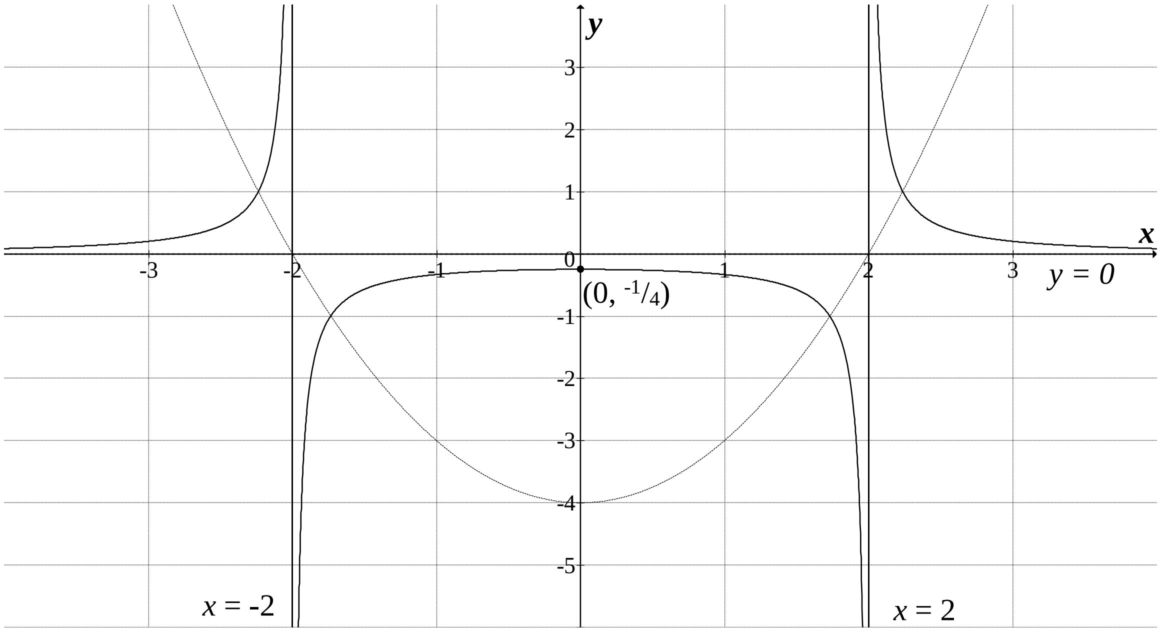
Worked solution

$$g(x) = \frac{1}{f(x)} = \frac{1}{x^2 - 4}$$

$g(x)$ has vertical asymptotes where $f(x) = 0$.
$\Rightarrow g(x)$ has vertical asymptotes at $x = 2$ and $x = -2$
and $g(x)$ has a horizontal asymptote at $y = 0$.
Since $f(x)$ has a local minimum stationary point at $(0, -4)$

$g(x)$ has a local maximum stationary point at $\left(0, -\frac{1}{4}\right)$.

$f(x) = g(x)$ at $y = \pm 1$



2 marks

Mark allocation

- 1 mark for sketching the curves correctly.
- 1 mark for finding the correct asymptotes and local maximum.

Tip

- *A function and its reciprocal intersect at* $y = \pm 1$.

End of Question 4

Question 5

a. Show that $z - 1$ is a factor of $z^3 - (1+i)z^2 + (2+i)z - 2$.

Worked solution

Let $P(z) = z^3 - (1 + i)z^2 + (2 + i)z - 2$

$$\begin{aligned} P(1) &= 1^3 - (1+i)\times 1^2 + (2+i)\times 1 - 2 \\ &= 1 - 1 - i + 2 + i - 2 \\ &= 0 \end{aligned}$$

$\therefore$ $z - 1$ is a factor of $P(z)$.

1 mark

Mark allocation

- 1 mark for substituting $z = 1$ into the expression and evaluating as zero.

b. Hence, or otherwise, find all the solutions of $z^3 - (1+i)z^2 + (2+i)z - 2 = 0$.

Worked solution

$z^3 - (1 + i)z^2 + (2 + i)z - 2 = 0$

Since $z - 1$ is a factor

$(z-1)(z^2 - iz + 2) = 0$

Hence $z^2 - iz + 2 = 0$, $z - 1 = 0$

$$z = \frac{i \pm \sqrt{i^2 - 8}}{2}, \quad z = 1$$

$$z = \frac{i \pm \sqrt{9i^2}}{2}$$

$$z = \frac{i \pm 3i}{2}$$

$$z = 2i, -i$$

$\therefore$ $z = -i, 2i$ and 1

3 marks

Mark allocation

- 1 mark for correctly factorising the expression.
- 1 mark for correctly solving the quadratic equation.
- 1 mark for the correct answer.

Tip

- *The expression* $z^3 - (1 + i)z^2 + (2 + i)z - 2$ *could also be factorised by first dividing it by* $z - 1$ *using long division.*

Total 1 + 3 = 4 marks

Question 5 – continued

Question 6

Three points, A, B and C, have coordinates $A(1, 1, 1)$, $B(2, 3, -6)$ and $C(5, -3, -3)$, respectively. If M is the midpoint of $\overrightarrow{AC}$, use a vector method to show that $\overrightarrow{MB}$ is perpendicular to $\overrightarrow{AC}$.

Worked solution

$$\overrightarrow{AB} = \underset{\sim}{i} + 2\underset{\sim}{j} - 7\underset{\sim}{k}$$

$$\overrightarrow{AC} = 4\underset{\sim}{i} - 4\underset{\sim}{j} - 4\underset{\sim}{k}$$

$$\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AC} = 2\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}$$

$$\overrightarrow{MB} = \overrightarrow{AB} - \overrightarrow{AM} = \underset{\sim}{i} + 2\underset{\sim}{j} - 7\underset{\sim}{k} - (2\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k})$$

$$= -\underset{\sim}{i} + 4\underset{\sim}{j} - 5\underset{\sim}{k}$$

$$\overrightarrow{AC}.\overrightarrow{MB} = (4\underset{\sim}{i} - 4\underset{\sim}{j} - 4\underset{\sim}{k}).(-\underset{\sim}{i} + 4\underset{\sim}{j} - 5\underset{\sim}{k})$$

$$= -4 - 16 + 20$$
$$= 0$$

$\Rightarrow \overrightarrow{AC}$ is perpendicular to $\overrightarrow{MB}$.

3 marks

Mark allocation

- 1 mark for correctly finding the vector $\overrightarrow{AC}$.
- 1 mark for correctly finding the vector $\overrightarrow{MB}$.
- 1 mark for showing the dot product $\overrightarrow{AC}.\overrightarrow{MB}$ is zero.

End of Question 6

Question 7

Find the value of k if $\int_0^k \frac{-1}{\sqrt{1-9x^2}}.dx = -\frac{\pi}{9}$.

Worked solution

$$\int_0^k \frac{-1}{\sqrt{1-9x^2}}.dx = -\frac{\pi}{9}$$

Let $u = 3x$

$\frac{du}{dx} = 3$ or $\frac{1}{3}\frac{du}{dx} = 1$

$x = 0 \Rightarrow u = 0$

$x = k \Rightarrow u = 3k$

$$\int_0^k \frac{-1}{\sqrt{1-9x^2}}.dx = \int_0^{3k} \frac{-1}{\sqrt{1-u^2}}.\frac{1}{3}.\frac{du}{dx}.dx$$

$$= \frac{1}{3}\int_0^{3k} \frac{-1}{\sqrt{1-u^2}}.du$$

$$\Rightarrow \frac{1}{3}\left[\cos^{-1} u\right]_0^{3k} = \frac{-\pi}{9}$$

$$\frac{1}{3}\left[\cos^{-1}(3k) - \cos^{-1} 0\right] = \frac{-\pi}{9}$$

$$\left[\cos^{-1}(3k) - \frac{\pi}{2}\right] = \frac{-\pi}{3}$$

$$\cos^{-1}(3k) = \frac{\pi}{2} - \frac{\pi}{3}$$

$$\cos^{-1}(3k) = \frac{\pi}{6}$$

$$\Rightarrow 3k = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\therefore k = \frac{\sqrt{3}}{6}$$

3 marks

Mark allocation

- 1 mark for correctly antidifferentiating the integrand.
- 1 mark for correctly evaluating the definite integral.
- 1 mark for the correct answer.

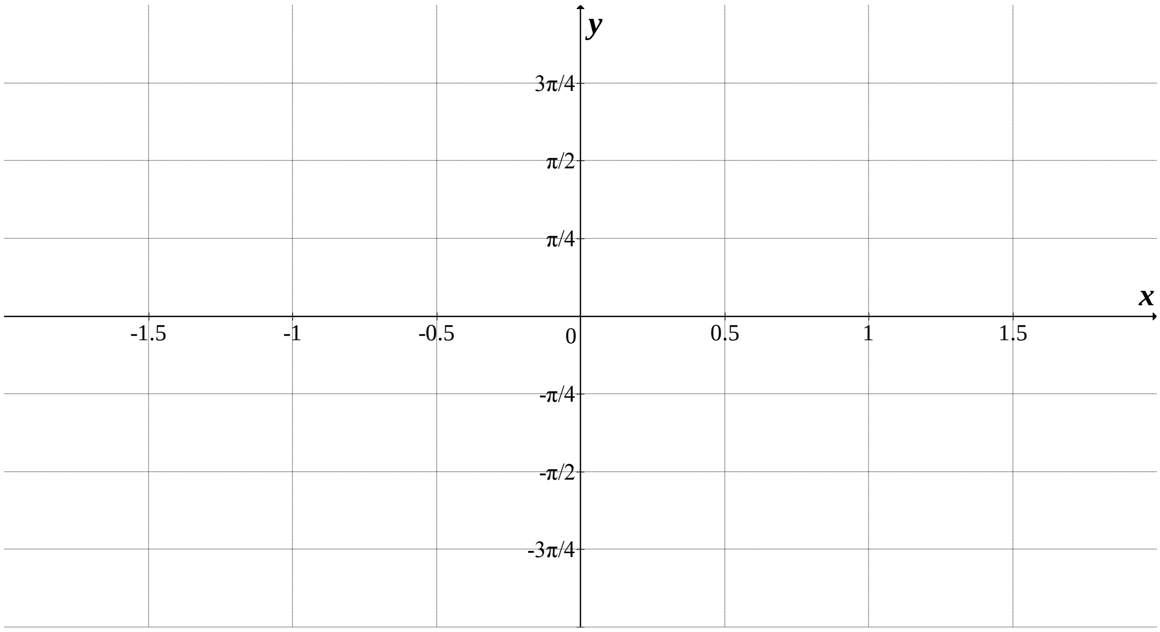
Tips

- *Express the integrand in the form* $\frac{-1}{\sqrt{1-u^2}}$ *so that it can be antidifferentiated by substitution.*
- *The equation could also be set up as* $\int_0^k \frac{1}{\sqrt{1-9x^2}}.dx = \frac{\pi}{9}$ *and antidifferentiated as* $\sin^{-1}(u)$.

End of Question 7

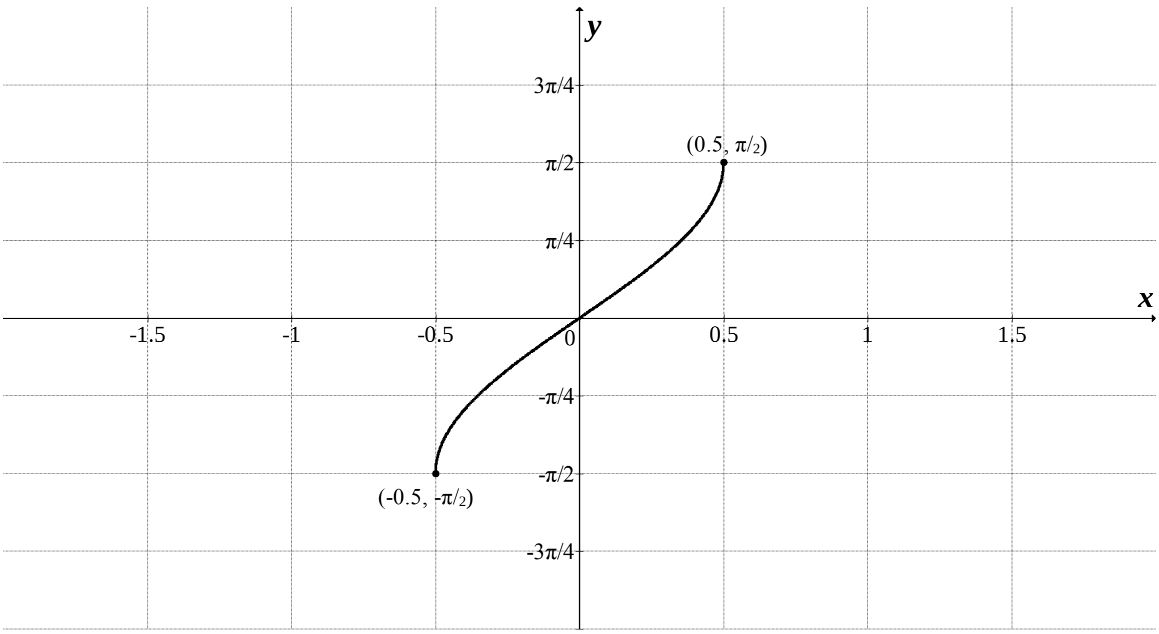
Question 8

a. Sketch the graph of $f(x) = \sin^{-1}(2x)$ on the set of axes below.
Clearly label the endpoints.



Worked solution

$$f\left(\frac{1}{2}\right) = \sin^{-1}(1) = \frac{\pi}{2}$$
$$f\left(-\frac{1}{2}\right) = \sin^{-1}(-1) = -\frac{\pi}{2}$$



2 marks

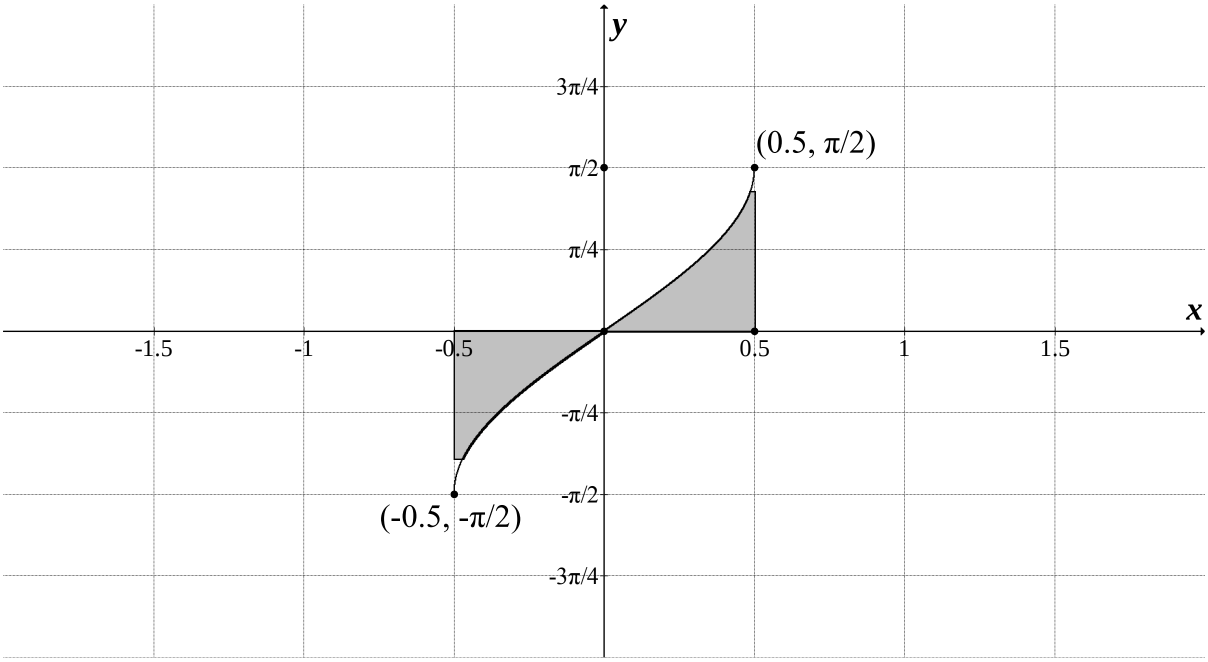
Mark allocation

- 1 mark for the correct curve.
- 1 mark for the correct endpoints.

Question 8 – continued

b. On the graph shown in part **a**, shade the area between $f(x)$ and the x-axis from $x = -\frac{1}{2}$ to $x = \frac{1}{2}$.

Worked solution



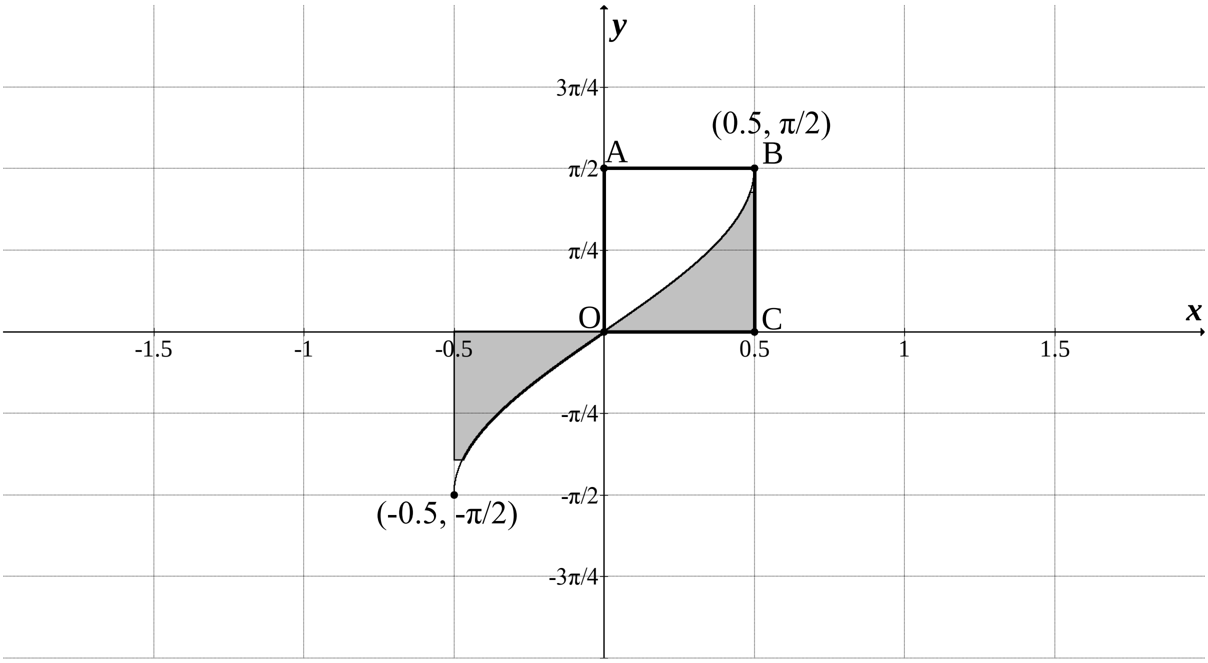
1 mark

c. Find the exact area between $f(x)$ and the x-axis from $x = -\frac{1}{2}$ to $x = \frac{1}{2}$.

Worked solution

Shade the required area and label the vertices of a rectangle *OABC*, as indicated below.

Question 8 – continued



Area required $= -\int_{-\frac{1}{2}}^{0} \sin^{-1}(2x).dx + \int_{0}^{\frac{1}{2}} \sin^{-1}(2x).dx$

$= 2\int_{0}^{\frac{1}{2}} \sin^{-1}(2x).dx$ using symmetry

= 2 × area of rectangle *OABC* – 2 × area between curve *OB* and the *y*-axis

= 2 × area of rectangle *OABC* – $2\int_{0}^{\frac{\pi}{2}} g(y).dy$

Since $y = \sin^{-1}(2x)$

then $2x = \sin(y)$

$x = \frac{1}{2}\sin(y) = g(y)$

Area required = 2 × $\frac{1}{2} \times \frac{\pi}{2} - 2\int_{0}^{\frac{\pi}{2}} \frac{1}{2}\sin y.dy$

$= \frac{\pi}{2} - \int_{0}^{\frac{\pi}{2}} \sin y.dy$

$= \frac{\pi}{2} + \left[\cos y\right]_{0}^{\frac{\pi}{2}}$

$= \frac{\pi}{2} + \cos\left(\frac{\pi}{2}\right) - \cos(0)$

$= \frac{\pi}{2} - 1$

The area is $\frac{\pi}{2} - 1$ square units.

3 marks

Question 8 – continued

Mark allocation

- 1 mark for correctly expressing x as a function of y.
- 1 mark for giving a correct expression for the area required.
- 1 mark for the correct answer.

Tip

- *Use symmetry to simplify the calculation of the area.*

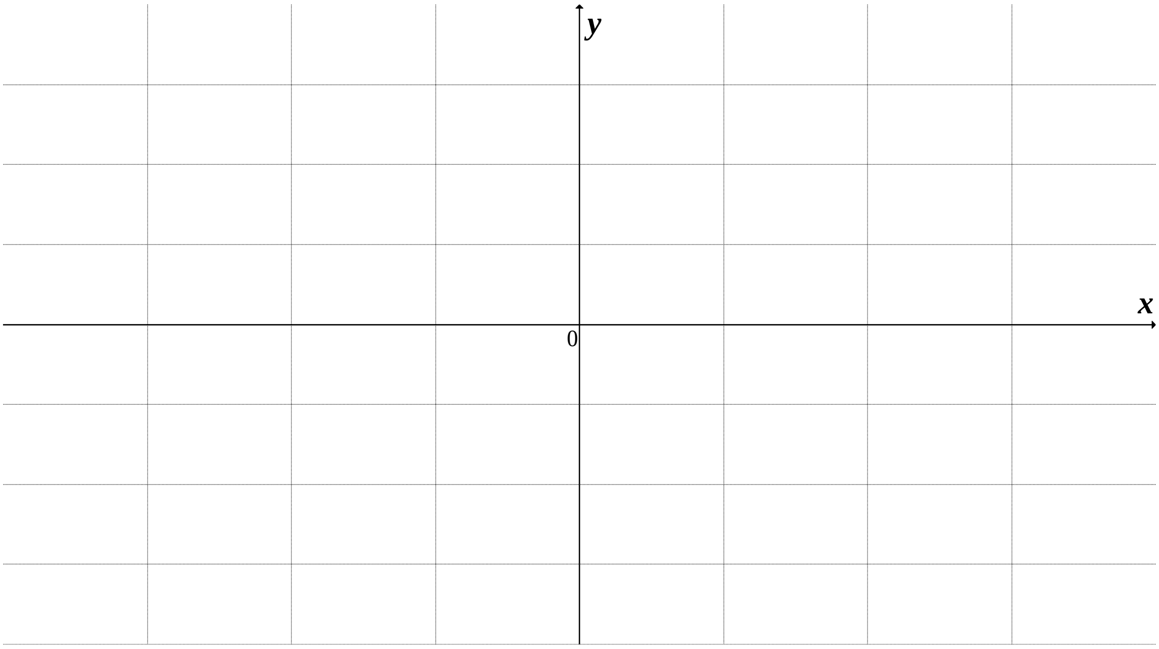
Total 2 + 1 + 3 = 6 marks

End of Question 8

Question 9

a. Sketch the graph of $f:\left[\frac{-\pi}{2},\frac{\pi}{2}\right]\to R,\ f(x)=\sec\left(2x-\frac{\pi}{2}\right)$ on the set of axes below.

Clearly label any asymptotes and stationary points.



Worked solution

$$f(x)=\sec\left(2x-\frac{\pi}{2}\right)=\frac{1}{\cos\left(2x-\frac{\pi}{2}\right)}$$

Vertical asymptotes: $\cos\left(2x-\frac{\pi}{2}\right)=0$

$$2x-\frac{\pi}{2}=.....,-\frac{3\pi}{2},-\frac{\pi}{2},\frac{\pi}{2},.....$$
$$2x=.....-\pi,\ 0,\ \pi,.....$$

So $x=-\frac{\pi}{2},0,\frac{\pi}{2}$ in the domain $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$.

Stationary points are where $\cos\left(2x-\frac{\pi}{2}\right)=\pm 1$.

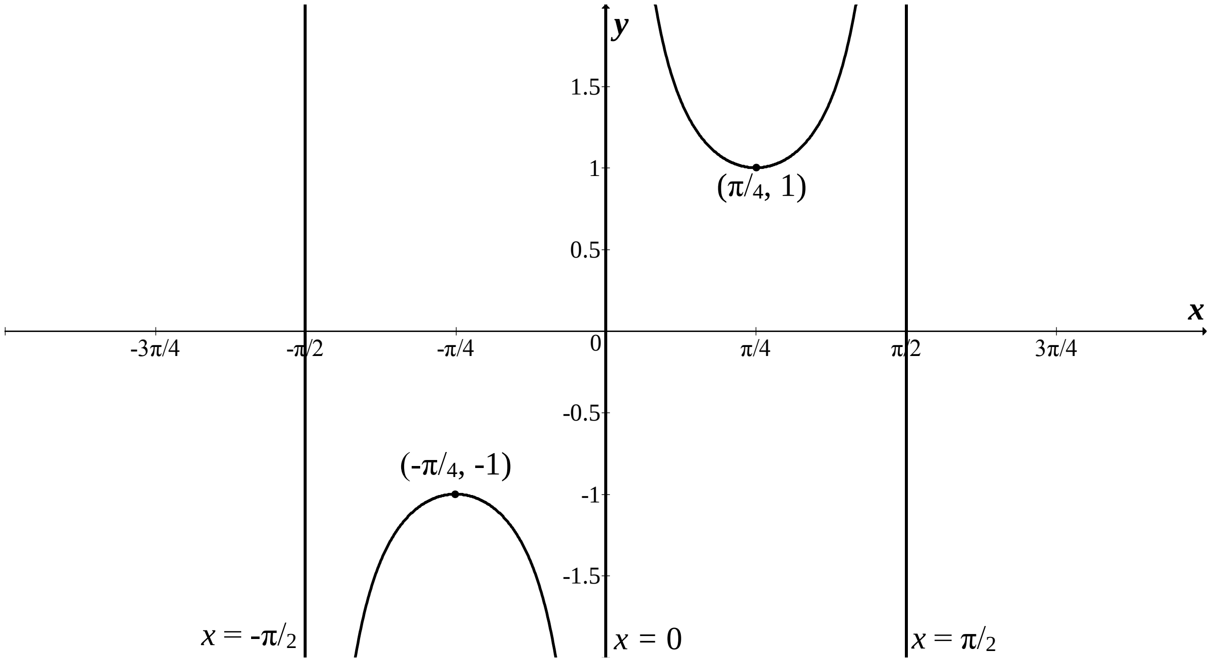
$$2x-\frac{\pi}{2}=-\pi,\ 0,\ \pi$$
$$2x=-\frac{\pi}{2},\frac{\pi}{2},\frac{3\pi}{2}$$
$$x=-\frac{\pi}{4},\frac{\pi}{4},\frac{3\pi}{4},......$$

So $x=-\frac{\pi}{4},\frac{\pi}{4}$ in the domain $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

$$f\left(-\frac{\pi}{4}\right)=\frac{1}{\cos(-\pi)}=\frac{1}{-1}=-1$$
$$f\left(\frac{\pi}{4}\right)=\frac{1}{\cos(0)}=\frac{1}{1}=1$$

Question 9 – continued

Hence, $\left(-\frac{\pi}{4},-1\right)$ and $\left(\frac{\pi}{4},\ 1\right)$ are the stationary points.



3 marks

Mark allocation

- 1 mark for correctly sketching the curve over the domain $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.
- 1 mark for correctly labelling the asymptotes.
- 1 mark for correctly labelling the local stationary points.

Tip

- *Express* $f(x)=\sec\left(2x-\frac{\pi}{2}\right)=\dfrac{1}{\cos\left(2x-\frac{\pi}{2}\right)}$ *to determine the asymptotes and stationary points.*

- *Sketch the graph of* $y=\cos 2(x-\frac{\pi}{4})$ *first, then sketch the reciprocal,* $f(x)=\sec\left(2x-\frac{\pi}{2}\right)$.

b. Calculate the exact volume generated when the area bounded by $f(x)$, the x- axis and the lines $x=\frac{\pi}{8}$ and $x=\frac{3\pi}{8}$ is rotated about the x-axis.

Worked solution

$$V=\pi\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}}\sec^2\left(2x-\frac{\pi}{2}\right).dx$$

$$=\frac{\pi}{2}\left[\tan\left(2x-\frac{\pi}{2}\right)\right]_{\frac{\pi}{8}}^{\frac{3\pi}{8}}$$

$$=\frac{\pi}{2}\left[\tan\left(\frac{\pi}{4}\right)-\tan\left(\frac{-\pi}{4}\right)\right]$$

$$=\frac{\pi}{2}(1+1)$$

$$=\pi$$

The volume is π cubic units.

2 marks

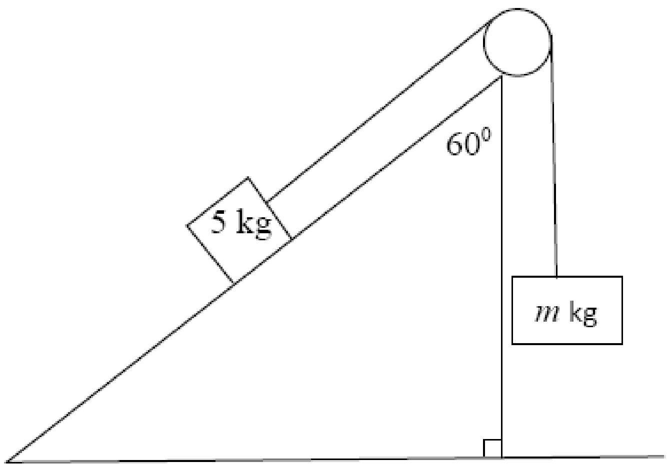
Mark allocation

- 1 mark for finding the correct integral to determine the volume required.
- 1 mark for the correct answer.

Total 3 + 2 = 5 marks

Question 10

A block of mass 5 kg rests on an incline which makes an angle of 60° to the vertical. The coefficient of friction between the block and the table surface is 0.4. The block is connected to another block of mass m kg by a light inextensible string over a smooth pulley at the edge of the incline. The mass, m, is hanging vertically.



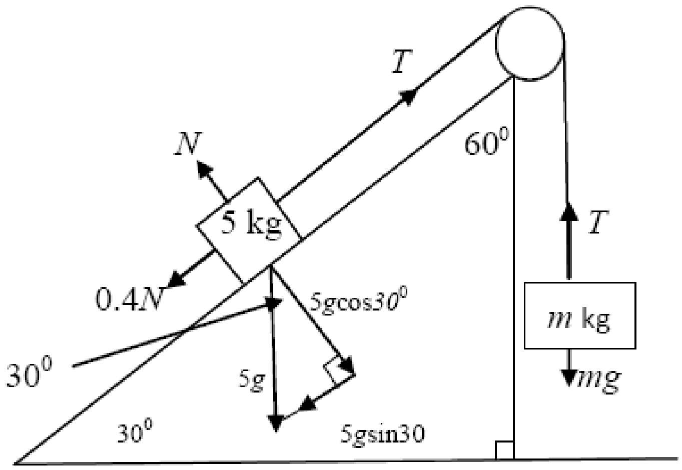
On the diagram above, label all the forces acting on the two masses.

Hence, find the **maximum** value of m for the system to remain in equilibrium.

Question 10 – continued

Worked solution

For the maximum value of *m*, the 5 kg mass is on the verge of moving up the plane.
Label on the diagram all the forces acting on the masses.



$$N = 5g\cos(30)$$
$$= 5g \times \frac{\sqrt{3}}{2}$$
$$= \frac{5g\sqrt{3}}{2}$$

$$R = mg - T + T - 5g\sin(30) - N\mu = 0$$
$$R = mg - 2.5g - 2.5g\sqrt{3} \times 0.4 = 0$$
$$R = mg - 2.5g - g\sqrt{3} = 0$$
$$\Rightarrow m - 2.5 - \sqrt{3} = 0$$
$$\Rightarrow m = 2.5 + \sqrt{3}$$

Therefore, the maximum value of *m* for equilibrium is $2.5 + \sqrt{3}$ kg.

3 marks

Mark allocation

- 1 mark for correctly labelling the force diagram.
- 1 mark for correctly evaluating the Normal reaction.
- 1 mark for the correct answer for the maximum value of mass *m*.

Tips

- *The weight force of the 5 kg mass should be resolved into the components parallel and perpendicular to the inclined plane.*
- *Resolve all forces in the direction of motion or intended motion.*

END OF SOLUTIONS BOOK