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# SPECIALIST MATHS TRIAL EXAMINATION 1 SOLUTIONS 2011

## Question 1

**a.** If $z+3$ is a factor then $z=-3$ is a solution. So $f(-3)=0$.

$$\begin{aligned}(-3)^3+(-3)^2-4\times(-3)+a&=0\\ -27+9+12+a&=0\\ a&=6 \text{ as required}\end{aligned}$$

**(1 mark)**

**b.**

$$\begin{array}{r|l} & z^2-2z+2 \\ \hline z+3 & z^3+z^2-4z+6 \\ & \underline{z^3+3z^2} \\ & -2z^2-4z \\ & \underline{-2z^2-6z} \\ & 2z+6 \\ & \underline{2z+6} \end{array}$$

$$\begin{aligned}f(z)&=(z+3)(z^2-2z+2)\\ &=(z+3)\{(z^2-2z+1)-1+2\}\\ &=(z+3)\{(z-1)^2+1\}\\ &=(z+3)\{(z-1)^2-i^2\}\\ &=(z+3)(z-1-i)(z-1+i)\end{aligned}$$

**(1 mark)**

**(1 mark)**

For $f(z)=0$,

$z=-3$ or $z=1\pm i$

**(1 mark)**

**Question 2**

**a.** $kx^2 - 2x^2y + y^3 - 6x = 2,\ k \in R$

$$2kx - 4xy - 2x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} - 6 = 0$$ **(1 mark)**

$$(-2x^2 + 3y^2)\frac{dy}{dx} = 4xy - 2kx + 6$$

$$\frac{dy}{dx} = \frac{4xy - 2kx + 6}{-2x^2 + 3y^2}$$

**(1 mark)**

**b.** Since the curve passes through the point $(1,0)$, we have

$$k - 0 + 0 - 6 = 2$$

$$k = 8$$ **(1 mark)**

So at $(1,0)$, $\dfrac{dy}{dx} = \dfrac{0 - 16 + 6}{-2 + 0}$

$$= 5$$

At $(1,0)$, the gradient is 5.

**(1 mark)**

**Question 3**

Method 1

$$|z - 1 - i| \le |z| \quad \text{where } z = x + yi$$

$$|x + yi - 1 - i| \le |x + yi|$$

$$\sqrt{(x-1)^2 + (y-1)^2} \le \sqrt{x^2 + y^2}$$

$$x^2 - 2x + 1 + y^2 - 2y + 1 \le x^2 + y^2$$

$$-2x - 2y + 2 \le 0$$

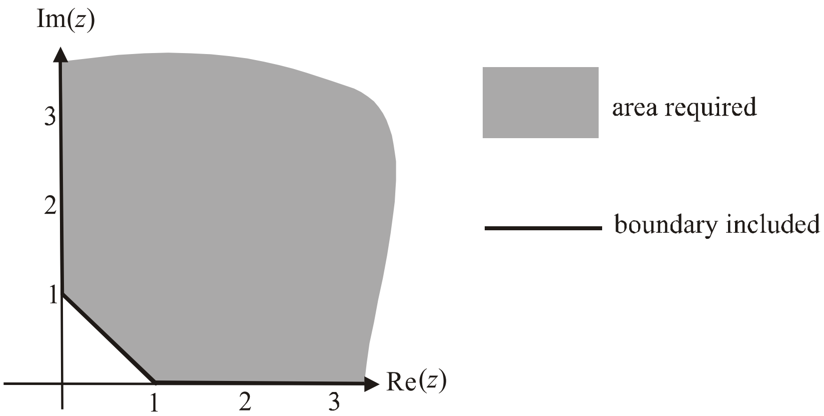
$$-2y \le 2x - 2$$

$$y \ge -x + 1$$

**(1 mark)**

Method 2

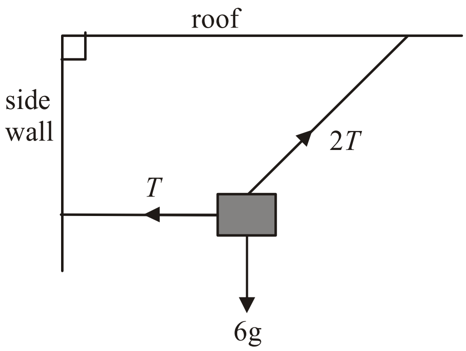
The equation $|z - 1 - i| = |z|$ describes the set of points that are equidistant from $0 + 0i$ and $1 + i$ This set of points forms a straight line that passes through $(0,1)$ and $(1,0)$. The inequation $|z - 1 - i| \le |z|$ describes the set of points closer to $1 + i$ than $0 + 0i$. This describes a half plane with the straight line passing through $(0,1)$ and $(1,0)$ as its boundary. **(1 mark)**




**(1 mark)** – correct area **(1 mark)** – correct boundaries including corner points

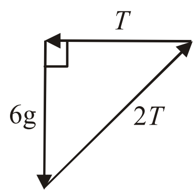
**Question 4**

**a.** Let $T$ be the tension in the horizontal wire.




**(1 mark)**

**b.**

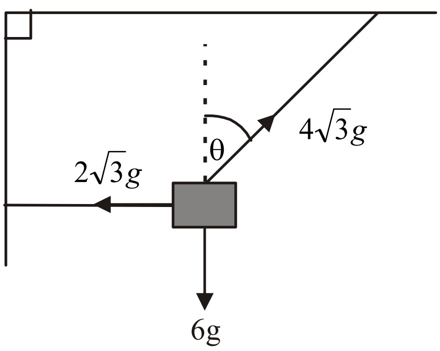



$$(2T)^2 = T^2 + 36g^2 \qquad \textbf{(1 mark)}$$
$$3T^2 = 36g^2$$
$$T^2 = 12g^2$$
$$T = 2\sqrt{3}g$$

Tension in horizontal wire is $2\sqrt{3}$ g newtons.
Tension in other wire is $4\sqrt{3}$ g newtons.

**(1 mark)**

**c.**




$4\sqrt{3}\text{g}\sin\theta = 2\sqrt{3}\text{g}$ OR $4\sqrt{3}\text{g}\cos\theta = 6\text{g}$

$$\sin\theta = \frac{2\sqrt{3}\text{g}}{4\sqrt{3}\text{g}} = \frac{1}{2}$$
$$\theta = \frac{\pi}{6}$$

$$\cos\theta = \frac{6\text{g}}{4\sqrt{3}\text{g}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$
$$\theta = \frac{\pi}{6}$$

**(1 mark)**

**Question 5**

**a.** For $g(x) = \arctan(x)$, $d_g = R$

For $f(x) = 2\arctan\left(\frac{2x+1}{3}\right) - 1$, $d_f = R$ **(1 mark)**

For $g(x) = \arctan(x)$, $r_g = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

For $f(x) = 2\arctan\left(\frac{2x+1}{3}\right) - 1$, $r_f = \left(\left(2 \times -\frac{\pi}{2}\right) - 1, \left(2 \times \frac{\pi}{2}\right) - 1\right)$

$$= (-\pi - 1, \pi - 1)$$

**(1 mark)**

**b.**

$$f(x) = 2\arctan\left(\frac{2x+1}{3}\right) - 1$$

Let $y = 2\arctan\left(\frac{2x+1}{3}\right) - 1$

$y = 2\arctan\left(\frac{u}{3}\right) - 1$ where $u = 2x + 1$

$\frac{dy}{du} = 2 \times \frac{3}{9+u^2}$ and $\frac{du}{dx} = 2$

$$= \frac{6}{u^2 + 9}$$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ (Chain rule) **(1 mark)** – use of the chain rule

$$= \frac{6}{u^2+9} \times 2$$

$$= \frac{12}{(2x+1)^2 + 9}$$

$$= \frac{12}{4x^2 + 4x + 10}$$

$$\frac{dy}{dx} = \frac{12}{2(2x^2 + 2x + 5)}$$

So $f'(x) = \frac{6}{2x^2 + 2x + 5}$

**(1 mark)** – correct answer

**Question 6**

$$a = \sqrt{9 - v^2}$$

$$\frac{dv}{dt} = \sqrt{9 - v^2}$$

$$\frac{dt}{dv} = \frac{1}{\sqrt{9 - v^2}}$$

$$t = \int \frac{1}{\sqrt{9 - v^2}} dv$$

$$t = \text{arsin}\left(\frac{v}{3}\right) + c$$

**(1 mark)**

When $t = \frac{\pi}{3}, \ v = \frac{3}{2}$

$$\frac{\pi}{3} = \text{arsin}\left(\frac{1}{2}\right) + c$$

$$c = \frac{\pi}{3} - \frac{\pi}{6}$$

$$= \frac{\pi}{6}$$

**(1 mark)**

$$t = \text{arsin}\left(\frac{v}{3}\right) + \frac{\pi}{6}$$

$$t - \frac{\pi}{6} = \text{arsin}\left(\frac{v}{3}\right)$$

$$\frac{v}{3} = \sin\left(t - \frac{\pi}{6}\right)$$

$$v = 3\sin\left(t - \frac{\pi}{6}\right)$$

**(1 mark)**

**Question 7**

**a.**

$$\text{Let } \frac{1}{x^2-2x-3} \equiv \frac{A}{(x-3)} + \frac{B}{(x+1)}$$
$$\equiv \frac{A(x+1)+B(x-3)}{(x-3)(x+1)}$$
$$\text{True iff } 1 \equiv A(x+1)+B(x-3)$$

Put $x=-1 \quad 1=-4B, \quad B=-\frac{1}{4}$

Put $x=3 \quad 1=4A, \quad A=\frac{1}{4}$

$$\int \frac{1}{x^2-2x-3}dx = \int\left(\frac{1}{4(x-3)} - \frac{1}{4(x+1)}\right)dx$$ **(1 mark)**

$$= \frac{1}{4}\log_e|x-3| - \frac{1}{4}\log_e|x+1| + c$$
$$= \frac{1}{4}\log_e\left|\frac{x-3}{x+1}\right| + c$$

**(1 mark)**

**b.** Method 1

$$\int_0^1 \frac{x}{2-x}\,dx$$

let $u=2-x$

$\frac{du}{dx}=-1$

$x=2-u$

$x=1,\ u=1$

$x=0,\ u=2$

$$= \int_2^1 (2-u)u^{-1} \times -1\frac{du}{dx}dx$$
$$= -\int_2^1 (2u^{-1}-1)du$$
$$= \int_1^2 (2u^{-1}-1)du$$

**(1 mark)** – correct integrand
**(1 mark)** – correct terminals

$$= \left[2\ln|u| - u\right]_1^2$$
$$= \{(2\ln(2)-2)-(2\ln(1)-1)\}$$
$$= 2\ln(2)-1$$

**(1 mark)**

Method 2

$$\int_0^1 \frac{x}{2-x}dx$$

$$\begin{array}{r} -1 \\ 2-x\,\big)\overline{x+0} \\ \underline{x-2} \\ 2 \end{array}$$

$$= \int_0^1\left(-1+\frac{2}{2-x}\right)dx$$ **(1 mark)**

$$= \left[-x-2\ln|2-x|\right]_0^1$$ **(1 mark)**

$$= \{(-1-2\ln(1))-(0-2\ln(2))\}$$
$$= -1+2\ln(2)$$

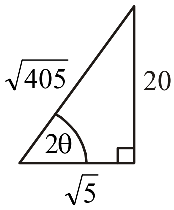
**(1 mark)**

**Question 8**

Method 1

$$\cot(2\theta)=\frac{\sqrt{5}}{20},\quad 0<\theta<\frac{\pi}{4}$$

$$\tan(2\theta)=\frac{20}{\sqrt{5}}$$




$$\cos(2\theta)=\frac{\sqrt{5}}{\sqrt{405}}$$

$$=\frac{1}{9}$$

**(1 mark)**

$$\cos(2\theta)=2\cos^2(\theta)-1 \quad \text{(formula sheet)}$$

**(1 mark)**

$$2\cos^2(\theta)=1+\frac{1}{9}$$

$$\cos^2(\theta)=\frac{5}{9}$$

$$\cos(\theta)=+\frac{\sqrt{5}}{3} \quad \text{since} \quad 0<\theta<\frac{\pi}{4}$$

**(1 mark)**

Method 2

$$\cot(2\theta)=\frac{\sqrt{5}}{20},\quad 0<\theta<\frac{\pi}{4}$$

$$\tan(2\theta)=\frac{20}{\sqrt{5}}$$

$$1+\tan^2(2\theta)=\sec^2(2\theta) \qquad \text{(formula sheet)}$$

$$1+\frac{400}{5}=\sec^2(2\theta)$$

**(1 mark)**

$$\sec(2\theta)=+\sqrt{81} \quad \text{since} \quad 0<\theta<\frac{\pi}{4}$$

$$\sec(2\theta)=9 \qquad\qquad 0<2\theta<\frac{\pi}{2}$$

$$\frac{1}{\cos(2\theta)}=9$$

**(1 mark)**

$$\cos(2\theta)=\frac{1}{9}$$

$$\cos(2\theta)=2\cos^2(\theta)-1 \qquad \text{(formula sheet)}$$

$$2\cos^2(\theta)=1+\frac{1}{9}$$

$$\cos^2(\theta)=\frac{5}{9}$$

$$\cos(\theta)=+\frac{\sqrt{5}}{3} \quad \text{since} \quad 0<\theta<\frac{\pi}{4}$$

**(1 mark)**

**Question 9**

**a.** $\underset{\sim}{r} = \arccos\left(\frac{t}{5}\right)\underset{\sim}{i} + 2t\,\underset{\sim}{j} \qquad t \in [0,5]$

$x = \arccos\left(\frac{t}{5}\right)$ and $y = 2t$ **(1 mark)**

$\frac{t}{5} = \cos(x) \qquad t = \frac{y}{2}$

$t = 5\cos(x)$

$\frac{y}{2} = 5\cos(x)$

$y = 10\cos(x)$

**(1 mark)**

**b.** To find the endpoints,

when $t = 0$, $x = \arccos(0) = \frac{\pi}{2}$, $y = 10\cos\left(\frac{\pi}{2}\right)$

$= 0$

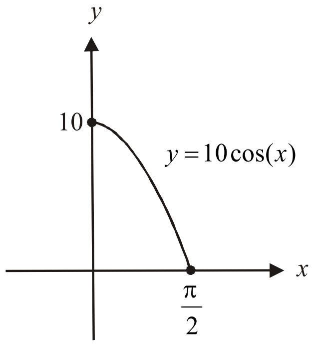
One endoint is $\left(\frac{\pi}{2},\ 0\right)$.

when $t = 5$, $x = \arccos(1) = 0$, $y = 10\cos(0)$

$= 10$

$(0,\ 10)$

The other endpoint is $(0,10)$.




**(1 mark)** – correct shape
**(1 mark)** – correct endpoints

**c.** $\underset{\sim}{r}(t) = \arccos\left(\frac{t}{5}\right)\underset{\sim}{i} + 2t\,\underset{\sim}{j}$

$\underset{\sim}{v}(t) = \frac{-1}{\sqrt{25 - t^2}}\underset{\sim}{i} + 2\,\underset{\sim}{j}$ **(1 mark)**

$|\underset{\sim}{v}(t)| = \sqrt{\frac{1}{25 - t^2} + 4}$

$|\underset{\sim}{v}(2)| = \sqrt{\frac{1}{21} + 4}$

$= \sqrt{\frac{85}{21}}$

**(1 mark)**

**Question 10**

<u>$x$-intercept</u> occurs when $y = 0$

$$\frac{x+1}{x^2+1} = 0$$

$$x = -1$$

$$\text{Area} = \int_{-1}^{0} \frac{x+1}{x^2+1}\,dx$$

**(1 mark)**

$$= \int_{-1}^{0} \frac{x}{x^2+1}\,dx + \int_{-1}^{0} \frac{1}{x^2+1}\,dx$$

$$= \int_{2}^{1} \frac{1}{2}\frac{du}{dx} u^{-1}\,dx + \left[\arctan(x)\right]_{-1}^{0}$$

where $u = x^2 + 1$

$$\frac{du}{dx} = 2x$$

$x = 0, \quad u = 1$

$x = -1, \quad u = 2$

$$= \frac{1}{2}\int_{2}^{1} u^{-1}\,du + \{\arctan(0) - \arctan(-1)\}$$

| S | A |
|---|---|
| T | C |

$$= \frac{1}{2}\left[\log_e|u|\right]_{2}^{1} + \left(0 - \frac{-\pi}{4}\right)$$

$$= \frac{1}{2}\{\log_e(1) - \log_e(2)\} + \frac{\pi}{4}$$

$$= \frac{-1}{2}\log_e(2) + \frac{\pi}{4}$$

$$= -\log_e\left(2^{\frac{1}{2}}\right) + \frac{\pi}{4}$$

$$= \frac{\pi}{4} - \log_e\left(\sqrt{2}\right)$$

**(1 mark)** – correct integration of $\dfrac{x}{x^2+1}$

**(1 mark)** correct integration of $\dfrac{1}{x^2+1}$

**(1 mark)** – correct answer in required form