



THE SCHOOL FOR EXCELLENCE
UNIT 3 & 4 SPECIALIST MATHEMATICS 2010
COMPLIMENTARY WRITTEN EXAMINATION 1 - SOLUTIONS

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MARKING SCHEME

- $(4 \times \frac{1}{2} \downarrow)$ means four answer half-marks rounded **down** to the next integer.
Rounding occurs at the end of a part of a question.
- **M1** = 1 Method mark.
- **A1** = 1 Answer mark (it **must** be this or its equivalent).
- **H1** = 1 consequential mark (**His/Her** mark...correct answer from incorrect statement or slip).

QUESTION 1

- (a) $u = e^x \therefore du = e^x dx, u^2 = e^{2x}$ M1
If $x = 0, \therefore u = 1$ and if $x = \ln 3, \therefore u = 3$ M1

$$\therefore \int_0^{\ln 3} \frac{e^x}{e^{2x} + 9} dx = \int_1^3 \frac{du}{u^2 + 9}$$

- (b) $\int_1^3 \frac{du}{u^2 + 9} = \left[\frac{1}{3} \tan^{-1} \left(\frac{u}{3} \right) \right]_1^3$ M1

$$= \left(\frac{1}{3} \tan^{-1} 1 \right) - \left(\frac{1}{3} \tan^{-1} \frac{1}{3} \right)$$

$$= \frac{1}{3} \times \frac{\pi}{4} - \frac{1}{3} \tan^{-1} \left(\frac{1}{3} \right)$$
 M1

$$= \frac{\pi - 4 \tan^{-1} \left(\frac{1}{3} \right)}{12} \text{ as required.}$$

Total = 4 marks

QUESTION 2

- (a) $\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ}, \overrightarrow{RP} = \overrightarrow{OP} - \overrightarrow{OR}$ A1

- (b) $\overrightarrow{OP} \perp \overrightarrow{QR} \therefore \overrightarrow{OP} \bullet \overrightarrow{QR} = 0$
 $\therefore \overrightarrow{OP} \bullet (\overrightarrow{OR} - \overrightarrow{OQ}) = 0$
 $\therefore \overrightarrow{OP} \bullet \overrightarrow{OR} = \overrightarrow{OP} \bullet \overrightarrow{OQ}$ M1

$$\overrightarrow{OQ} \perp \overrightarrow{RP} \therefore \overrightarrow{OQ} \bullet \overrightarrow{RP} = 0$$
$$\therefore \overrightarrow{OQ} \bullet (\overrightarrow{OP} - \overrightarrow{OR}) = 0$$
$$\therefore \overrightarrow{OQ} \bullet \overrightarrow{OP} = \overrightarrow{OQ} \bullet \overrightarrow{OR}$$
 M1

$$\therefore \overrightarrow{OP} \bullet \overrightarrow{OR} = \overrightarrow{OQ} \bullet \overrightarrow{OR}$$

$$\therefore \overrightarrow{OR} \bullet (\overrightarrow{OP} - \overrightarrow{OQ}) = 0$$

$$\therefore \overrightarrow{OR} \bullet \overrightarrow{QP} = 0$$

Therefore $\overrightarrow{OR}$ perpendicular to $\overrightarrow{QP}$, as required. M1

Total = 4 marks

QUESTION 3

(a) $x = 1 \therefore e^y - y^2 \log_e 1 = e$

$$\therefore e^y = e$$

$$\therefore y = 1$$

$$\therefore a = 1$$

A1

(b)

$$\frac{d}{dx}(e^{xy}) = e^{xy} \left(x \frac{dy}{dx} + y \right)$$

$$\frac{d}{dx}(y^2 \log_e x) = \frac{y^2}{x} + 2y \frac{dy}{dx} \log_e x$$

M1

$$\frac{d}{dx}(e) = 0$$

$$\therefore e^{xy} \left(x \frac{dy}{dx} + y \right) - \frac{y^2}{x} - 2y \frac{dy}{dx} \log_e x = 0$$

Substitute $x = 1$ and $y = 1$:

$$e \left(\frac{dy}{dx} + 1 \right) - 1 = 0$$

M1

$$\frac{dy}{dx} + 1 = \frac{1}{e}$$

$$\frac{dy}{dx} = \frac{1}{e} - 1$$

A1

Total = 4 marks

QUESTION 4

(a)

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{2x}{x^2+1}$$

M1

$$\frac{1}{2}v^2 = \int \frac{2x}{x^2+1} dx = \ln|x^2+1| + c$$

Substitute $v = 2$, $x = 1 \therefore 2 = \ln 2 + c \therefore c = 2 - \ln 2$

M1

$$\therefore \frac{1}{2}v^2 = \ln|x^2+1| - \ln 2 + 2$$

$$\therefore v^2 = 2 + \ln\left|\frac{x^2+1}{2}\right|$$

$$\therefore v = \sqrt{2 + \ln\left|\frac{x^2+1}{2}\right|}$$

Positive root only since velocity must be positive

A1

(b) $x = 5 \therefore v = \sqrt{2 + \ln 13}$

Therefore $a = 13$ and $b = 2$.

A1

Total = 4 marks

QUESTION 5

(a) $z = \frac{1}{2} \operatorname{cis}\left(\frac{2\pi}{3}\right) = \frac{1}{2}\left(\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)\right)$

$$w = \frac{1}{2} \operatorname{cis}\left(\frac{\pi}{4}\right) = \frac{1}{2}\left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right)$$

A2

(b) $zw = \frac{1}{2} \times \frac{1}{2} \times \operatorname{cis}\left(\frac{2\pi}{3} + \frac{\pi}{4}\right) = \frac{1}{4} \operatorname{cis}\left(\frac{11\pi}{12}\right) = \frac{1}{4}\left(\cos\left(\frac{11\pi}{12}\right) + i \sin\left(\frac{11\pi}{12}\right)\right)$

M1

(c) (i) $zw = \frac{(-1+i\sqrt{3})(\sqrt{2}+i\sqrt{2})}{16} = \frac{-\sqrt{2}-i\sqrt{2}+i\sqrt{6}-\sqrt{6}}{16} = \left(\frac{-\sqrt{2}-\sqrt{6}}{16}\right) + i\left(\frac{\sqrt{6}-\sqrt{2}}{16}\right)$

A1

(ii) $\cos\left(\frac{11\pi}{12}\right) = \frac{-\sqrt{2}-\sqrt{6}}{4}$ and $\sin\left(\frac{11\pi}{12}\right) = \frac{-\sqrt{2}+\sqrt{6}}{4}$

A1

Total = 5 marks

QUESTION 6

(a) $\frac{dx}{dt} = -60 \text{ kmh}^{-1}$

A1

$$\frac{dy}{dt} = -70 \text{ kmh}^{-1}$$

(b) $z^2 = x^2 + y^2$, $x = 0.8$ and $y = 0.6 \therefore z = 1$

M1

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

M1

$$z \frac{dz}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt}$$

$$\frac{dz}{dt} = 0.8 \times (-60) + 0.6 \times (-70) = -90 \text{ kmh}^{-1}$$

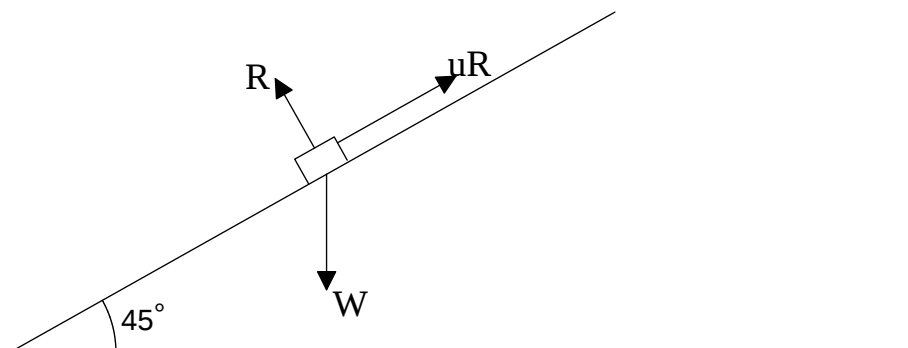
Therefore z is decreasing at the rate of 90 kmh^{-1} .

A1

Total = 4 marks

QUESTION 7

(a)



A1

(b) Resolving parallel to plane: $W \sin 45^\circ - \mu R = 0 \therefore W \sin 45^\circ = \mu R$

Resolving perpendicular to plane: $R - W \cos 45^\circ = 0 \therefore R = W \cos 45^\circ$

$$W \sin 45^\circ = \mu W \cos 45^\circ$$

$$\mu = \frac{\sin 45^\circ}{\cos 45^\circ} = 1$$

A1

(c) (i) $\mu R = 1 \times 2g = 19.6 \text{ Newtons}$ but only require friction of 9.8 Newtons to stop the motion. Therefore friction = $9.8 \text{ Newtons South}$.

A1

(ii) Acceleration = 0

A1

$$(d) \quad \Sigma \vec{F} = m\vec{a} \therefore 29.4 - 19.6 = 2a \therefore a = 4.9 \text{ms}^{-2}$$

$$\therefore \text{uniform acceleration, } v^2 = u^2 + 2as, u = 0, v = 7, a = 4.9$$

$$\therefore s = \frac{49}{9.8} = 5 \text{ metres}$$

Therefore object travels 5 metres.

A1

Total = 5 marks

QUESTION 8

$$\vec{r} = x\vec{i} + y\vec{j} \therefore \vec{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}$$

M1

$$\frac{dx}{dt} = \frac{1}{3}, \quad \frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = (6x - 3x^2) \times \frac{1}{3} = 2x - x^2$$

If velocity is horizontal, then $\frac{dy}{dt} = 0$.

$$\therefore 2x - x^2 = 0$$

$$\therefore x(2 - x) = 0$$

$$\therefore x = 0 \text{ or } x = 2$$

M1

$$\vec{v} = \frac{1}{3}\vec{i} + (2x - x^2)\vec{j}$$

$$\therefore \vec{a} = \frac{d\vec{v}}{dt} = 0\vec{i} + \frac{d}{dt}(2x - x^2)\vec{j} = \frac{d}{dx}(2x - x^2) \times \frac{dx}{dt}\vec{j} = (2 - 2x) \times \frac{1}{3}\vec{j} = \left(\frac{2}{3} - \frac{2x}{3}\right)\vec{j}$$

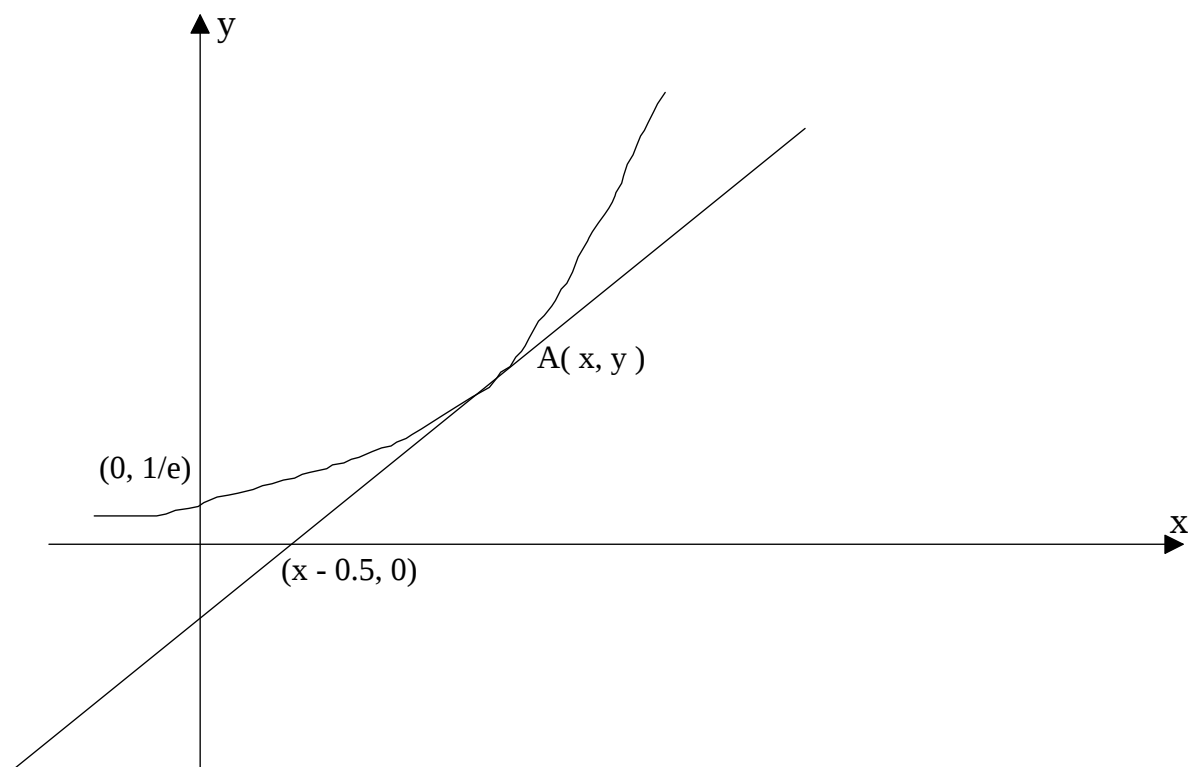
$$\text{If } x = 0, \text{ then } \vec{a} = \frac{2}{3}\vec{j}$$

$$\text{If } x = 2, \text{ then } \vec{a} = -\frac{2}{3}\vec{j}$$

A1

Total = 3 marks

QUESTION 9



Gradient of tangent is $\frac{y}{\frac{1}{2}} = 2y = \frac{dy}{dx}$

$$\therefore \frac{dx}{dy} = \frac{1}{2y} \quad \text{M1}$$

$$\therefore x = \frac{1}{2} \ln|y| + c$$

$$\text{when } x = 0, y = \frac{1}{e} \therefore c = -\frac{1}{2} \ln\left|\frac{1}{e}\right| = -\frac{1}{2} \times -1 = \frac{1}{2}$$

M1

$$\therefore x = \frac{1}{2} \ln|y| + \frac{1}{2}$$

$$\therefore 2x - 1 = \ln|y|$$

$$\therefore y = e^{2x-1} \text{ hence equation of curve is } f(x) = e^{2x-1}.$$

A1

Total = 3 marks

QUESTION 10

- (a) Sum of angles in $\triangle ABC = \alpha + \alpha + \alpha + 2\alpha = 5\alpha$

$$\therefore 5\alpha = 180^\circ$$

$$\therefore \alpha = 36^\circ$$

M1

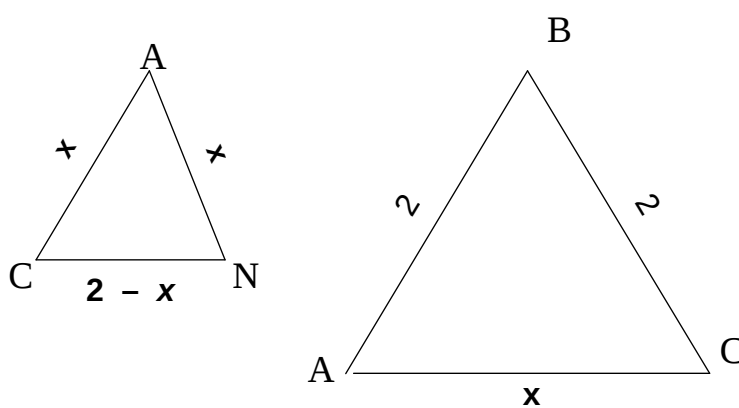
- (b) $\triangle ABC$ has two angles of 72° , therefore isosceles.

$$\text{In } \triangle ANC, \angle ANC = 180^\circ - 3\alpha = 180^\circ - 108^\circ = 72^\circ$$

Therefore $\triangle ANC$ also has two angles of 72° .

M1

- (c) $\triangle ABC$ is similar to $\triangle ANC$ because 3 pairs of equal corresponding angles.



Using ratios of corresponding sides:

$$\frac{x}{2-x} = \frac{2}{x} \quad \therefore x^2 = 4 - 2x \quad \therefore x^2 + 2x - 4 = 0$$

$$\therefore x = -1 \pm \sqrt{5}$$

But $x > 0$, therefore $x = -1 + \sqrt{5}$.

A1

M is midpoint of AB, because $\triangle ABN$ is isosceles. Therefore $AM = 1$

$$AN = AC = x = -1 + \sqrt{5}$$

$$\text{and } \therefore \cos \alpha = \frac{1}{-1 + \sqrt{5}} = \frac{1}{-1 + \sqrt{5}} \times \frac{1 + \sqrt{5}}{1 + \sqrt{5}} = \frac{1 + \sqrt{5}}{4}$$

$$\therefore \cos\left(\frac{\pi}{5}\right) = \frac{1 + \sqrt{5}}{4}$$

A1

Total = 4 marks

END OF SOLUTIONS TO EXAMINATION 1