Neap:

Trial Examination 2010

VCE Specialist Mathematics Units 3 & 4

Written Examination 2

Suggested Solutions

SECTION 1

Question	Answer	Question	Answer
1	A	12	C
2	D	13	C
3	C	14	E
4	D	15	B
5	E	16	E
6	B	17	A
7	B	18	A
8	C	19	E
9	A	20	C
10	A	21	B
11	D	22	D

SECTION 1

Question 1 **A**

Given $f(x) = ax + \frac{b}{x^2}$ where $a > 0$ and $b < 0$.

So $f'(x) = a - \frac{2b}{x^3}$

Solving $f'(x) = 0$ for x gives $x = \left(\frac{2b}{a}\right)^{\frac{1}{3}}$ or equivalent.

So $f''(x) = \frac{6b}{x^4}$

$$f''\left(\left(\frac{2b}{a}\right)^{\frac{1}{3}}\right) = 3\left(\frac{a^4}{2b}\right)^{\frac{1}{3}}$$

$3\left(\frac{a^4}{2b}\right)^{\frac{1}{3}} < 0$ for $a > 0$ and $b < 0$.

Therefore f has a local maximum.

The y-axis ($x = 0$) is a vertical asymptote.

As $x \to \pm\infty$, $\frac{b}{x^2} \to 0$ and so $f(x) \to ax$.

Thus $y = ax$ is an oblique asymptote.

Question 2 **D**

For an ellipse, we require $9 - m > 0$ and $m - 4 > 0$.

Hence $m < 9$ and $m > 4$, i.e. $4 < m < 9$.

Question 3 **C**

$\sec(3t) = \dfrac{1}{\cos(3t)}$

Vertical asymptotes occur for values of t such that $\cos(3t) = 0$ for $0 \le t \le \pi$.

Let $\theta = 3t$.

Solving $\cos(\theta) = 0$ for $0 \le \theta \le 3\pi$ we obtain $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}$.

So $t = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}$.

Question 4 **D**

$g(1) = 1 - 2\tan^{-1}(1) = 1 - \dfrac{\pi}{2}$ and so **A** is true.

The basic graph of $y = \tan^{-1}(x)$ is dilated by a factor of 2 in the y-direction, then reflected in the x-axis and translated 1 unit in the positive y-direction.

The maximal domain of g is the same as that for $y = \tan^{-1}(x)$ i.e. $x \in R$. So **B** is true.

The range of g is $\left(\left(-\dfrac{\pi}{2} \times 2\right) + 1, \left(\dfrac{\pi}{2} \times 2\right) + 1\right)$ i.e. $(-\pi + 1, \pi + 1)$. So **C** is true.

$g'(x) = -\dfrac{2}{x^2 + 1}$ and $g''(x) = \dfrac{4x}{(x^2 + 1)^2}$

Solving $g''(x) = 0$ for x gives $x = 0$.

$g(0) = 1$

As $g'(0) = -2\ (\neq 0)$, the point (0, 1) is a non-stationary point of inflection. Hence **D** is not true.

$g'(x) = -\dfrac{2}{x^2 + 1}$ and so $g'(x) < 0$ for $x \in R$. So option **E** is true.

Question 5 **E**

The parametric equations are $x = e^t \quad (1)$

$y = e^{-2t} \quad (2)$

From (1), if $t \ge 0$ then $x \ge 1$.

Squaring both sides of (1) we obtain $x^2 = e^{2t}$.

Substituting the above into $y = \dfrac{1}{e^{2t}}$ we obtain $y = \dfrac{1}{x^2}$, $x \ge 1$.

Question 6 B

From de Moivre's theorem, if $z = \text{cis}(\theta)$ then $z^n = \text{cis}(n\theta) = \cos(n\theta) + i\sin(n\theta)$.

Similarly, $\frac{1}{z^n} = \text{cis}(-n\theta) = \cos(n\theta) - i\sin(n\theta)$ since $\cos(-n\theta) = \cos(n\theta)$ and $\sin(-n\theta) = -\sin(n\theta)$.

$$\begin{aligned} z^n - \frac{1}{z^n} &= (\cos(n\theta) + i\sin(n\theta)) - (\cos(n\theta) - i\sin(n\theta)) \\ &= 2i\sin(n\theta) \end{aligned}$$

Question 7 B

As $i^3 = -i$, we have $w = i^3 z$.

Multiplication by i corresponds to a rotation of $\frac{\pi}{2}$ anticlockwise about the origin.

As $i^3 = i \times i \times i$, multiplication by i^3 i.e. $-i$, corresponds to a rotation of $\frac{3\pi}{2}$ anticlockwise about the origin.

This can be confirmed by noting that $i^3 = -i = \text{cis}\left(\frac{3\pi}{2}\right)$ and thus if $z = \text{cis}(\theta)$, $-iz = \text{cis}\left(\theta + \frac{3\pi}{2}\right)$.

Question 8 C

If $z = ai$ is one root of $P(z) = 0$ then $z = -ai$ is also a root from the conjugate root theorem.

So $P(z)$ must have $z^2 + a^2$ as a factor.

The only option where $z^2 + a^2$ is a factor is option **C** i.e. $z^3 + a^2 z = z(z^2 + a^2)$.

Question 9 A

If $z = r\text{cis}(\theta)$, then $\bar{z} = r\text{cis}(-\theta)$.

Now $\frac{1}{\bar{z}} = \frac{1}{r}\text{cis}(\theta)$ and so $\left(\frac{1}{\bar{z}}\right)^2 = \frac{1}{r^2}\text{cis}(2\theta)$.

Question 10 A

The differential equation $\frac{dy}{dx} = f(x)$ with $y = b$ when $x = a$ has solution $y = \int_a^x f(t)dt + b$.

Using the above definition, we obtain $y = \int_0^3 e^{t^3} dt + 2$.

Question 11 **D**

Let $u = \tan(x)$ and so $\frac{du}{dx} = \sec^2(x)$.

When $x = 0$, $u = 0$ and when $x = \frac{\pi}{6}$, $u = \frac{1}{\sqrt{3}}$.

$$\int_0^{\frac{\pi}{6}} \tan^2(x)\sec^2(x)dx = \int_0^{\frac{\pi}{6}} u^2 \frac{du}{dx}\, dx$$

$$= \int_0^{\frac{1}{\sqrt{3}}} u^2 du$$

$$= -\int_{\frac{1}{\sqrt{3}}}^{0} u^2 du$$

Question 12 **C**

From the direction field, it appears that all solutions to the differential equation $\frac{dv}{dt} = f(v)$ have a limiting value of 60 m/s as $t \to \infty$.

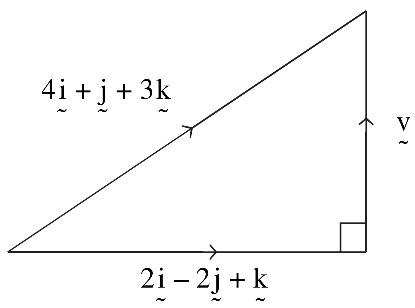
Question 13 **C**

Let $\underset{\sim}{v}$ be the vector component of $4\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$ perpendicular to $2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$.

$$(2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}) + \underset{\sim}{v} = 4\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$$

$$\underset{\sim}{v} = 4\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k} - (2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k})$$

$$= 2\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$$



Question 14 E

$\underset{\sim}{u} + \underset{\sim}{w} = (2+m)\underset{\sim}{i} + (2+n)\underset{\sim}{j} + \underset{\sim}{k}$

If $\underset{\sim}{u} + \underset{\sim}{w}$ is parallel to $\underset{\sim}{v}$ then $\underset{\sim}{u} + \underset{\sim}{w} = \lambda \underset{\sim}{v}$ where $\lambda \in R$.

$(2+m)\underset{\sim}{i} + (2+n)\underset{\sim}{j} + \underset{\sim}{k} = 2\lambda \underset{\sim}{j} + 2\lambda \underset{\sim}{k}$

By equating the $\underset{\sim}{i}$ components, $2 + m = 0$ i.e. $m = -2$.

By equating the $\underset{\sim}{k}$ components, $2\lambda = 1$ i.e. $\lambda = \frac{1}{2}$.

By equating the $\underset{\sim}{j}$ components, $2 + n = 2 \times \frac{1}{2}$ i.e. $n = -1$.

So $m = -2$ and $n = -1$.

Question 15 B

Option **B** is a necessary and sufficient condition.

$\overrightarrow{MN} = \overrightarrow{PO}$ i.e. one pair of opposite sides are equal and parallel meaning that *MNOP* is a parallelogram and $|\overrightarrow{MN}| = |\overrightarrow{MP}|$ means that *MNOP* has adjacent sides of equal length.

Question 16 E

$$\overrightarrow{DM} = \overrightarrow{DC} + \overrightarrow{CM}$$
$$= \underset{\sim}{i} + \frac{1}{2}\underset{\sim}{j}$$

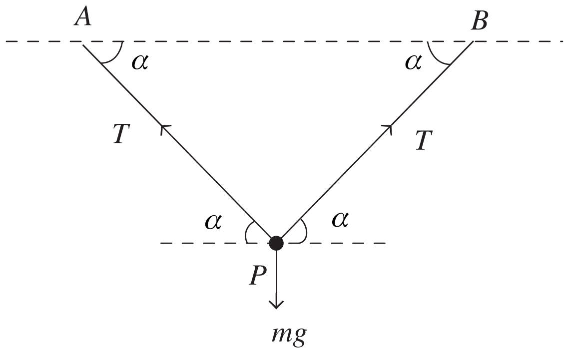
$$\overrightarrow{DN} = \overrightarrow{DA} + \overrightarrow{AN}$$
$$= \underset{\sim}{j} + \frac{1}{3}\underset{\sim}{i}$$
$$= \frac{1}{3}\underset{\sim}{i} + \underset{\sim}{j}$$

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{DM} \cdot \overrightarrow{DN}}{|\overrightarrow{DM}||\overrightarrow{DN}|}\right)$$
$$= \cos^{-1}\left(\frac{\left(\underset{\sim}{i} + \frac{1}{2}\underset{\sim}{j}\right) \cdot \left(\frac{1}{3}\underset{\sim}{i} + \underset{\sim}{j}\right)}{\left|\underset{\sim}{i} + \frac{1}{2}\underset{\sim}{j}\right|\left|\frac{1}{3}\underset{\sim}{i} + \underset{\sim}{j}\right|}\right)$$
$$= \cos^{-1}\left(\frac{\frac{1}{3} + \frac{1}{2}}{\frac{\sqrt{5}}{2} \times \frac{\sqrt{10}}{3}}\right)$$
$$= \cos^{-1}\left(\frac{\frac{5}{6}}{\frac{5\sqrt{2}}{6}}\right)$$
$$= \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$
$$= \frac{\pi}{4}$$

Question 17 A

$$\Sigma\underset{\sim}{F} = m\underset{\sim}{a}$$
$$\underset{\sim}{N} + \underset{\sim}{T} + \underset{\sim}{R} = 10\underset{\sim}{a}$$
$$(-2\underset{\sim}{i} - \underset{\sim}{j}) + (8\underset{\sim}{i} - 3\underset{\sim}{j}) + (3\underset{\sim}{i} + 16\underset{\sim}{j}) = 10\underset{\sim}{a}$$
$$9\underset{\sim}{i} + 12\underset{\sim}{j} = 10\underset{\sim}{a}$$
$$\underset{\sim}{a} = \frac{1}{10}(9\underset{\sim}{i} + 12\underset{\sim}{j})$$
$$|\underset{\sim}{a}| = \frac{1}{10}\sqrt{9^2 + 12^2}$$
$$= 1.5$$

Question 18 A



Resolving forces in the vertical direction we have $T\sin(\alpha) + T\sin(\alpha) = mg$.

$2T\sin(\alpha) = mg$

So, $T = \dfrac{mg}{2\sin(\alpha)}$

Question 19 E

Let the distance between A and B be x metres.

So the distance midway between A and B is $\dfrac{x}{2}$ metres.

The speed of the particle midway between A and B is v_m m/s.

Using $v^2 = u^2 + 2as$ with $v = v_m$ and $s = \dfrac{x}{2}$ we obtain $v_m{}^2 = u^2 + ax$.

Using $v^2 = u^2 + 2as$ with $u = v_m$ and $s = \dfrac{x}{2}$ we obtain $v^2 = v_m{}^2 + ax$.

Rearranging $v^2 = v_m{}^2 + ax$ we obtain $v_m{}^2 = v^2 - ax$.

$v_m{}^2 = u^2 + ax \qquad (1)$

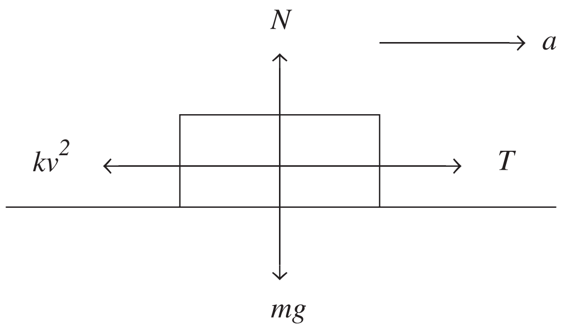
$v_m{}^2 = v^2 - ax \qquad (2)$

(1) + (2) gives:

$2v_m{}^2 = u^2 + v^2$ and so $v_m = \sqrt{\dfrac{u^2 + v^2}{2}}$.

So the particle's speed midway between A and B is $\sqrt{\dfrac{u^2 + v^2}{2}}$ (m/s).

Question 20 C



The train's equation of motion in the horizontal direction is $T - kv^2 = ma$.

So $a = \dfrac{T - kv^2}{m}$.

We can find *k* by setting $a = 0$ when $v = V$.

$\dfrac{1}{m}(T - kv^2) = 0$

Hence $k = \dfrac{T}{V^2}$.

So $a = \dfrac{1}{m}\left(T - \dfrac{Tv^2}{V}\right)$ i.e. $a = \dfrac{T}{mV^2}(V^2 - v^2)$.

Question 21 B

The initial momentum (p_i) is 2.5×10 i.e. 25 (kg m/s).

The final momentum (p_f) is 2.5×6 i.e. 15 (kg m/s).

Change in momentum $(\Delta p) = p_f - p_i$

$= -10$ (kg m/s)

Alternatively:

$\Delta p = m\Delta v$ where Δv is the change in velocity

$= 2.5(6 - 10)$

$= -10$ (kg m/s)

Question 22 D

At $t = 0$, $v = 2$ and so option **E** is true.

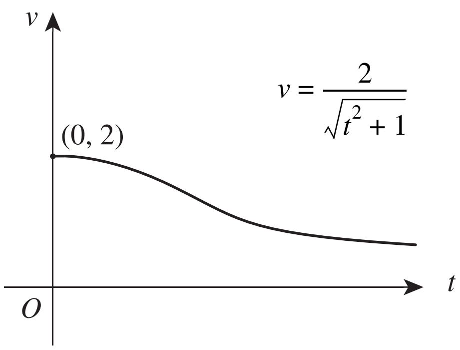
Calculating the acceleration we obtain $\frac{dv}{dt} = -\frac{2t}{(t^2 + 1)^{\frac{3}{2}}}$.

When $t = 0$, $\frac{dv}{dt} = 0$ and so **A** is true.

The expression for $\frac{dv}{dt}$ confirms that the acceleration is always negative. Hence **B** is true.

As $\frac{dx}{dt} = \frac{2}{\sqrt{t^2 + 1}}$, then the distance, x metres, travelled by the particle in the first 3 seconds of motion is given by $x = \int_0^3 \frac{2}{\sqrt{t^2 + 1}} dt$. Hence **C** is true.

Note: $x = \int_0^3 \frac{2}{\sqrt{t^2 + 1}} dt$ gives distance since $\frac{dx}{dt} > 0$ for $t \geq 0$.

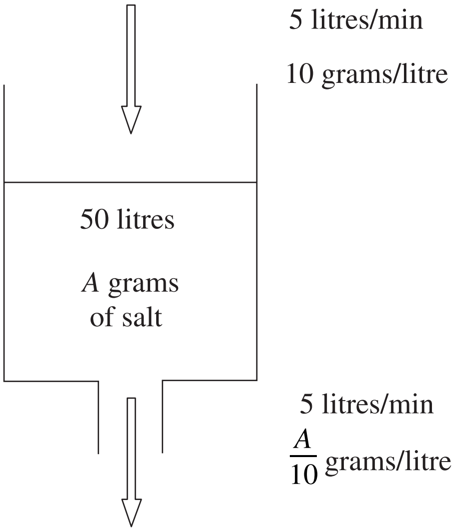


Referring to the velocity–time graph, after the initial velocity of 2 m/s, the particle slows down, fairly quickly at first and then more slowly. Hence option **D** is false.

SECTION 2

Question 1

a.



$\frac{dA}{dt}$ = rate of inflow – rate of outflow M1

$\frac{dA}{dt} = 5 \times 2 - 5 \times \frac{A}{50} = 10 - \frac{A}{10}$ A1

b. **i.** $\frac{dA}{dt} = 10 - \frac{A}{10}$

$\frac{dA}{dt} = \frac{100 - A}{10}$ M1

$\frac{dt}{dA} = \frac{10}{100 - A}$

$\int_0^t dt = 10\int_{10}^{50} \frac{1}{100 - A} dA$ and so $t = 10\int_{10}^{50} \frac{1}{100 - A} dA$ A1

ii. $t = 10\log_e\left(\frac{9}{5}\right)$ A1

c. Let k grams be the amount of salt in the tank after 15 minutes.

$10\int_{10}^{k} \left(\frac{1}{100 - A}\right) dA = 15$

Attempting to solve the above equation. M1

$k = 79.9$ (grams) or 120.1 (grams)

After a long time, the amount of salt in the tank approaches, but never exceeds, 100 grams because the concentration of the salt solution approaches 2 grams/litre. A1

Hence $k = 79.9$ (grams) (correct to the nearest tenth of a gram). A1

Question 2

a. Method 1:

The maximum height occurs when the vertical component of the velocity is zero.

Solving $30\sin(50°) - 9.8t = 0$ for t gives $t = 2.345\ldots$ (s). M1

Substituting $t = 2.345\ldots$ into $30\sin(50°)t - 4.9t^2$ gives $y = 26.9$ m (correct to the nearest tenth of a metre). A1

Use alternatively Method 2:

The maximum height occurs when the vertical component of the velocity is zero.

Substituting $u = 30\sin(50°)$, $v = 0$ and $g = -9.8$ into $v^2 = u^2 + 2gy$. M1

Solving $(30\sin(50°))^2 + 2(-9.8)y = 0$ for y gives $y = 26.9$ (m) (correct to the nearest tenth of a metre). A1

b. Method 1:

Solving $30\sin(50°)t - 4.9t^2 = 2$ for t gives $t = 0.0887\ldots$ (s) or $t = 4.601\ldots$ (s). M1

Rejecting the smaller t-value, we obtain $t = 4.6$ (s) (correct to one decimal place). A1

Use alternatively Method 2:

It takes $2.345\ldots$ seconds for the cricket ball to reach its maximum height. Now we find the remaining time of flight before the cricket ball is caught.

Solving $4.9t^2 = 26.946\ldots - 2$ for t with $t > 0$ gives $t = 2.256\ldots$ (s). M1

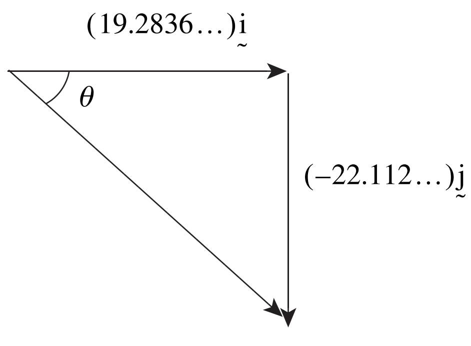
So $2.345\ldots + 2.256\ldots = 4.6$ (s) (correct to one decimal place). A1

c. $\underset{\sim}{\text{r}}'(t) = 30\cos(50°)\underset{\sim}{i} + (30\sin(50°) - 9.8t)\underset{\sim}{j}$ A1

$|\underset{\sim}{\text{r}}'(4.601\ldots)| = \sqrt{(30\cos(50°))^2 + (30\sin(50°) - 9.8 \times 4.601\ldots)^2}$ M1

Hence the cricket ball's speed is 29.3 (m/s) (correct to one decimal place). A1

d. Let θ be the angle between the direction of the cricket ball's motion and the horizontal.

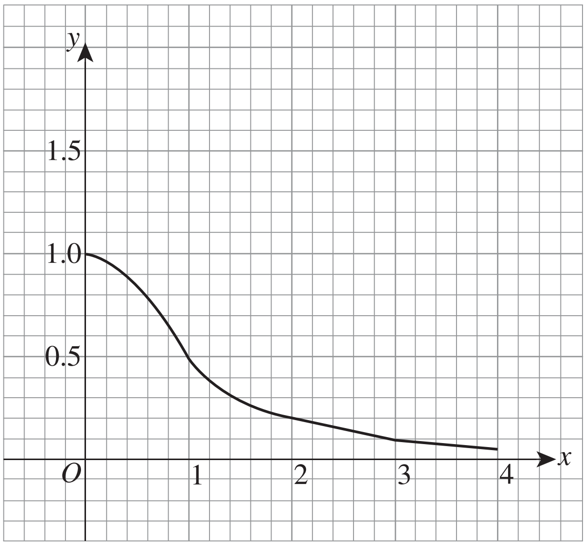


$\tan(\theta) = \left|\dfrac{30\sin(50°) - 9.8 \times 4.601\ldots}{30\cos(50°)}\right| = \dfrac{22.112\ldots}{19.2836\ldots}$ M1 A1

Hence $\theta = 48.9°$ (correct to the nearest tenth of a degree). A1

Question 3

a.



An intercept at (0, 1) and $y = 0$ is a horizontal asymptote. A1

Correct shape and scale. A1

b. **i.** $g(x) = \dfrac{1}{x^2 + 1}$

$g'(x) = -\dfrac{2x}{(x^2 + 1)^2}$

$g''(x) = \dfrac{2(3x^2 - 1)}{(x^2 + 1)^3}$ (or equivalent) A1

ii. Solving $\dfrac{2(3x^2 - 1)}{(x^2 + 1)^3} = 0$ for x with $x \geq 0$ gives $x = \dfrac{\sqrt{3}}{3}$. M1 A1

By calculating $g'\left(\dfrac{\sqrt{3}}{3} \pm a\right)$ M1

$g'\left(\dfrac{\sqrt{3}}{3}\right) = -\dfrac{3\sqrt{3}}{8}(-0.6495\ldots)$

e.g. $g'\left(\dfrac{\sqrt{3}}{3} - 0.05\right) = -0.6456\ldots\left(> g'\left(\dfrac{\sqrt{3}}{3}\right)\right)$ and

$g'\left(\dfrac{\sqrt{3}}{3} + 0.05\right) = -0.6460\ldots\left(> g'\left(\dfrac{\sqrt{3}}{3}\right)\right)$ A1

Hence $x = \dfrac{\sqrt{3}}{3}$ is where the largest negative gradient occurs.

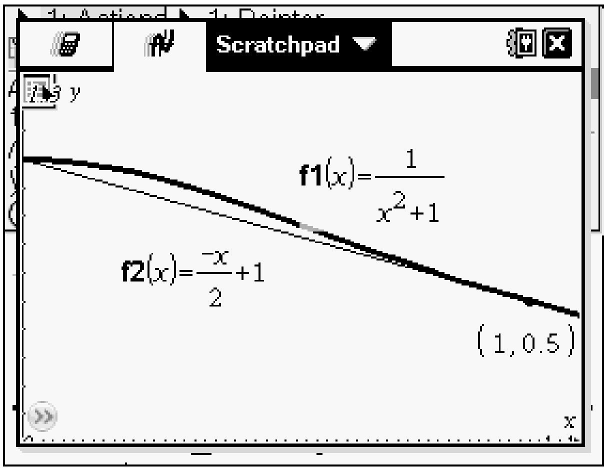
OR calculating $g''\left(\dfrac{\sqrt{3}}{3} \pm a\right)$ M1

e.g. $g''\left(\dfrac{\sqrt{3}}{3} - 0.05\right) = -0.1587\ldots(< 0)$ and

$g''\left(\dfrac{\sqrt{3}}{3} + 0.05\right) = 0.1335\ldots(> 0)$ A1

Hence $x = \dfrac{\sqrt{3}}{3}$ is where the largest negative gradient occurs.

c. **i.**



V_1 is the volume of solid of revolution formed by rotating the graph of g by 360° about the y-axis for $0 \le x \le 1$.

Rearranging $y = \dfrac{1}{x^2 + 1}$ to express x^2 in terms of y we obtain $x^2 = \dfrac{1}{y} - 1$. M1

Hence $V_1 = \pi \displaystyle\int_{\frac{1}{2}}^{1} \left(\frac{1}{y} - 1\right) dy$. A1

V_2 is the volume of solid of revolution formed by rotating the tangent line by 360° about the y-axis for $0 \le x \le 1$.

Rearranging $y = -\dfrac{x}{2} + 1$ to express x in terms of y we obtain $x = 2(1 - y)$. M1

Hence $V_2 = \pi \displaystyle\int_{\frac{1}{2}}^{1} (2(1 - y))^2 dy$. A1

ii. $V_1 = \pi\left(\log_e(2) - \dfrac{1}{2}\right)$ A1

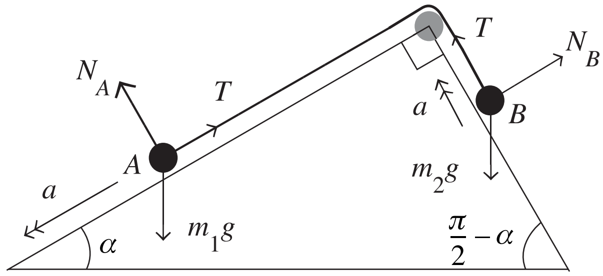
$V_2 = \dfrac{\pi}{6}$ A1

Given that $V_1 > V_2$, we obtain $\pi\left(\log_e(2) - \dfrac{1}{2}\right) > \dfrac{\pi}{6}$.

So $\log_e(2) - \dfrac{1}{2} > \dfrac{1}{6}$ i.e. $\log_e(2) > \dfrac{2}{3}$. A1

Question 4

a.



Particle A: normal reaction force N_A; weight force m_1g; tension T A1

Particle B: normal reaction force N_B; weight force m_2g; tension T A1

b. Particle A: $m_1g\sin(\alpha) - T = m_1a$ (1) A1

Particle B: $T - m_2g\sin\left(\frac{\pi}{2}-\alpha\right) = m_2a$ or $T - m_2g\cos(\alpha) = m_2a$ (2) A1

c. (From **Question 4 b**) (1) + (2) gives $a = \dfrac{m_1g\sin(\alpha) - m_2g\cos(\alpha)}{m_1 + m_2}$ M1

Particle A will slide down the inclined plane if, and only if, $a > 0$.
i.e. $m_1g\sin(\alpha) - m_2g\cos(\alpha) > 0$. M1

$m_1g\sin(\alpha) > m_2g\cos(\alpha)$

$\dfrac{m_1g\sin(\alpha)}{g\cos(\alpha)} > \dfrac{m_2g\cos(\alpha)}{g\cos(\alpha)}$ $(g\cos(\alpha) \neq 0)$

$m_1\tan(\alpha) > m_2$

$\tan(\alpha) > \dfrac{m_2}{m_1}$ and so $\alpha > \tan^{-1}\left(\dfrac{m_2}{m_1}\right)$ A1

d. **i.** $\dfrac{da}{d\alpha} = \dfrac{g(2\cos^3(\alpha) - \sin^3(\alpha))}{(2\cos(\alpha) + \sin(\alpha))^2}$ (or equivalent e.g. see below) A1

$\dfrac{da}{d\alpha} = \dfrac{(2 + \tan(\alpha))g\cos(\alpha) - 2\sec^2(\alpha)g\sin(\alpha)}{(2 + \tan(\alpha))^2}$

Solving $\dfrac{da}{d\alpha} = 0$ for α gives $\alpha = 0.8999\ldots$. M1 A1

So the two base angles are 0.90 radians (51.56°) and 0.67 radians (38.44°) (correct to two decimal places). A1

ii. Substituting $\alpha = 0.8999\ldots$ into $a = \dfrac{g\sin(\alpha)}{\tan(\alpha) + 2}$ we obtain $a = 2.35$ (m/s^2) (correct to two decimal places). A1

Question 5

a. $|z+5-i| = \sqrt{2}$ where $z = x + yi$

$|x+5+(y-1)i| = \sqrt{2}$ A1

$\sqrt{(x+5)^2+(y-1)^2} = \sqrt{2}$

Squaring both sides we obtain $(x+5)^2+(y-1)^2 = 2$. A1

b. Method 1:

$\text{Arg}(z) = \frac{3\pi}{4}$ is the half-line emanating from O, but not including O, which makes an angle of $\frac{3\pi}{4}$ with the positive Re(z) direction.

This half-line has a cartesian equation given by $y = -x$. A1

$\text{Arg}(z+2i) = \frac{3\pi}{4}$ is the half-line of $\text{Arg}(z) = \frac{3\pi}{4}$ translated –2 units in the Im(z) direction.

Hence L has a cartesian equation given by $y = -x-2, x < 0$. A1

Use alternatively Method 2:

Let $z = x + yi$

So $z + 2i = x + (y+2)i$.

$\tan\left(\frac{3\pi}{4}\right) = \frac{y+2}{x}$, $y > -2$ and $x < 0$. A1

As $\tan\left(\frac{3\pi}{4}\right) = -1$, we obtain $\frac{y+2}{x} = -1$, $x < 0$.

Hence L has a cartesian equation given by $y = -x-2, x < 0$. A1

c. Point B has coordinates (–4, 2).

Substituting $x = -4$ into $y = -x-2, x < 0$ gives $y = 2$ and substituting $x = -4$ and $y = 2$ into $(x+5)^2+(y-1)^2$ we obtain $(-4+5)^2+(2-1)^2 = 2$. A1

Hence point B lies on L and also lies on C.

d. Method 1:

$$\frac{d}{dx}((x+5)^2) + \frac{d}{dx}((y-1)^2) = \frac{d}{dx}(2)$$

$$2(x+5) + 2(y-1)\frac{dy}{dx} = 0$$ M1

$$\frac{dy}{dx} = \frac{-(x+5)}{y-1}$$ A1

At (–4, 2), the gradient of both C and L is –1. So L touches C. A1

Use alternatively Method 2:

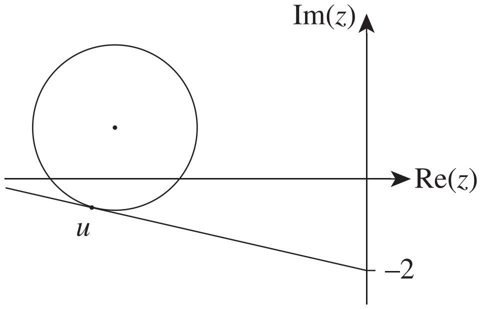
The gradient of the line (radius) joining (–5, 1) and (–4, 2) is $m = \frac{2-1}{-4-(-5)} = 1$. A1

L has a gradient of –1.

The product of the two gradients is –1. A1

Hence the radius is perpendicular to L and therefore is a tangent. So L touches C. A1

e.



Method 1:

Let the cartesian equation of the movable half-line be $y = mx - 2$, $x < 0$.

Let $u = x + yi$.

Substituting $y = mx - 2$ into $(x+5)^2 + (y-1)^2 = 2$ gives $(m^2+1)x^2 + (10-6m)x + 34 = 2$.

Solving $(m^2+1)x^2 + (10-6m)x + 34 = 2$ for x gives $x = \dfrac{3m - 5 \pm \sqrt{-23m^2 - 30m - 7}}{m^2+1}$. M1

Solving $-23m^2 - 30m - 7 = 0$ for m gives $m = -1$ or $m = -\dfrac{7}{23}$. Reject $m = -1$. A1

Solving $(x+5)^2 + (y-1)^2 = 2$ and $y = -\dfrac{7}{23}x - 2$ for x and y gives $x = -\dfrac{92}{17}$ and $y = -\dfrac{6}{17}$. M1 A1

Hence $u = -\dfrac{92}{17} - \dfrac{6}{17}i$ and so $u + 2i = -\dfrac{92}{17} + \dfrac{28}{17}i$. A1

$\text{Arg}\left(-\dfrac{92}{17} + \dfrac{28}{17}i\right) = \pi - \tan^{-1}\left(\dfrac{7}{23}\right)$ A1

Use alternatively Method 2:

Let the cartesian equation of the movable half-line be $y = mx - 2, x < 0$.

Let $u = x + yi$.

Substituting $y = mx - 2$ into $(x+5)^2 + (y-1)^2 = 2$ gives $(m^2+1)x^2 + (10-6m)x + 32 = 0$.

$\Delta = (10-6m)^2 - 4 \times 32 \times (m^2+1)$ M1

Solving $(10-6m)^2 - 4 \times 32 \times (m^2+1) = 0$ (or equivalent) for m gives $m = -1$ or $m = -\dfrac{7}{23}$. A1

Reject $m = -1$.

Solving $(x+5)^2 + (y-1)^2 = 2$ and $y = -\dfrac{7}{23}x - 2$ for x and y gives $x = -\dfrac{92}{17}$ and $y = -\dfrac{6}{17}$. M1 A1

Hence $u = -\dfrac{92}{17} - \dfrac{6}{17}i$ and so $u + 2i = -\dfrac{92}{17} + \dfrac{28}{17}i$. A1

$\text{Arg}\left(-\dfrac{92}{17} + \dfrac{28}{17}i\right) = \pi - \tan^{-1}\left(\dfrac{7}{23}\right)$ A1

Method 3:

Let $u = x + yi$.

$$\begin{aligned}\text{Arg}(u + 2i) &= \text{Arg}(x + (y + 2)i)\\ &= \pi - \tan^{-1}\left|\frac{y+2}{x}\right| \text{ if } u \text{ lies on } C\end{aligned}$$ M1

If u lies on C then $y = 1 \pm \sqrt{2 - (x + 5)^2}$.

The maximum value of $\text{Arg}(u + 2i)$ occurs when the line joining $(0, -2)$ to $\left(x, 1 - \sqrt{2 - (x + 5)^2}\right)$ is a tangent to $y = 1 - \sqrt{2 - (x + 5)^2}$.

Using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, we obtain $m = \dfrac{3 - \sqrt{2 - (x + 5)^2}}{x}$. A1

We need to find the value of x such that $m = \dfrac{d}{dx}(1 - \sqrt{2 - (x + 5)^2})$.

Solving $\dfrac{3 - \sqrt{2 - (x + 5)^2}}{x} = \dfrac{d}{dx}(1 - \sqrt{2 - (x + 5)^2})$ for x gives $x = -\dfrac{92}{17}$. M1

Substituting $x = -\dfrac{92}{17}$ into $y = 1 - \sqrt{2 - (x + 5)^2}$, we obtain $y = -\dfrac{6}{17}$. A1

Hence $u = -\dfrac{92}{17} - \dfrac{6}{17}i$ and so $u + 2i = -\dfrac{92}{17} + \dfrac{28}{17}i$. A1

$$\begin{aligned}\text{Arg}(u + 2i) &= \text{Arg}\left(-\frac{92}{17} + \frac{28}{17}i\right)\\ &= \pi - \tan^{-1}\left(\frac{7}{23}\right)\end{aligned}$$ A1

Note: Other solution approaches are possible.