

The Mathematical Association of Victoria

SPECIALIST MATHEMATICS 2010

Trial Written Examination 1--SOLUTIONS

Question 1

$$3x^2 + 4y^2 = 48$$

$$\frac{d}{dx}(3x^2) + \frac{d}{dx}(4y^2) = \frac{d}{dx}(48) \quad [\text{M1}]$$

$$6x + \frac{d}{dy}(4y^2) \frac{dy}{dx} = 0$$

$$6x + 8y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{3x}{4y} \quad [\text{A1}]$$

$$\text{When } x = 2, \quad 12 + 4y^2 = 48$$

$$y = \pm 3$$

$$\text{When } x = 2, y = 3, \quad \frac{dy}{dx} = -\frac{6}{12} = -\frac{1}{2}$$

$$\text{When } x = 2, y = -3, \quad \frac{dy}{dx} = -\frac{6}{-12} = \frac{1}{2} \quad [\text{A1}]$$

Total 3 marks

Question 2

$$y = \arcsin(2x)$$

$$y = \sin^{-1}(u) \quad \text{where } u = 2x$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \times 2$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}} \quad [\text{A1}]$$

$$\frac{dy}{dx} = 2(1-4x^2)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = 2 \times -\frac{1}{2} (1-4x^2)^{-\frac{3}{2}} \times -8x \quad [\text{A1}]$$

$$\frac{d^2 y}{dx^2} = 8x(1-4x^2)^{\frac{3}{2}}$$

$$\frac{d^2 y}{dx^2} = \frac{8x}{\sqrt{(1-4x^2)^3}}$$

[A1]

Total 3 marks

Question 3

$$\tan(2\theta) = \sqrt{3}, \text{ where } \theta \in \left(-\pi, -\frac{\pi}{2}\right)$$

$$\tan(2\theta) = \frac{2 \tan(\theta)}{1 - \tan^2(\theta)}$$

$$\sqrt{3}(1 - \tan^2(\theta)) = 2 \tan(\theta) \quad [\text{M1}]$$

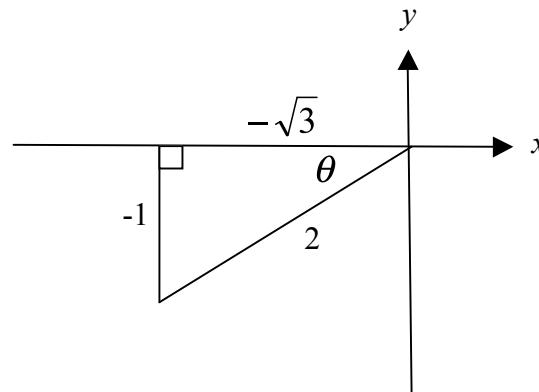
$$\sqrt{3} \tan^2(\theta) + 2 \tan(\theta) - \sqrt{3} = 0$$

$$\tan(\theta) = \frac{-2 \pm \sqrt{4 - 4(\sqrt{3})(-\sqrt{3})}}{2\sqrt{3}} \quad \text{quadratic formula} \quad [\text{M1}]$$

$$= \frac{-2 \pm 4}{2\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}, \frac{-3}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}, \text{ since } \theta \in \left(-\pi, -\frac{\pi}{2}\right)$$



$$\text{From diagram, } \sin(\theta) = -\frac{1}{2}, \cos(\theta) = -\frac{\sqrt{3}}{2} \quad [\text{M1}]$$

$$\text{cis}(\theta) = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$

$$= -\frac{1}{2}(\sqrt{3} + i) \quad [\text{A1}]$$

Total 4 marks

Question 4**a.**

$$-2 + 2\sqrt{3}i = 4\text{cis}\left(\frac{2\pi}{3}\right)$$

Let $z = r\text{cis}(\theta)$ be a square root of $-2 + 2\sqrt{3}i$

$$\text{Hence } z^2 = r^2\text{cis}(2\theta) = 4\text{cis}\left(\frac{2\pi}{3} + 2k\pi\right), \text{ where } k \in \mathbb{J} \quad [\text{M1}]$$

$$r = 2, \quad 2\theta = \frac{2\pi}{3} + 2k\pi$$

$$\text{Let } k = 0, \quad \theta = \frac{\pi}{3}$$

$$\text{Let } k = 1, \quad 2\theta = \frac{2\pi}{3} + 2\pi$$

$$\theta = \frac{4\pi}{3}$$

$$z = 2\text{cis}\left(\frac{\pi}{3}\right),$$

$$= 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$$

$$= 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$= 1 + \sqrt{3}i, \quad -1 - \sqrt{3}i$$

$$z = 2\text{cis}\left(-\frac{2\pi}{3}\right)$$

$$= 2\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$$

$$= 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

[A1]

[A1]

Alternative solution

Let $z = a + bi$ be a square root of $-2 + 2\sqrt{3}i$, where $a, b \in \mathbb{R}$

$$z^2 = a^2 + 2abi - b^2 = -2 + 2\sqrt{3}i \quad [\text{M1}]$$

Equating real coefficients and equating complex coefficients

$$a^2 - b^2 = -2 \quad \dots(1) \qquad 2ab = 2\sqrt{3}$$

$$b = \frac{\sqrt{3}}{a} \quad \dots(2)$$

Substituting (2) into (1)

$$a^2 - \frac{3}{a^2} = -2$$

$$a^4 - 3 = -2a^2$$

$$a^4 + 2a^2 - 3 = 0$$

$$a^4 - 3 = -2a^2$$

$$(a^2 + 3)(a^2 - 1) = 0 \quad [A1]$$

$$a^2 = -3 \text{ (no real solutions)} \quad a^2 = 1$$

$$a = \pm 1$$

$$\text{If } a = 1, b = \sqrt{3} \text{ and if } a = -1, b = -\sqrt{3}$$

$$\text{Hence } z = 1 + \sqrt{3}i, -1 - \sqrt{3}i \quad [A1]$$

b.

$$z^2 + (\sqrt{3} - i)z + (1 - \sqrt{3}i) = 0$$

$$z = \frac{-\sqrt{3} + i \pm \sqrt{(\sqrt{3} - i)^2 - 4(1 - \sqrt{3}i)}}{2} \quad [M1]$$

$$= \frac{-\sqrt{3} + i \pm \sqrt{3 - 2\sqrt{3}i - 1 - 4 + 4\sqrt{3}i}}{2}$$

$$= \frac{-\sqrt{3} + i \pm \sqrt{-2 + 2\sqrt{3}i}}{2} \quad [A1]$$

$$= \frac{-\sqrt{3} + i + (\sqrt{3}i)}{2}, \frac{-\sqrt{3} + i - (\sqrt{3}i)}{2}, \text{ from part a.}$$

$$\frac{(-\sqrt{3}) + (\sqrt{3}i)}{2}, \frac{(-1 - \sqrt{3}) + (-\sqrt{3}i)}{2} \quad [A1]$$

Total 6 marks

Question 5

$$y = \frac{x^3 - 1}{x}$$

$$y = x^2 - \frac{1}{x}$$

Sketch $y_1 = x^2$ and $y_2 = -\frac{1}{x}$ and use addition of ordinates.

$$x\text{-intercepts } (y = 0) \quad \frac{x^3 - 1}{x} = 0$$

$$x^3 - 1 = 0$$

$$x = 1$$

x -intercept (1, 0)

[A1]

$$\text{Stationary points: } \frac{dy}{dx} = 0 \quad \frac{dy}{dx} = 2x + \frac{1}{x^2}$$

$$2x + \frac{1}{x^2} = 0$$

$$2x^3 + 1 = 0$$

$$x^3 = -\frac{1}{2}$$

$$x = -\frac{1}{\sqrt[3]{2}}, \quad y = \frac{-\frac{3}{2}}{-\frac{1}{\sqrt[3]{2}}}$$

[A1]

$$\text{Minimum: } \left(-\frac{1}{\sqrt[3]{2}}, \frac{3\sqrt[3]{2}}{2} \right)$$

$$\text{Point of inflexion: } \frac{d^2y}{dx^2} = 0 \quad \frac{dy}{dx} = 2x + x^{-2}$$

$$\frac{d^2y}{dx^2} = 2 - 2x^{-3}$$

$$2 - \frac{2}{x^3} = 0$$

$$2x^3 - 2 = 0$$

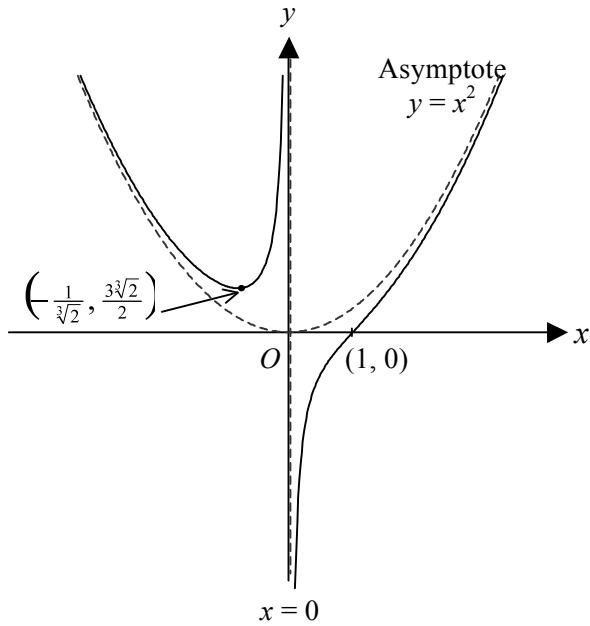
$$x^3 = 1$$

Point of inflexion (1, 0)

[A1]

Asymptotes: $y = x^2$ and $x = 0$ (y -axis)

[A1]



Correct graph [A1]

Total 5 marks

Question 6

$$\int_m^{\frac{\pi}{2}} \sec^2\left(\frac{x}{2}\right) dx = \frac{2}{3} (3 - \sqrt{3})$$

$$\left[2 \tan\left(\frac{x}{2}\right) \right]_m^{\frac{\pi}{2}} = \frac{2}{3} (3 - \sqrt{3})$$

[M1]

$$2 \left(\tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{m}{2}\right) \right) = \frac{2}{3} (3 - \sqrt{3})$$

$$1 - \tan\left(\frac{m}{2}\right) = \frac{3 - \sqrt{3}}{3}$$

[A1]

$$1 - \tan\left(\frac{m}{2}\right) = 1 - \frac{\sqrt{3}}{3}$$

$$\tan\left(\frac{m}{2}\right) = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

$$\frac{m}{2} = \frac{\pi}{6}$$

$$m = \frac{\pi}{3}$$

[A1]

Total 3 marks

Question 7

$$\int \frac{x}{\sqrt[3]{3x^2 + 1}} dx$$

$$= \frac{1}{6} \int \frac{6x}{(3x^2 + 1)^{\frac{1}{3}}} dx$$

Let $u = 3x^2 + 1$

$$\frac{du}{dx} = 6x \quad [\text{M1}]$$

$$\Rightarrow du = 6x dx$$

$$= \frac{1}{6} \int u^{-\frac{1}{3}} du \quad [\text{A1}]$$

$$= \frac{1}{6} \times \frac{3}{2} u^{\frac{2}{3}} + c$$

$$= \frac{1}{4} (3x^2 + 1)^{\frac{2}{3}} + c \quad [\text{A1}]$$

Total 3 marks

Question 8

$$a = v^3 + \pi^2 v$$

$$v \frac{dv}{dx} = v^3 + \pi^2 v$$

$$\frac{dv}{dx} = v^2 + \pi^2$$

$$\frac{dx}{dv} = \frac{1}{v^2 + \pi^2} \quad [\text{A1}]$$

$$x = \frac{1}{\pi} \int \frac{\pi}{v^2 + \pi^2} dv$$

$$x = \frac{1}{\pi} \arctan\left(\frac{v}{\pi}\right) + c \quad [\text{A1}]$$

Particle starts from rest at the origin:

At $x = 0$, $v = 0$ $0 = \frac{1}{\pi} \arctan\left(\frac{0}{\pi}\right) + c$

$$\Rightarrow c = 0$$

$$x = \frac{1}{\pi} \arctan\left(\frac{v}{\pi}\right)$$

$$v = \pi \tan(\pi x) \quad [\text{A1}]$$

At $x = 0.25 \text{ m}$ $v = \pi \tan(0.25\pi)$

$$v = \pi \tan\left(\frac{\pi}{4}\right)$$

$$v = \pi \text{ m/s} \quad [\text{A1}]$$

Total 4 marks

Question 9

The paths of the particles will meet when the particles are in the same position, but this may be at different times. Let t_A and t_B be the time variables for each particle.

$$\underline{r}_A(t) = (t_A^2 + 1)\underline{i} + 2t_A\underline{j}$$

$$\underline{r}_B(t) = (7t_B - 5)\underline{i} + (t_B + 6)\underline{j}$$

Equating $\underline{i}$ components:

$$t_A^2 + 1 = 7t_B - 5 \quad \dots (1)$$

Equating $\underline{j}$ components:

$$2t_A = t_B + 6$$

$$t_B = 2t_A - 6 \quad \dots (2)$$

Substitute (2) into (1)

$$t_A^2 + 1 = 7(2t_A - 6) - 5 \quad \text{[M1]}$$

$$t_A^2 + 1 = 14t_A - 42 - 5$$

$$t_A^2 - 14t_A + 48 = 0$$

$$(t_A - 6)(t_A - 8) = 0$$

$$t_A = 6, 8 \text{ seconds} \quad \text{[A1]}$$

When $t_A = 6$ seconds, $t_B = 2 \times 6 - 6 = 6$ seconds

The particles will be in the same position at the same time and so will collide.

Finding this position: $\underline{r}_A(6) = (6^2 + 1)\underline{i} + 2 \times 6\underline{j} = 37\underline{i} + 12\underline{j}$

$$\underline{r}_B(6) = (7 \times 6 - 5)\underline{i} + (6 + 6)\underline{j} = 37\underline{i} + 12\underline{j}$$

The particles collide at the point (37, 12) [A1]

When $t_A = 8$ seconds, $t_B = 2 \times 8 - 6 = 10$ seconds.

$$\underline{r}_A(8) = (8^2 + 1)\underline{i} + 2 \times 8\underline{j} = 65\underline{i} + 16\underline{j} \quad \text{[A1]}$$

$$\underline{r}_B(8) = (7 \times 10 - 5)\underline{i} + (10 + 6)\underline{j} = 65\underline{i} + 16\underline{j}$$

The paths meet at the point (65, 16), but this is not a collision because the particles will be in this position at different times. [A1]

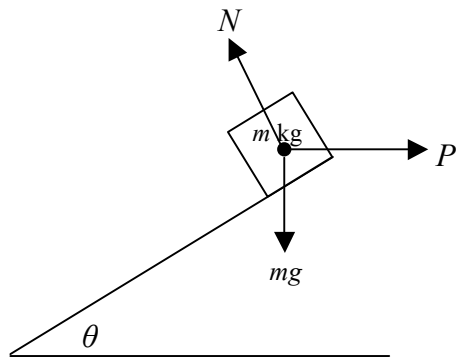
Total 5 marks

Question 10

Let N newtons be the reaction force of the inclined plane on the box.

The weight force acting on the stationary box is mg newtons.

The plane is smooth.



N = Normal reaction
 mg = Weight force

Resolving forces parallel to the plane

$$mg \sin(\theta) - P \cos(\theta) = 0$$

$$P = mg \tan(\theta) \dots\dots (1) \quad [A1]$$

Resolving forces perpendicular to the plane

$$N - P \sin(\theta) - mg \cos(\theta) = 0$$

$$N = P \sin(\theta) + mg \cos(\theta) \dots\dots(2) \quad [A1]$$

Substitute (1) into (2) to eliminate P

$$N = mg \tan(\theta) \sin(\theta) + mg \cos(\theta) \text{ newtons} \quad [A1]$$

$$N = mg \left(\frac{\sin^2(\theta)}{\cos(\theta)} + \cos(\theta) \right)$$

$$N = mg \left(\frac{\sin^2(\theta) + \cos^2(\theta)}{\cos(\theta)} \right)$$

$$N = mg \left(\frac{1}{\cos(\theta)} \right) \quad [A1]$$

$$N = mg \sec(\theta) \text{ newtons, as required.}$$

Total 4 marks
 TOTAL MARKS: 40