

Year 2010

VCE

Specialist Mathematics

Solutions

Trial Examination 1



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Question 1

$$x^2 + 2xy + 3y^2 = 18$$

taking $\frac{d}{dx}$ of each term (implicit differentiation)

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(2xy) + \frac{d}{dx}(3y^2) = \frac{d}{dx}(18)$$

product rule in the second term

$$2x + 2y + 2x \frac{dy}{dx} + 6y \frac{dy}{dx} = 0 \quad \text{M1}$$

$$(2x + 6y) \frac{dy}{dx} = -2x - 2y$$

$$\frac{dy}{dx} = \frac{-(x + y)}{x + 3y} \quad \text{A1}$$

when the tangent is horizontal $\frac{dy}{dx} = 0 \Rightarrow y = -x$

$$x^2 - 2x^2 + 3x^2 = 2x^2 = 18 \Rightarrow x^2 = 9 \quad x = \pm 3 \Rightarrow y = \mp 3$$

coordinates are $(3, -3)$ and $(-3, 3)$ A1

Question 2

$$\int \frac{4 - 3x}{\sqrt{16 - 9x^2}} dx$$

$$= \int \frac{4}{\sqrt{16 - 9x^2}} dx - \int \frac{3x}{\sqrt{16 - 9x^2}} dx$$

$$\text{let } u = 3x \quad \frac{du}{dx} = 3 \quad \text{let } v = 16 - 9x^2 \quad \frac{dv}{dx} = -18x \quad \text{M1}$$

$$= \frac{4}{3} \int \frac{1}{\sqrt{16 - u^2}} du + \frac{1}{6} \int v^{-\frac{1}{2}} dv \quad \text{A1}$$

$$= \frac{4}{3} \sin^{-1}\left(\frac{u}{4}\right) + \frac{1}{3} v^{\frac{1}{2}} + c$$

$$= \frac{4}{3} \sin^{-1}\left(\frac{3x}{4}\right) + \frac{1}{3} \sqrt{16 - 9x^2} + c \quad \text{A1}$$

since an antiderivative is required, any value of c , including zero is acceptable

Question 3

i. $(a+bi)^2 = 3+4i$ where $a, b \in R$

$$a^2 + 2abi + b^2i^2 = (a^2 - b^2) + 2abi = 3 + 4i$$

equating real and imaginary parts

real (1) $a^2 - b^2 = 3$ M1

imaginary (2) $2ab = 4$ from (2) $b = \frac{2}{a}$ substitute into (1)

$$a^2 - \left(\frac{2}{a}\right)^2 = 3$$
 M1

$$a^2 - \frac{4}{a^2} - 3 = 0 \quad \text{multiply by } a^2$$

$$a^4 - 3a^2 - 4 = 0$$

$$(a^2 - 4)(a^2 + 1) = 0$$

$$a = \pm 2 \quad a = \pm i \quad \text{but } a \in R \quad \text{so } a = \pm 2 \quad \text{only } b = \pm 1$$
 A1

$$(\pm(2+i))^2 = 3+4i$$

alternative methods such as using polar and DeMoivre's Theorem are acceptable.

ii. $z^2 + iz - 1 - i = 0$

using the quadratic formulae with $a=1$ $b=i$ $c=-1-i$

$$\Delta = b^2 - 4ac = i^2 - 4(-1-i) = -1 + 4 + 4i = 3 + 4i$$
 M1

$$z = \frac{-b \pm \sqrt{\Delta}}{2a}$$

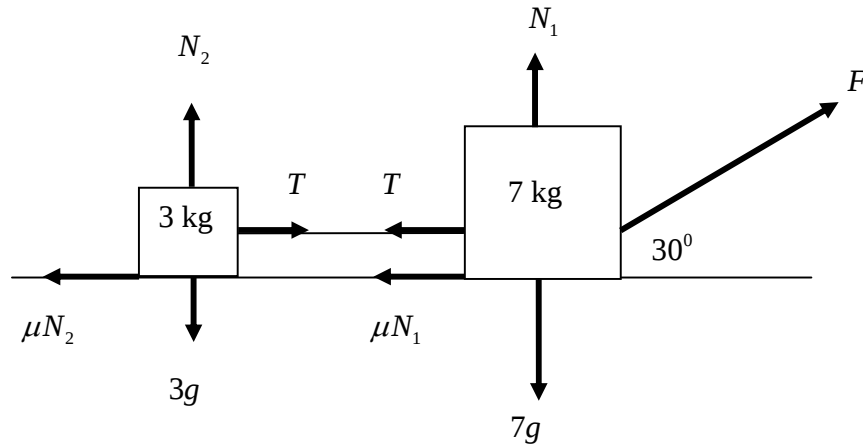
$$z = \frac{-i \pm (2+i)}{2} = \frac{-i+2+i}{2}, \frac{-i-2-i}{2}$$

$$z = 1 \quad \text{and} \quad -1-i$$
 A1

Question 4

i. all the forces correct

A1



$$F = 40 \text{ newtons} \quad \mu = \frac{\sqrt{3}}{4}$$

ii. resolving around the 7 kg box

$$\text{horizontally} \quad (1) \quad F \cos(30^\circ) - T - \mu N_1 = 7a$$

$$\text{vertically} \quad (2) \quad F \sin(30^\circ) + N_1 - 7g = 0$$

M1

$$(2) \Rightarrow N_1 = 7g - F \sin(30^\circ) \text{ substitute into (1)}$$

$$F \cos(30^\circ) - T - \mu(7g - F \sin(30^\circ)) = 7a$$

$$F(\cos(30^\circ) + \mu \sin(30^\circ)) - T - 7\mu g = 7a \quad (3)$$

resolving around the 3 kg box

$$\text{horizontally} \quad (4) \quad T - \mu N_2 = 3a$$

M1

$$\text{vertically} \quad (5) \quad N_2 - 3g = 0 \Rightarrow N_2 = 3g \text{ substitute into (4)}$$

$$(4) \quad T - 3\mu g = 3a \text{ adding this to (3) to eliminate } T$$

$$F(\cos(30^\circ) + \mu \sin(30^\circ)) - 10\mu g = 10a$$

$$40 \left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4} \times \frac{1}{2} \right) - \frac{10\sqrt{3} \times 9.8}{4} = 10a$$

$$\frac{\sqrt{3}}{4} (80 + 20 - 98) = \frac{\sqrt{3}}{2} = 10a$$

$$a = \frac{\sqrt{3}}{20} \text{ m/s}^2$$

A1

Question 5

$$A(-2+\sqrt{3}, 3), B(2\sqrt{3}-2, 0) \text{ and } C(-2, 2)$$

$$\overrightarrow{OA} = (-2+\sqrt{3})\underline{i} + 3\underline{j} \quad \overrightarrow{OB} = (2\sqrt{3}-2)\underline{i} \quad \overrightarrow{OC} = -2\underline{i} + 2\underline{j}$$

$$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC} = \sqrt{3}\underline{i} + \underline{j} \quad |\overrightarrow{CA}| = \sqrt{3+1} = 2$$

$$\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC} = 2\sqrt{3}\underline{i} - 2\underline{j} \quad |\overrightarrow{CB}| = \sqrt{12+4} = 4$$

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = 6 - 2 = 4 \quad \text{M1}$$

$$\cos(\angle ACB) = \frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{|\overrightarrow{CA}| |\overrightarrow{CB}|} = \frac{1}{2}$$

$$\angle ACB = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ \quad \left(\text{ or } \frac{\pi}{3} \right) \quad \text{A1}$$

Question 6

a. using $\tan(2A) = \frac{2 \tan(A)}{1 - \tan^2(A)}$ with $A = \frac{\pi}{8}$ and $2A = \frac{\pi}{4}$

$$\tan\left(\frac{\pi}{4}\right) = 1 = \frac{2 \tan\left(\frac{\pi}{8}\right)}{1 - \tan^2\left(\frac{\pi}{8}\right)} \Rightarrow 1 - \tan^2\left(\frac{\pi}{8}\right) = 2 \tan\left(\frac{\pi}{8}\right) \quad \text{M1}$$

$$\tan^2\left(\frac{\pi}{8}\right) + 2 \tan\left(\frac{\pi}{8}\right) = 1 \Rightarrow \tan^2\left(\frac{\pi}{8}\right) + 2 \tan\left(\frac{\pi}{8}\right) + 1 = 2$$

$$\left(\tan\left(\frac{\pi}{8}\right) + 1\right)^2 = 2 \Rightarrow \tan\left(\frac{\pi}{8}\right) + 1 = \pm\sqrt{2}$$

$$\tan\left(\frac{\pi}{8}\right) = -1 \pm \sqrt{2} \quad \text{since } \tan\left(\frac{\pi}{8}\right) > 0 \quad \text{A1}$$

$$\tan\left(\frac{\pi}{8}\right) = \sqrt{2} - 1 \quad \text{shown}$$

b. let $z = 1 + (\sqrt{2} - 1)i$ $\text{Arg}(z) = \tan^{-1}(\sqrt{2} - 1) = \frac{\pi}{8}$ A1

$$\text{now } iz = i + (\sqrt{2} - 1)i^2 = 1 - \sqrt{2} + i$$

$$\text{Arg}(iz) = \text{Arg}(i) + \text{Arg}(z) = \frac{\pi}{2} + \text{Arg}(z)$$

$$\text{Arg}(1 - \sqrt{2} + i) = \frac{\pi}{2} + \frac{\pi}{8} \quad \text{A1}$$

$$\text{Arg}(1 - \sqrt{2} + i) = \frac{5\pi}{8}$$

Question 7

a. $y = \frac{x^2 + 4}{2x} = \frac{x}{2} + \frac{2}{x}$

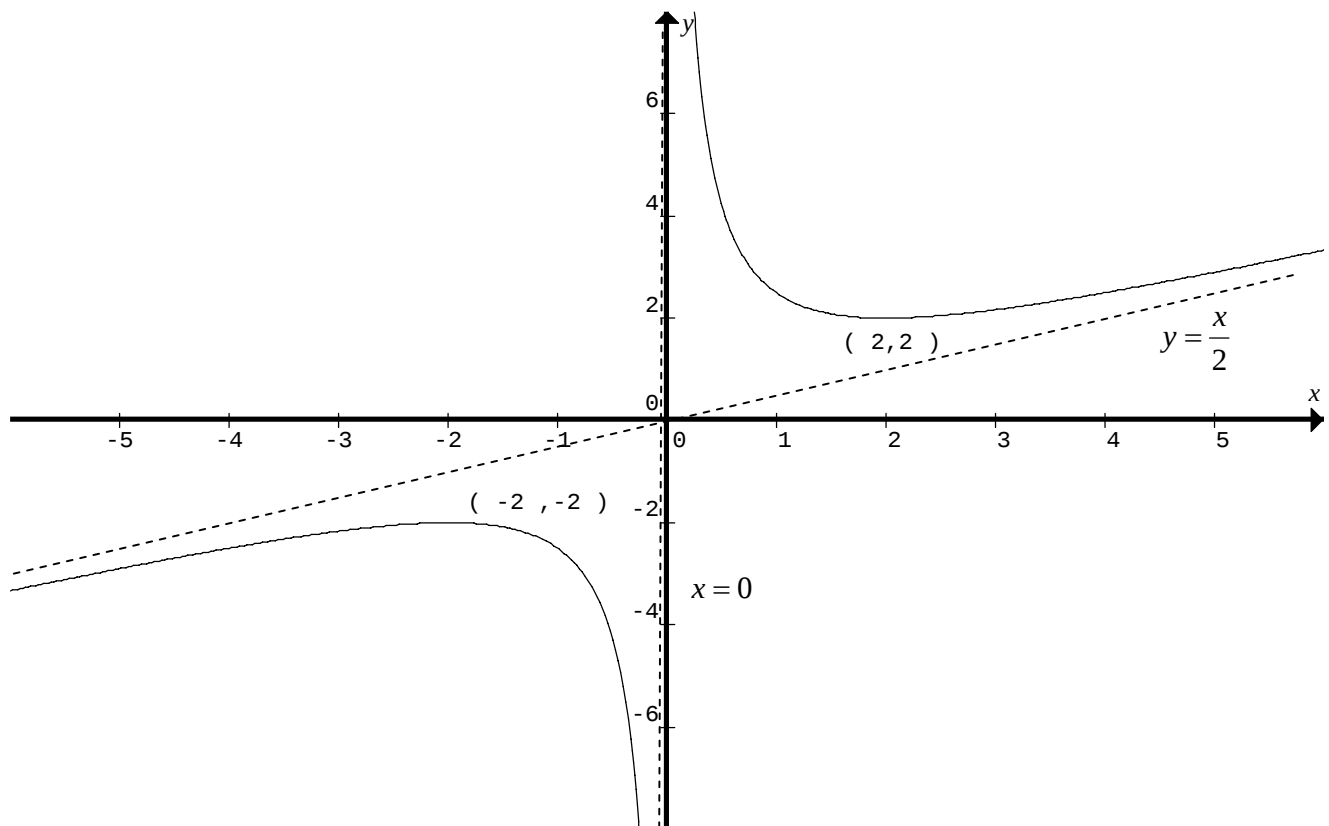
$y = \frac{x}{2}$ is an asymptote and $x = 0$, y-axis is a vertical asymptote A1

the graph does not cross the x or y-axis,

for turning points, $\frac{dy}{dx} = \frac{1}{2} - \frac{2}{x^2} = 0 \Rightarrow x^2 = 4$ so $x = \pm 2$ M1

$(2, 2)$ local min $(-2, -2)$ local max A1

correct graph, with asymptotes G1



b. $\underline{r}(t) = 2 \tan(t) \underline{i} + 2 \operatorname{cosec}(2t) \underline{j}$

$$x = 2 \tan(t)$$

$$\frac{x^2 + 4}{2x} = \frac{4 \tan^2(t) + 4}{4 \tan(t)}$$

$$\frac{x^2 + 4}{2x} = \frac{4(1 + \tan^2(t))}{4 \tan(t)} \quad \text{M1}$$

$$\frac{x^2 + 4}{2x} = \frac{\sec^2(t)}{\tan(t)} = \frac{1}{\cos^2(t)} \times \frac{\cos(t)}{\sin(t)}$$

$$\frac{x^2 + 4}{2x} = \frac{1}{\sin(t) \cos(t)} = \frac{2}{2 \sin(t) \cos(t)} = \frac{2}{\sin(2t)} \quad \text{A1}$$

$$y = 2 \operatorname{cosec}(2t) \quad \text{shown}$$

Question 8

i. $\ddot{x} = 18x^3 + 18x^2 + 4x$ use $\ddot{x} = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 18x^3 + 18x^2 + 4x$$

$$\frac{1}{2}v^2 = \int (18x^3 + 18x^2 + 4x) dx \quad \text{M1}$$

$$\frac{1}{2}v^2 = \frac{9}{2}x^4 + 6x^3 + 2x^2 + C_1$$

now when $x = 1$ $v = -5$

$$\frac{25}{2} = \frac{9}{2} + 6 + 2 + C_1 \Rightarrow C_1 = 0$$

$$v^2 = 9x^4 + 12x^3 + 4x^2$$

$$v^2 = x^2(9x^2 + 12x + 4) \quad \text{A1}$$

$$v^2 = x^2(3x + 2)^2$$

- ii. $v = \pm x(3x+2)$ since when $t=0$, $x=1$, $v=-5$ take the negative
since $t > 0$ and $v < 0$

$$v = \frac{dx}{dt} = -x(3x+2)$$

$$\frac{dt}{dx} = \frac{-1}{x(3x+2)}$$

by partial fractions

$$\frac{A}{x} + \frac{B}{3x+2} = \frac{A(3x+2) + Bx}{x(3x+2)} = \frac{x(B+3A) + 2A}{x(3x+2)} \quad \text{M1}$$

$$B+3A=0 \quad \text{and} \quad 2A=-1 \Rightarrow A=-\frac{1}{2} \quad B=-3A=\frac{3}{2}$$

$$t = \frac{1}{2} \int \left(\frac{3}{3x+2} - \frac{1}{x} \right) dx$$

$$2t = \log_e(3x+2) - \log_e(x) + C_2 \quad \text{A1}$$

now when $t=0$ $x=1$

$$0 = \log_e(5) + C_2 \Rightarrow C_2 = -\log_e(5)$$

$$2t = \log_e \left(\frac{3x+2}{5x} \right)$$

$$\frac{3x+2}{5x} = e^{2t} \quad \text{M1}$$

$$\frac{3x+2}{x} = 5e^{2t}$$

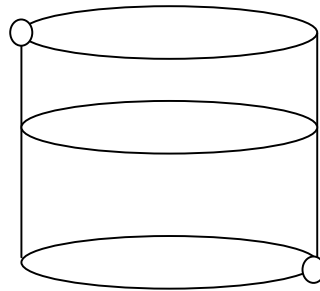
$$3 + \frac{2}{x} = 5e^{2t}$$

$$\frac{2}{x} = 5e^{2t} - 3$$

$$x = x(t) = \frac{2}{5e^{2t} - 3} \quad \text{A1}$$

Question 9**i.**

inflow 0.8
kg/litre
at 5 litre/min



outflow at 3 litre/min

Now $\frac{dQ}{dt} = \text{inflow} - \text{outflow}$

and the volume $V(t)$ of the tank at a time t , $V(t) = 100 + (5 - 3)t = 100 + 2t$

$$\frac{dQ}{dt} = 5 \times 0.8 - \frac{3Q}{100 + 2t} = 4 - \frac{3Q}{100 + 2t} \quad \text{A1}$$

$$\text{ii.} \quad Q = \frac{4}{5}(100 + 2t) + C(100 + 2t)^n$$

$$\text{differentiating LHS } \frac{dQ}{dt} = \frac{8}{5} + 2nC(100 + 2t)^{n-1} \quad \text{M1}$$

$$\begin{aligned} \text{RHS } 4 - \frac{3Q}{100 + 2t} &= 4 - \frac{3}{100 + 2t} \left(\frac{4}{5}(100 + 2t) + C(100 + 2t)^n \right) \\ &= 4 - \frac{12}{5} - 3C(100 + 2t)^{n-1} \\ &= \frac{8}{5} - 3C(100 + 2t)^{n-1} \end{aligned}$$

$$\text{therefore } n = -\frac{3}{2} \quad \text{A1}$$

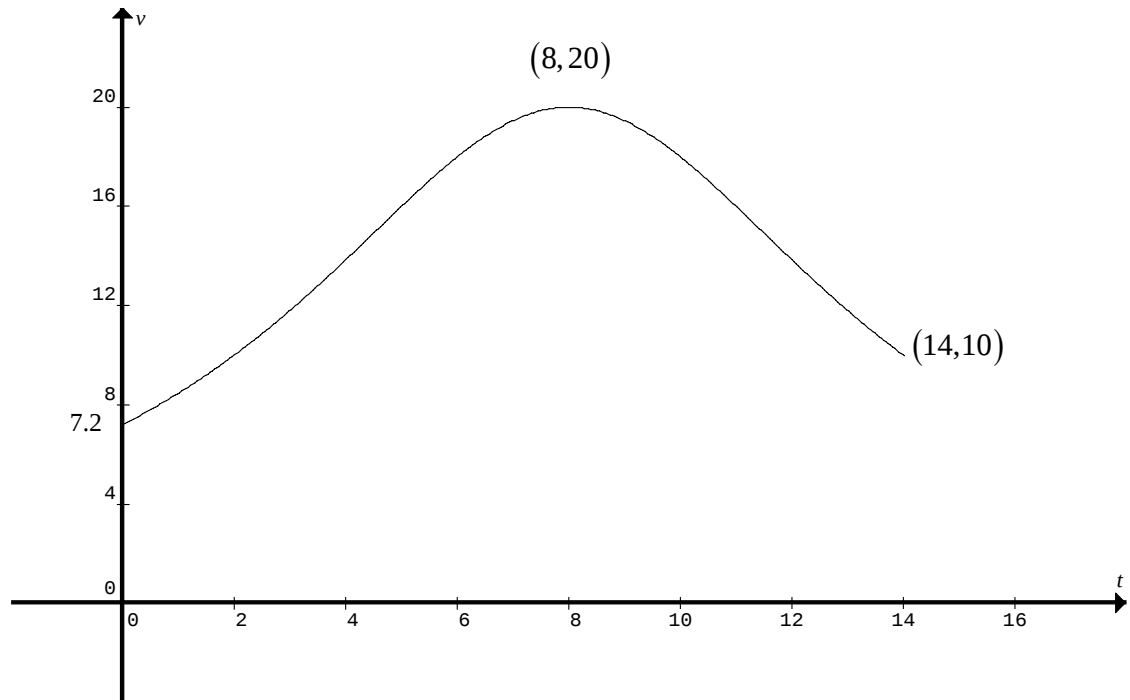
Question 10

$$\text{i.} \quad v(t) = \frac{720}{t^2 - 16t + 100} = \frac{720}{t^2 - 16t + 64 + 36} = \frac{720}{(t-8)^2 + 36}$$

$$v(0) = 7.2 \quad v(8) = 20 \quad v(14) = 10$$

The maximum velocity occurs when $t = 8$ and is 20 m/s,
graph over $t \in [0, 14]$

A1



$$\text{ii.} \quad v = \frac{dx}{dt} = \frac{720}{(t-8)^2 + 36}$$

$$x = \int_8^{14} \left(\frac{720}{(t-8)^2 + 36} \right) dt$$

A1

$$x = \left[\frac{720}{6} \tan^{-1} \left(\frac{t-8}{6} \right) \right]_8^{14}$$

A1

$$x = \frac{720}{6} \left(\tan^{-1}(1) - \tan^{-1}(0) \right) = 120 \left(\frac{\pi}{4} - 0 \right)$$

$$x = 30\pi \text{ m}$$

A1

END OF SUGGESTED SOLUTIONS