

2010 VCAA Specialist Mathematics Exam 1 Solutions
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Q1 $f(z) = z^3 + 9z^2 + 28z + 20$ and $f(-1) = 0$, $\therefore z+1$ is a factor of $f(z)$.

$$f(z) = (z+1)(z^2 + pz + 20).$$

Expand and compare coefficients, $p+1=9$, $\therefore p=8$.

$$\therefore f(z) = (z+1)(z^2 + 8z + 20) = (z+1)((z+4)^2 + 2^2) \\ = (z+1)(z+4-2i)(z+4+2i)$$

Q2a $a = \frac{R}{m}$, $\frac{dv}{dt} = \frac{v-4}{2}$ and $v=0$ at $t=0$.

Note: The body starts from rest, $\therefore$ the acceleration is in the negative direction and the body moves in the negative direction, i.e. its velocity is negative for $t > 0$.

Q2b $\frac{dv}{dt} = \frac{v-4}{2}$, $\frac{dt}{dv} = \frac{2}{v-4}$, $t = \int \frac{2}{v-4} dv = 2 \log_e |v-4| + c$.
 $v=0$ at $t=0$, $\therefore c = -2 \log_e 4$ and

$$t = 2 \log_e |v-4| - 2 \log_e 4 = 2 \log_e \left| \frac{v-4}{4} \right|, \therefore \left| \frac{v-4}{4} \right| = e^{\frac{t}{2}}$$

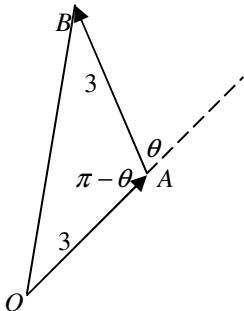
Since the velocity is negative for $t > 0$, $\frac{v-4}{4} = -e^{\frac{t}{2}}$.

Hence $v = 4(1 - e^{\frac{t}{2}})$.

Q3a $\vec{AB} = \vec{OB} - \vec{OA} = (-\tilde{i} + 3\tilde{j} + 4\tilde{k}) - (\tilde{i} + 2\tilde{j} + 2\tilde{k}) \\ = -2\tilde{i} + \tilde{j} + 2\tilde{k}$

Q3b $\vec{OA} \cdot \vec{AB} = |\vec{OA}| |\vec{AB}| \cos \theta$, $\therefore 4 = 9 \cos \theta$, $\cos \theta = \frac{4}{9}$.

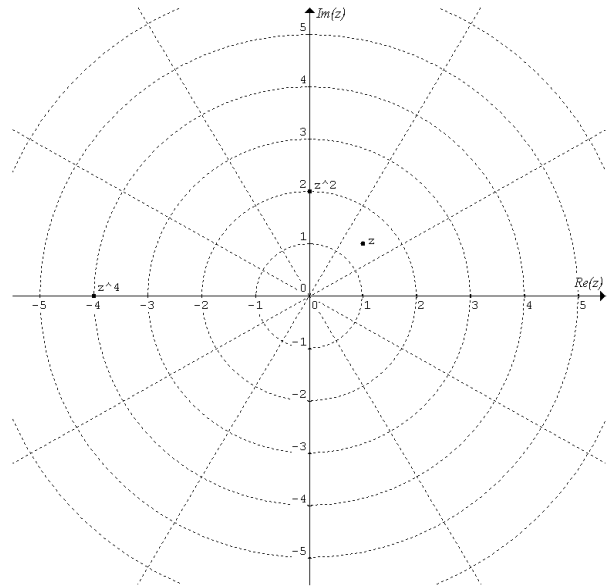
Q3c



$$\cos \theta = \frac{4}{9}, \sin \theta = \sqrt{1 - \cos^2 \theta} = \frac{\sqrt{65}}{9}$$

$$\text{Area of } \triangle OAB = \frac{1}{2} \times 3 \times 3 \times \sin(\pi - \theta) = \frac{9}{2} \sin \theta = \frac{\sqrt{65}}{2} \text{ unit}^2$$

Q4



Q5 $f(x) = \tan^{-1}(2x)$, $f'(x) = \frac{2}{1+4x^2}$, $f''(x) = -\frac{16x}{(1+4x^2)^2}$,

$$f''\left(\frac{\pi}{2}\right) = -\frac{16 \times \frac{\pi}{2}}{\left(1+4\left(\frac{\pi}{2}\right)^2\right)^2} = -\frac{8\pi}{(1+\pi^2)^2}$$

Q6 Let $u = \cos(2x)$, $\frac{du}{dx} = -2\sin(2x)$, $\therefore \sin(2x) = -\frac{1}{2} \frac{du}{dx}$.

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \cos^2(2x) \sin(2x) dx = \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} -\frac{1}{2} u^2 \frac{du}{dx} dx = \int_{-1}^0 -\frac{1}{2} u^2 du \\ = \left[-\frac{u^3}{6} \right]_{-1}^0 = -\frac{1}{6}$$

Q7 $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{2}{1-x^2} \right) = \frac{4x}{(1-x^2)^2}$, $-1 < x < 1$.

$$\therefore \frac{dy}{dx} = \frac{2}{1-x^2} + c.$$

Given $\frac{dy}{dx} = 3$ when $x=0$, $\therefore c=1$ and $\frac{dy}{dx} = \frac{2}{1-x^2} + 1$.

$$y = \int \left(\frac{2}{1-x^2} + 1 \right) dx = \int \left(\frac{1}{1+x} + \frac{1}{1-x} + 1 \right) dx \quad (\text{partial fractions}) \\ = \log_e(1+x) - \log_e(1-x) + x + C \quad (-1 < x < 1) \\ = \log_e \left(\frac{1+x}{1-x} \right) + x + C$$

Given $y=4$ when $x=0$, $\therefore C=4$ and $y = \log_e \left(\frac{1+x}{1-x} \right) + x + 4$.

Q8a $y = -t \cos(t) = 0, \therefore t = 0$

or $\cos(t) = 0$, i.e. $t = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$

Second time at $t = \frac{3\pi}{2}$.

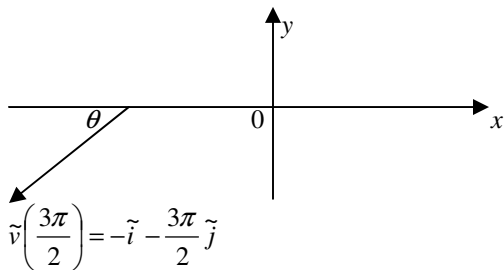
Q8b $\tilde{r}(t) = t \sin(t) \tilde{i} - t \cos(t) \tilde{j}, t \geq 0$.

$$\tilde{v}(t) = \frac{d\tilde{r}}{dt} = (\sin(t) + t \cos(t)) \tilde{i} - (\cos(t) - t \sin(t)) \tilde{j}$$

$$\therefore \tilde{v}\left(\frac{3\pi}{2}\right) = -\tilde{i} - \frac{3\pi}{2} \tilde{j}$$

$$\text{Speed} = |\tilde{v}| = \sqrt{(-1)^2 + \left(-\frac{3\pi}{2}\right)^2} = \frac{\sqrt{4+9\pi^2}}{2}$$

Q8c



$$\tan \theta = \frac{3\pi}{2}$$

Q10 x -intercepts: $(x^2 - 1)\sqrt{x+1} = 0, \therefore x^2 - 1 = 0, x = \pm 1$.

Let $u = x + 1, \therefore x = u - 1$

$$\therefore x^2 - 1 = (u - 1)^2 - 1 = u^2 - 2u \text{ and } \frac{du}{dx} = 1.$$

$$\text{Area} = -\int_{-1}^1 (x^2 - 1)\sqrt{x+1} dx = -\int_{-1}^1 (u^2 - 2u)\sqrt{u} \frac{du}{dx} dx$$

$$= -\int_0^2 \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}}\right) du$$

$$= -\left[\frac{2u^{\frac{7}{2}}}{7} - \frac{4u^{\frac{5}{2}}}{5}\right]_0^2$$

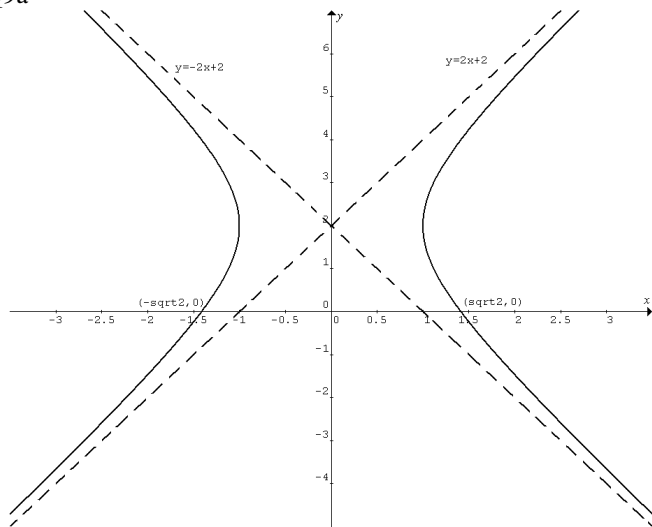
$$= -\frac{2^{\frac{9}{2}}}{7} + \frac{2^{\frac{9}{2}}}{5}$$

$$= 2^{\frac{9}{2}} \left(-\frac{1}{7} + \frac{1}{5}\right)$$

$$= 2^{\frac{9}{2}} \left(\frac{2}{35}\right) = \frac{2^{\frac{11}{2}}}{35} = \frac{2^5 \sqrt{2}}{35} = \frac{32\sqrt{2}}{35}$$

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors

Q9a



Q9b $x^2 - \frac{(y-2)^2}{4} = 1$

When $x = 2, y = 2 - 2\sqrt{3} \ (y < 0)$

Implicit differentiation: $2x - \frac{1}{2}(y-2)\frac{dy}{dx} = 0, \therefore \frac{dy}{dx} = \frac{4x}{y-2}$.

At $(2, 2 - 2\sqrt{3}), \frac{dy}{dx} = \frac{4 \times 2}{(2 - 2\sqrt{3}) - 2} = -\frac{4\sqrt{3}}{3}$.