

2010 Specialist Maths Trial Exam 2 Solutions

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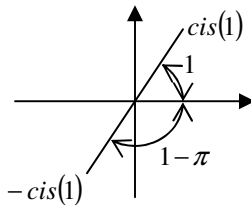
Section 1

1	2	3	4	5	6	7	8	9	10	11
C	C	C	B	B	A	A	E	C	C	C
12	13	14	15	16	17	18	19	20	21	22
D	E	E	B	A	E	C	A	D	D	B

$$Q1 \quad z^2 - z + 1 = 0, \quad z = \frac{-1 \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3}i}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$Q2 \quad z = -cis(1)$$



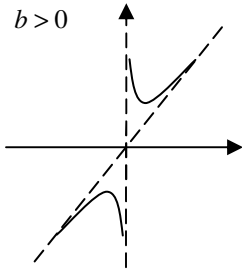
$$|z| = 1, \quad \text{Arg}(z) = 1 - \pi$$

Q3 The polynomial has three linear factors, not solutions or roots.

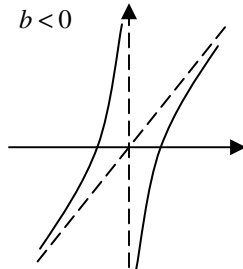
Q4 Any complex number on the straight line is equidistant from $z = 0$ and $z = a + ai$, i.e. $|z - a - ai| = |z|$.

$$\therefore |z - a - ai| - |z| = 0$$

Q5 $b > 0$



$b < 0$



Q6 $4x^2 + px + q^2$ cannot be a perfect square for

$$y = \frac{2}{4x^2 + px + q^2} \text{ to have a stationary point, i.e.}$$

$$\Delta = p^2 - 16q^2 \neq 0. \therefore p^2 \neq 16q^2, \quad p \neq \pm 4q.$$

Only choice A ($p < -4q$) satisfies this requirement.

$$Q7 \quad \sin\left(a + \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = \sin\left(a + \frac{5\pi}{6}\right)$$

$$= \sin a \cos\left(\frac{5\pi}{6}\right) + \cos a \sin\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} \sin a + \frac{1}{2} \cos a$$

$$Q8 \quad f(x) = \sin^{-1}\left(\frac{x}{a} + b\right) + c, \text{ where } a, b, c \in (-\infty, 0), \text{ i.e. } a < 0, \\ b < 0 \text{ and } c < 0.$$

$$-1 \leq \frac{x}{a} + b \leq 1, \quad -1 - b \leq \frac{x}{a} \leq 1 - b, \quad a(-1 - b) \geq x \geq a(1 - b),$$

$$\text{i.e. } a(1 - b) \leq x \leq -a(1 + b)$$

$$Q9 \quad \tan^{-1} b = 0.2, \therefore b = \tan 0.2 > 0$$

$$b = \tan 0.2 = \tan 2(0.1) = \frac{2 \tan 0.1}{1 - \tan^2 0.1}$$

$$\therefore b(1 - \tan^2 0.1) = 2 \tan 0.1$$

$$\text{Simplify to: } b \tan^2 0.1 + 2 \tan 0.1 - b = 0$$

$$\therefore \tan 0.1 = \frac{\sqrt{1+b^2}-1}{b} \text{ but not } \frac{-\sqrt{1+b^2}-1}{b} \text{ because}$$

$$\tan 0.1 > 0.$$

Q10 The range of $\cos^{-1} x$ is $[0, \pi]$.

$$\cos^{-1}(\cos \theta) = \cos^{-1}(\cos(-\theta)) = \cos^{-1}(\cos(2\pi - \theta)) = 2\pi - \theta$$

Q11 $3\tilde{i} - 4\tilde{j}$ cannot be expressed as a linear combination of

$$-2\tilde{i} + 3\tilde{k} \text{ and } \tilde{j} - 2\tilde{k}$$

$$Q12 \quad \overrightarrow{AO} \cdot \overrightarrow{BO} = \frac{1}{2}, \therefore \cos \angle AOB = \frac{1}{2}, \quad \angle AOB = \frac{\pi}{3},$$

$$\angle APB = \frac{1}{2} \angle AOB = \frac{\pi}{6}, \therefore \cos \angle APB = \frac{\sqrt{3}}{2}.$$

$$\overrightarrow{AP} \cdot \overrightarrow{BP} = \sqrt{3}, \quad |\overrightarrow{AP}| |\overrightarrow{BP}| \cos \angle APB = \sqrt{3}, \therefore |\overrightarrow{AP}| |\overrightarrow{BP}| = 2$$

$$Q13 \quad \overrightarrow{OQ} = \frac{3\overrightarrow{OP} + 2\overrightarrow{OR}}{2+3} = \frac{(-3, 0, 6) + (4, 2, -4)}{5} = \frac{(1, 2, 2)}{5}$$

$$|\overrightarrow{OQ}| = \frac{1}{5} \sqrt{1^2 + 2^2 + 2^2} = \frac{3}{5}$$

$$Q14 \quad \left| \frac{\sqrt{3}}{2} \tilde{i} - \frac{1}{\sqrt{2}} \tilde{j} - \frac{1}{2} \tilde{k} \right| = \sqrt{\frac{3}{4} + \frac{1}{2} + \frac{1}{4}} = \frac{\sqrt{3}}{\sqrt{2}}$$

$$\text{Angle with the } x\text{-axis} = \cos^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{\sqrt{2}}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^\circ$$

$$\text{Angle with the } y\text{-axis} = \cos^{-1}\left(\frac{-\frac{1}{\sqrt{2}}}{\frac{\sqrt{3}}{\sqrt{2}}}\right) = \cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$

$$\text{Angle with the } z\text{-axis} = \cos^{-1}\left(\frac{-\frac{1}{2}}{\frac{\sqrt{3}}{\sqrt{2}}}\right) = \cos^{-1}\left(-\frac{1}{\sqrt{6}}\right)$$

Q15 $\tilde{v} = \cos^{-1}(t)(2\tilde{i} - 3\tilde{j} + \tilde{k})$. $2\tilde{i} - 3\tilde{j} + \tilde{k}$ is a constant vector, $\therefore$ the particle moves in the same direction, i.e. in a straight line.

$$Q16 \quad \frac{dy}{dh} = \sqrt{y(2-y)} \frac{dx}{dh}, \quad \frac{dy}{dx} = \sqrt{y(2-y)}, \quad \frac{dx}{dy} = \frac{1}{\sqrt{y(2-y)}}$$

$$\therefore \frac{dx}{dy} = \frac{1}{\sqrt{1-(y-1)^2}}, \quad x = \int \frac{1}{\sqrt{1-(y-1)^2}} dy = \sin^{-1}(y-1) + c$$

$$\therefore y = \sin(x-c) + 1$$

A

$$Q17 \quad \text{At } t = 8, \quad s = -2, \quad x = 2 + (-2) = 0$$

E

Q18 The particle starts from rest and travels in the negative direction with increasing speed, then slows down to a stop at $t = 7$. During $7 < t \leq 9$ the particle travels in the positive direction with increasing speed. In the first 7 s the particle has the greatest displacement (area between the curve and the t -axis) from its initial position.

C

$$Q19 \quad \int_0^{\frac{\pi}{2}} \cos^3 x \sqrt{1 - \sin x} dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^2 x \sqrt{1 - \sin x} \cos x dx$$

$$= \int_0^{\frac{\pi}{2}} (1 - \sin^2 x) \sqrt{1 - \sin x} \cos x dx$$

$$= \int_1^0 -(1 - (1-u)^2) \sqrt{u} du$$

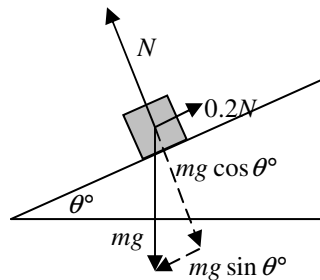
$$= \int_0^1 (2u - u^2) \sqrt{u} du$$

$$= \int_0^1 \left(2u^{\frac{3}{2}} - u^{\frac{5}{2}} \right) du$$

Let $u = 1 - \sin x$,
 $\sin x = 1 - u$,
 $\frac{du}{dx} = -\cos x$
 When $x = 0$, $u = 1$;
 when $x = \frac{\pi}{2}$, $u = 0$.

A

Q20



$$N - mg \cos \theta = 0$$

$$mg \sin \theta - 0.2N = 0$$

$$\therefore \sin \theta = 0.2 \cos \theta, \quad \tan \theta = 0.2, \quad \theta = \tan^{-1}(0.2) \approx 11^\circ$$

D

$$Q21 \quad v = -\sqrt{100 - x}$$

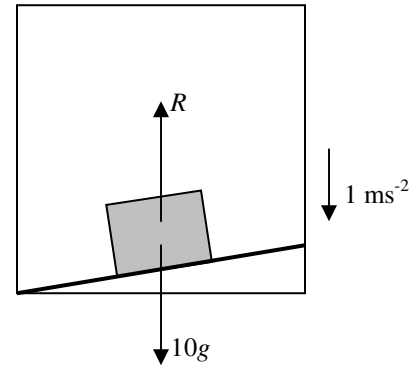
The negative sign indicates that the particle moves in the negative direction.

$$v^2 = 100 - x, \quad a = \frac{d}{dx} \left(\frac{1}{2} v^2 \right) = -\frac{1}{2}$$

The acceleration is constant and it is in the negative direction, i.e. in the same direction as the velocity. $\therefore$ the particle speeds up and maintains the same direction of motion.

D

Q22



$$10g - R = 10 \times 1, \quad 98 - R = 10$$

$$\therefore R = 88 \text{ N}$$

B

Section 2

$$Q1a \quad \tan 75^\circ = \tan(45^\circ + 30^\circ) = \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}$$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = 2 + \sqrt{3}$$

$$\therefore y = \pm(2 + \sqrt{3})x$$

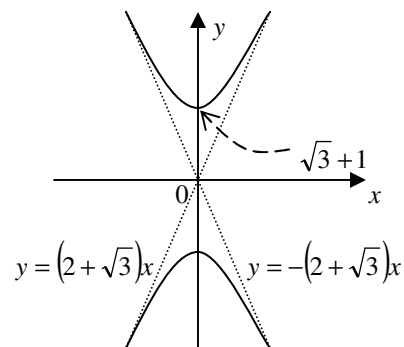
$$Q1b \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

$$\text{When } x = 0, \quad y = \pm b = \pm(\sqrt{3} + 1). \quad \therefore b = \sqrt{3} + 1$$

$$\text{Gradient of asymptote} = \frac{b}{a} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1}. \quad \therefore a = \sqrt{3} - 1$$

$$\therefore \frac{x^2}{(\sqrt{3} - 1)^2} - \frac{y^2}{(\sqrt{3} + 1)^2} = -1$$

Q1c



$$Q1di \quad \text{On the positive-gradient asymptote: } x = a, \quad y = a \tan 75^\circ = (2 + \sqrt{3})a \quad \therefore \overrightarrow{OA} = a\tilde{i} + (2 + \sqrt{3})a\tilde{j}$$

$$\text{On the negative-gradient asymptote: } x = b, \quad y = -b \tan 75^\circ = -(2 + \sqrt{3})b \quad \therefore \overrightarrow{OB} = b\tilde{i} - (2 + \sqrt{3})b\tilde{j}$$

$$Q1dii \quad \overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}) = \left(\frac{a+b}{2} \right)\tilde{i} + (2 + \sqrt{3})\left(\frac{a-b}{2} \right)\tilde{j}$$

$$\text{Q1diii } \overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = (a-b)\tilde{i} + (2+\sqrt{3})(a+b)\tilde{j}$$

$$\therefore |\overrightarrow{BA}|^2 = (a-b)^2 + (2+\sqrt{3})^2(a+b)^2 = 4$$

$$\text{Q1div } \overrightarrow{OM} = \left(\frac{a+b}{2}\right)\tilde{i} + (2+\sqrt{3})\left(\frac{a-b}{2}\right)\tilde{j}$$

$$\text{Coordinates of point } M \text{ are } x = \frac{a+b}{2} \text{ and } y = (2+\sqrt{3})\left(\frac{a-b}{2}\right).$$

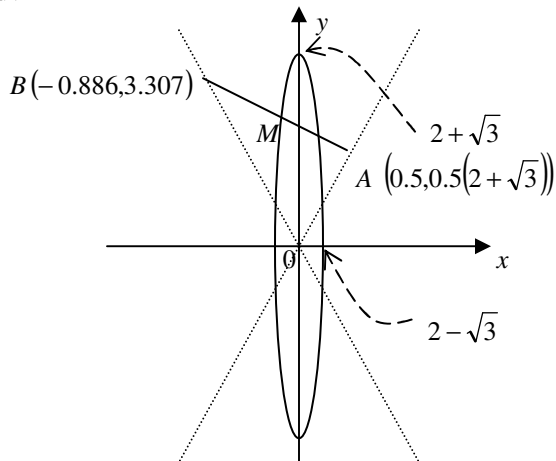
$$\therefore a+b = 2x \text{ and } a-b = \frac{2y}{2+\sqrt{3}}$$

$$\text{Substitute into } (a-b)^2 + (2+\sqrt{3})^2(a+b)^2 = 4$$

$$\left(\frac{2y}{2+\sqrt{3}}\right)^2 + (2+\sqrt{3})^2(2x)^2 = 4$$

$$\text{Simplify: } \frac{x^2}{(2-\sqrt{3})^2} + \frac{y^2}{(2+\sqrt{3})^2} = 1, \text{ an ellipse.}$$

Q1dv



$$\text{Q1dvi } (a-b)^2 + (2+\sqrt{3})^2(a+b)^2 = 4$$

When $a = 0.5$, $b = -0.886$ or 0.020 by calculator.

$$A: \text{ When } x = a = 0.5, y = (2+\sqrt{3})0.5$$

$$B: \text{ When } x = b \approx -0.886, y = -(2+\sqrt{3})(-0.886) \approx 3.307$$

$$\text{Q2ai } f(z) = z^3 - 6iz^2 - 12z + 7i$$

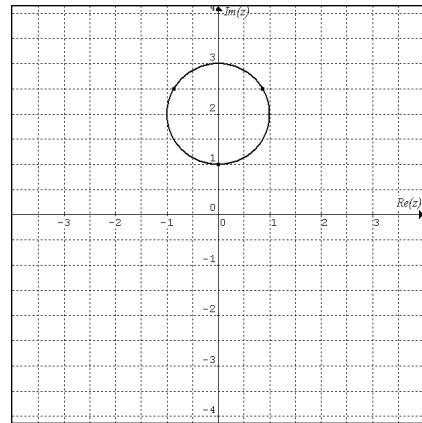
$$= (z-a)^3 + b = z^3 - 3az^2 + 3a^2z - a^3 + b$$

$$\therefore 3a = 6i \text{ and } -a^3 + b = 7i$$

$$\therefore a = 2i \text{ and } b = -i$$

$$\text{Hence } f(z) = (z-2i)^3 - i$$

Q2aii, iii



$$\text{Q2aiv } f(z) = 0, (z-2i)^3 - i = 0, (z-2i)^3 = i$$

$$\therefore (z-2i)^3 = \text{cis}\left(\frac{\pi}{2} + 2n\pi\right), z-2i = \text{cis}\left(\frac{\pi}{6} + \frac{2n\pi}{3}\right)$$

$$n=0, z-2i = \frac{\sqrt{3}}{2} + \frac{1}{2}i, \therefore z = \frac{\sqrt{3}}{2} + \frac{5}{2}i$$

$$n=-1, z-2i = -i, \therefore z = i$$

$$n=1, z-2i = -\frac{\sqrt{3}}{2} + \frac{1}{2}i, \therefore z = -\frac{\sqrt{3}}{2} + \frac{5}{2}i$$

$$\text{Q2b } -iz^3 - 6z^2 + 12iz + 7 = 0$$

$$-iz^3 - 6z^2 + 12iz + 7 = 0$$

$$-i(z^3 - 6iz^2 - 12z + 7i) = 0$$

$$\therefore z^3 - 6iz^2 - 12z + 7i = 0$$

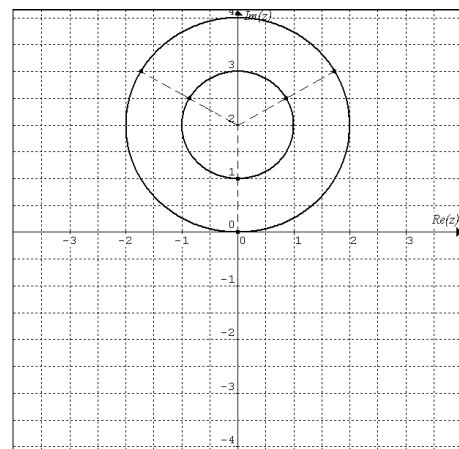
$$\therefore z = \pm \frac{\sqrt{3}}{2} + \frac{5}{2}i \text{ or } z = i$$

$$\text{Q2c } g(z) = f(z) - 7i = (z-2i)^3 - 8i = 0$$

$$\therefore (z-2i)^3 = 8i = 8\text{cis}\left(\frac{\pi}{2} + 2n\pi\right)$$

$$\therefore z-2i = 2\text{cis}\left(\frac{\pi}{6} + \frac{2n\pi}{3}\right)$$

$\therefore$ the roots of $g(z) = 0$ are on a circle of centre $(0,2)$ and radius of 2 units. For $f(z) = 0$ the roots are on a circle of the same centre $(0,2)$ and radius of 1 unit.

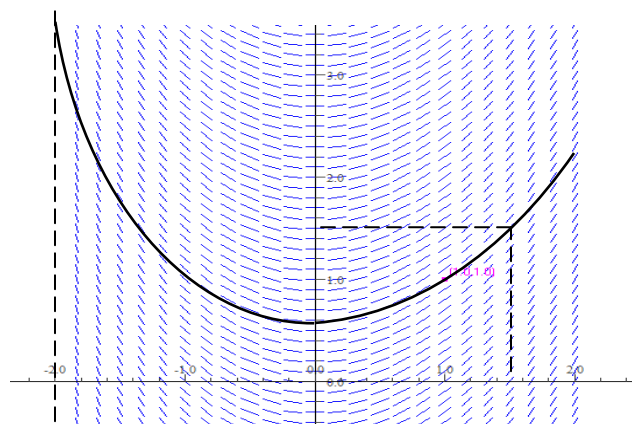


Q3a $-1 \leq \frac{x}{2} \leq 1$ and $\frac{x+2}{2} > 0$

$\therefore -2 \leq x \leq 2$ and $x > -2$

$\therefore x \in (-2, 2]$

Q3bi



Q3bii When $x = 1.5$, $y \approx 1.5$

Q3c $\frac{dy}{dx} = \sin^{-1}\left(\frac{x}{2}\right) + \log_e\left(\frac{x+2}{2}\right)$

$x = 1, \quad y = 1, \quad \frac{dy}{dx} = 0.92906$

$x = 1.25, \quad y = 1 + 0.25 \times 0.92906 = 1.23227, \quad \frac{dy}{dx} = 1.16064$

$x = 1.5, \quad y = 1.23227 + 0.25 \times 1.16064 \approx 1.52$

Q3di $\int_1^{1.5} \left(\sin^{-1}\left(\frac{x}{2}\right) + \log_e\left(\frac{x+2}{2}\right) \right) dx$

Q3dii By calculator $\int_1^{1.5} \left(\sin^{-1}\left(\frac{x}{2}\right) + \log_e\left(\frac{x+2}{2}\right) \right) dx \approx 0.58$

$\therefore y \approx 1 + 0.58 = 1.58$

Q4a $f(x) = \frac{x}{\sqrt{p^2 - x^2}} + q$

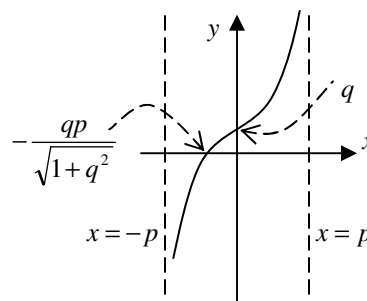
Asymptotes: $p^2 - x^2 = 0, \quad x = \pm p$

y-intercept: $x = 0, \quad y = q$

x-intercept: $y = 0, \quad \frac{x}{\sqrt{p^2 - x^2}} + q = 0, \quad x = -q\sqrt{p^2 - x^2}$

$x^2 = q^2(p^2 - x^2), \quad x^2 = q^2 p^2 - q^2 x^2, \quad x^2(1 + q^2) = q^2 p^2$

$x = \pm \frac{qp}{\sqrt{1 + q^2}}$



Q4b $p = \sqrt{3}, \quad q = \sqrt{3}, \quad \therefore x = -\frac{qp}{\sqrt{1+q^2}} = -\frac{3}{2}$

Area = $\int_{-\frac{3}{2}}^{\frac{3}{2}} \left(\frac{x}{\sqrt{3-x^2}} + \sqrt{3} \right) dx = \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{x}{\sqrt{3-x^2}} dx + \int_{-\frac{3}{2}}^{\frac{3}{2}} \sqrt{3} dx$

$= \int_{-\frac{3}{2}}^{\frac{3}{2}} \left(-\frac{1}{2\sqrt{u}} \right) du + \left[\sqrt{3}x \right]_{-\frac{3}{2}}^{\frac{3}{2}} \quad (\text{where } u = 3 - x^2)$

$= 0 + 3\sqrt{3} = 3\sqrt{3}$

Q4ci $V = \int_{-\frac{3}{2}}^{\frac{3}{2}} \pi \left(\frac{x}{\sqrt{3-x^2}} + \sqrt{3} \right)^2 dx$

Q4cii By calculator $V \approx 33.2$

Q4d $V = \int_{-\frac{3}{2}}^h \pi \left(\frac{x}{\sqrt{3-x^2}} + \sqrt{3} \right)^2 dx$

$= \pi \int_{-\frac{3}{2}}^h \left(\frac{x^2}{3-x^2} + \frac{2\sqrt{3}x}{\sqrt{3-x^2}} + 3 \right) dx$

$= \pi \int_{-\frac{3}{2}}^h \left(\frac{3}{3-x^2} - 1 + \frac{2\sqrt{3}x}{\sqrt{3-x^2}} + 3 \right) dx$

$= \pi \int_{-\frac{3}{2}}^h \left(\frac{3}{(\sqrt{3})^2 - x^2} + 2 + \frac{2\sqrt{3}x}{\sqrt{3-x^2}} \right) dx$

$= \pi \int_{-\frac{3}{2}}^h \left(\frac{\sqrt{3}}{2} \left(\frac{1}{\sqrt{3}+x} + \frac{1}{\sqrt{3}-x} \right) + 2 + \frac{2\sqrt{3}x}{\sqrt{3-x^2}} \right) dx$

$= \frac{\pi\sqrt{3}}{2} \int_{-\frac{3}{2}}^h \left(\frac{1}{\sqrt{3}+x} + \frac{1}{\sqrt{3}-x} \right) dx + 2\pi \int_{-\frac{3}{2}}^h dx + 2\sqrt{3}\pi \int_{-\frac{3}{2}}^h \frac{x}{\sqrt{3-x^2}} dx$

$= \frac{\pi\sqrt{3}}{2} \int_{-\frac{3}{2}}^h \left(\frac{1}{\sqrt{3}+x} + \frac{1}{\sqrt{3}-x} \right) dx + 2\pi \int_{-\frac{3}{2}}^h dx - \pi\sqrt{3} \int_{\frac{3}{4}}^{3-h^2} \frac{1}{\sqrt{u}} du$

$= \frac{\pi\sqrt{3}}{2} \left[\log_e \frac{\sqrt{3}+x}{\sqrt{3}-x} \right]_{-\frac{3}{2}}^h + \pi[2x]_{-\frac{3}{2}}^h - \pi\sqrt{3} \left[2\sqrt{u} \right]_{\frac{3}{4}}^{3-h^2}$

$= \pi \left(\frac{\sqrt{3}}{2} \log_e \frac{\left(\sqrt{3} + \frac{3}{2} \right) \left(\sqrt{3} + h \right)}{\left(\sqrt{3} - \frac{3}{2} \right) \left(\sqrt{3} - h \right)} + 2h - 2\sqrt{3}\sqrt{3-h^2} + 6 \right)$

$$\begin{aligned} \text{Q4e } \frac{dV}{dt} &= \frac{dV}{dh} \times \frac{dh}{dt} \\ 0.5 &= \pi \left(\frac{h}{\sqrt{3-h^2}} + \sqrt{3} \right)^2 \frac{dh}{dt} \\ \therefore \frac{dh}{dt} &= \frac{1}{2\pi \left(\frac{h}{\sqrt{3-h^2}} + \sqrt{3} \right)^2} \end{aligned}$$

When $h=0$, $\frac{dh}{dt} = \frac{1}{6\pi}$ cm per second.

$$\text{Q5a } v = \frac{1}{2} \tan^{-1}(5t + 2.5) + \frac{\pi}{2}, \quad a = \frac{dv}{dt} = \frac{5}{2(1 + (5t + 2.5)^2)}$$

At $t=0$, $v = 2.166 \text{ ms}^{-1}$, $a = 0.345 \text{ ms}^{-2}$.

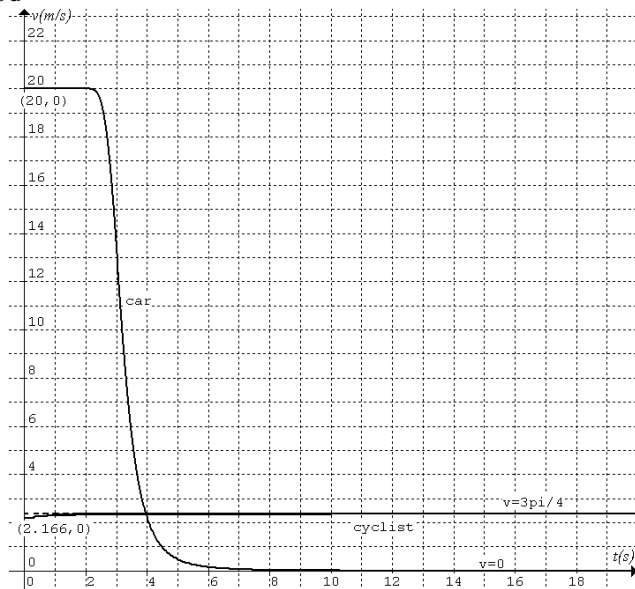
Q5b Resultant force $R = ma = 85 \times 0.345 \approx 29 \text{ N}$

Q5c Cyclist: Momentum $p = mv = 85 \times 2.166 \approx 184.11 \text{ kg ms}^{-1}$

Car: Momentum $p = mv = 1200 \times 20 = 24000 \text{ kg ms}^{-1}$

Difference $= 24000 - 184.11 \approx 23816 \text{ kg ms}^{-1}$

Q5d



$$\text{Cyclist: } v = \frac{1}{2} \tan^{-1}(5t + 2.5) + \frac{\pi}{2}$$

$$\text{As } t \rightarrow \infty, \tan^{-1}(5t + 2.5) \rightarrow \frac{\pi}{2}, \therefore v \rightarrow \frac{3\pi}{4}.$$

$$\text{Asymptote is } v = \frac{3\pi}{4}$$

$$\text{Car: As } t \rightarrow \infty, v = \frac{20}{1 + 0.5(t-2)^4} \rightarrow 0.$$

$$\text{Asymptote is } v = 0$$

Q5e At $t=10$ s, distance by car = area under graph (0 to 10 s) $= 20 \times 2 + 26.39 = 66.39 \text{ m}$; distance by cyclist $= 23.26 \text{ m}$.
 $\therefore$ car is ahead by $66.39 - 23.26 \approx 43 \text{ m}$

Q5f At $t=10$ s, car speed ≈ 0 and cyclist speed $\approx \frac{3\pi}{4} \approx 2.356$.

Let T be the time when the car and the cyclist are next to each other.

$$(T-10) \times 2.356 \approx 43, \therefore T \approx 28 \text{ s}$$

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