



INSIGHT
Trial Exam Paper

2010

**SPECIALIST
MATHEMATICS**

Written examination 2

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions.
- mark allocations

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SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.

Choose the response that is **correct** for the question.

A correct answer scores 1, an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

Take the **acceleration due to gravity** to have magnitude $g \text{ m/s}^2$, where $g = 9.8$.

Question 1

The hyperbola with equation $\frac{(x+2)^2}{25} - \frac{(y-3)^2}{16} = 1$ has asymptotes given by

- A. $5y - 4x = 23$ and $5y - 4x = 1$
- B. $5y + 4x = 23$ and $5y + 4x = 7$
- C. $5y + 4x = 15$ and $4y + 5x = 16$
- D. **$5y - 4x = 23$ and $5y + 4x = 7$**
- E. $5y - 4x = 23$ and $5x + 4y = 7$

Answer is D.

Worked solution

$$(y - 3) = \pm \frac{4}{5}(x + 2)$$

$$\begin{array}{ll} \therefore 5y - 15 = 4x + 8 & \text{or} \quad 5y - 15 = -4x - 8 \\ 5y - 4x = 23 & \text{or} \quad 5y + 4x = 7 \end{array}$$

Question 2

The equation $x^2 + 2x + y^2 + 4my + 10 = 0$, where m is a real constant, will represent a circle if

- A. $m > \frac{3}{2}$ or $m < -\frac{3}{2}$
- B. $m = \pm \frac{5}{2}$
- C. $m > \frac{5}{2}$ or $m < -\frac{5}{2}$
- D. $-\frac{3}{2} < m < \frac{3}{2}$
- E. $-\frac{5}{2} < m < \frac{5}{2}$

Answer is A.

Worked solution

$$x^2 + 2x + y^2 + 4my + 10 = 0$$

$$x^2 + 2x + 1 - 1 + y^2 + 4my + 4m^2 - 4m^2 + 10 = 0$$

$$(x+1)^2 - 1 + (y+2m)^2 - 4m^2 + 10 = 0$$

$$(x+1)^2 + (y+2m)^2 = 4m^2 - 9$$

$$4m^2 - 9 > 0 \text{ (must be positive)}$$

$$\text{So } m^2 > \frac{9}{4}$$

$$\therefore m > \frac{3}{2} \text{ or } m < -\frac{3}{2}$$

Question 3

What is the largest subset X of R such that $f^{-1} : X \rightarrow R$, and $f^{-1}(x) = \arccos\left(\frac{2x+a}{3}\right) + b$, where a and b are positive real constants?

- A. $-\frac{a}{2} \leq x \leq \frac{1-a}{2}$
- B. $\frac{a+3}{2} \leq x \leq \frac{a-3}{2}$
- C. $-\frac{a}{2} \leq x \leq \frac{3-b}{2}$
- D. $-\frac{a-3}{2} \leq x \leq \frac{a+3}{2}$
- E. $-\frac{3-a}{2} \leq x \leq \frac{3-a}{2}$

Answer is E.

Worked solution

$$-1 \leq \frac{2x+a}{3} \leq 1$$

$$-3 \leq 2x+a \leq 3$$

$$-3-a \leq 2x \leq 3-a$$

$$-\frac{3-a}{2} \leq x \leq \frac{3-a}{2}$$

Question 4

Which one of the followings can be correct for x if $\sin(x) + \sqrt{3} \cos(x) = n$, where $-2 \leq n \leq 2$?

- A. $\arcsin(n) + \frac{\pi}{3}$
 B. $\arcsin\left(\frac{n}{2}\right) - \frac{\pi}{3}$
 C. $\arcsin(2n)$
 D. $\arcsin\left(\frac{n}{2}\right) + \frac{\pi}{3}$
 E. $\arcsin(n) - \frac{\pi}{3}$

Answer is B.

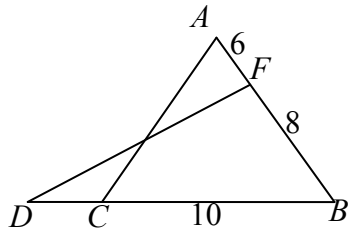
Worked solution

$$\begin{aligned}\sin(x) + \sqrt{3} \cos(x) &= n \\ \frac{\sin(x)}{2} + \frac{\sqrt{3} \cos(x)}{2} &= \frac{n}{2} \\ \frac{1}{2} \sin(x) + \frac{\sqrt{3}}{2} \cos(x) &= \frac{n}{2} \\ \cos\left(\frac{\pi}{3}\right) \sin(x) + \sin\left(\frac{\pi}{3}\right) \cos(x) &= \frac{n}{2} \\ \sin\left(x + \frac{\pi}{3}\right) &= \frac{n}{2} \\ x + \frac{\pi}{3} &= \arcsin\left(\frac{n}{2}\right) \\ x &= \arcsin\left(\frac{n}{2}\right) - \frac{\pi}{3}\end{aligned}$$

Alternatively

Using the Cas calculator: Solve $(\sin(x) + \sqrt{3} \cos(x) = n, x)$, $x = \arcsin\left(\frac{n}{2}\right) + \left(\frac{6k-1}{3}\right)\pi$.

If $k = 0$, $x = \arcsin\left(\frac{n}{2}\right) - \frac{\pi}{3}$.

Question 5

The area of $\triangle ABC$ is 20 cm^2 , where $d(\overline{AF}) = 6 \text{ cm}$, $d(\overline{BF}) = 8 \text{ cm}$, $d(\overline{CB}) = 10 \text{ cm}$.

Hence, $\cos(2B)$ is

- A. $\frac{39}{49}$
- B. $\frac{40}{49}$
- C. $\frac{41}{49}$
- D. $\frac{42}{49}$
- E. $\frac{43}{49}$

Answer is C.

Worked solution

$$\text{Area of } \triangle ABC = \frac{1}{2} \times 10 \times 14 \times \sin B = 20$$

$$\text{So } \sin B = \frac{2}{7}$$

$$\cos(2B) = 1 - 2 \sin^2(B)$$

$$= 1 - 2 \left(\frac{4}{49} \right)$$

$$= \frac{41}{49}$$

Question 6

The position vector of a particle at time $t \geq 0$ is given by $\underline{r} = (3 - 2t)\underline{i} + (5 + 2t)\underline{j}$. The path of the particle has equation

- A. $y = -x + 8$
- B. $y = x + 8$
- C. $y = -x - 8$
- D. $y = x + 2$
- E. $y = x - 8$

Answer is A.

Worked solution

$$\therefore x = 3 - 2t$$

$$y = 5 + 2t$$

$$x + y = 8$$

$$\therefore y = -x + 8$$

Question 7

$(a + i)(1 - ai) = 4 + (b - 1)i$, $i^2 = -1$, and $a, b \in R$. Hence, $a \times b$ is

- A. 4
- B. 2
- C. 0
- D. -2
- E. -4

Answer is E.

Worked solution

$$(a + i)(1 - ai) = 4 + (b - 1)i$$

$$a - a^2i + i - ai^2 = 4 + (b - 1)i$$

$$2a - a^2i + i = 4 + (b - 1)i$$

$$2a + (1 - a^2)i = 4 + (b - 1)i$$

$$\therefore 2a = 4 \text{ (real parts of both sides)}$$

$$\therefore a = 2 \text{ and } 1 - a^2 = b - 1, 1 - 4 = b - 1$$

$$\therefore a \times b = -4$$

Question 8

$z = x + 1 + xi$, $x \in R$ and $|\overline{z - iz}| = \sqrt{10}$. The sum of all the values of x is

- A. -2
- B. 0
- C. -1
- D. 1
- E. 2

Answer is C.

Worked solution

$$|\overline{z - iz}| = \sqrt{10}.$$

$$|z - iz| = \sqrt{10}$$

$$|z||1 - i| = \sqrt{10}$$

$$|z|\sqrt{1+1} = \sqrt{10}$$

$$\therefore |z| = \sqrt{5}$$

$$|x + 1 + xi| = \sqrt{5}$$

$$\sqrt{(x+1)^2 + x^2} = \sqrt{5}$$

$$(x+1)^2 + x^2 = 5$$

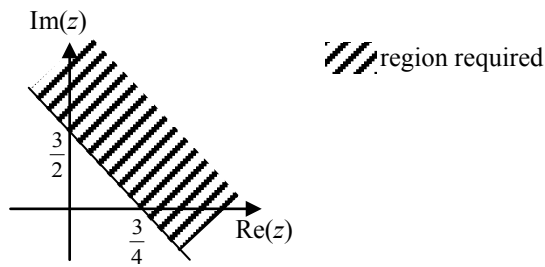
$$2x^2 + 2x - 4 = 0$$

$$x^2 + x - 2 = 0$$

$$\therefore (x+2)(x-1) = 0, x = 1 \text{ or } x = -2$$

$$\therefore -2 + 1 = -1$$

Question 9



The region represented on the Argand diagram above, could be defined by

- A. $|z + 1| \geq |z + 2|$
- B. $\left|z - \frac{2}{3}\right| \leq \frac{3}{4}$
- C. $|z - 3i| < |z - 2|$
- D. $|z + i| \geq |z - 2|$
- E. $|z + 2i| = |z - 3|$

Answer is D.

Worked solution

$$|z + i| \geq |z - 2| \text{ will be } x^2 + (y + 1)^2 \geq (x - 2)^2 + y^2.$$

$$\therefore x^2 + y^2 + 2y + 1 \geq x^2 - 4x + 4 + y^2$$

$$2y \geq -4x + 3$$

$$\therefore y \geq -2x + \frac{3}{2}$$

Hence, the equation of the line is $\frac{x}{\frac{3}{4}} + \frac{y}{\frac{3}{2}} = 1$:

$$4x + 2y = 3$$

$$y = -2x + \frac{3}{2}$$

Question 10

The polynomial equation $p(z) = 0$ has real coefficients and $p(3) = 0$. If

$p(z) = z^3 - az^2 + bz - ab$ and $a, b \in R$, which one of the following is **not** possible?

- A. $a = 3$ and $b = -9$
- B. $a = 3$ and $b \in R$
- C. $a \in R$ and $b = -9$
- D. $a = 0$ and $b = -9$
- E. $a = -3$ and $b = 9$

Answer is E.

Worked solution

$$\begin{aligned} p(z) &= z^3 - az^2 + bz - ab \\ &= z^2(z - a) + b(z - a) \end{aligned}$$

$$p(z) = (z - a)(z^2 + b)$$

$$p(3) = 0, \quad p(3) = (3 - a)(9 + b) = 0$$

$$\therefore a = 3 \text{ or } b = -9$$

Question 11

Which one of the following is $\text{Im}(z - \bar{z})^{2n-1}$, where $z = x + iy$ and n is an even natural number?

- A. 0
- B. $(2x)^{2n-1}$
- C. $(2y)^{2n-1}$
- D. $-(2y)^{2n-1}$
- E. $-(2x)^{2n-1}$

Answer is D.

Worked solution

Let $z = x + yi$:

$$\text{Im}(z - \bar{z})^{2n-1} = \text{Im}(x + yi - (x - yi))^{2n-1}$$

$$= \text{Im}(2yi)^{2n-1}, \quad 2n - 1 \text{ is an odd number (as } n \text{ is an even number),}$$

which means $i^{2n-1} = -i$.

$$\therefore \text{Im}[2yi]^{2n-1} = \text{Im}[(2y)^{2n-1} \times -i]$$

$$= -(2y)^{2n-1}$$

Question 12

The slope of the curve $y^2 = x^2 + \sin(xy)$ for any given point is

- A. $\frac{2x + y \cos(xy)}{2y - x \cos(xy)}$
- B. $\frac{2y + \cos(xy)}{2x - y \cos(xy)}$
- C. $\frac{2 + y \sin(xy)}{2y + 2x}$
- D. $\frac{2y - x \cos(xy)}{2x - y \cos(xy)}$
- E. $\frac{2y + x \cos(xy)}{2x - y \cos(xy)}$

Answer is A.

Worked solution

$$y^2 = x^2 + \sin(xy) \quad (\text{Apply implicit differentiation.})$$

$$2y \frac{dy}{dx} = 2x + \left(1y + \frac{dy}{dx}x\right) \cos(xy)$$

$$2y \frac{dy}{dx} = 2x + y \cos(xy) + x \cos(xy) \frac{dy}{dx}$$

$$2y \frac{dy}{dx} - x \cos(xy) \frac{dy}{dx} = 2x + y \cos(xy)$$

$$\frac{dy}{dx} (2y - x \cos(xy)) = 2x + y \cos(xy)$$

$$\therefore \frac{dy}{dx} = \frac{2x + y \cos(xy)}{2y - x \cos(xy)}$$

Question 13

Using a suitable substitution, $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\tan^3 x + \tan x) dx$ can be expressed in terms of u as

A. $\int_1^{\sqrt{3}} (u^3 + u) du$

B. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} u du$

C. $\int_1^{\sqrt{3}} (u^2 + 1) du$

D. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (u^2 + 1) du$

E. $\int_1^{\sqrt{3}} u du$

Answer is E.

Worked solution

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\tan^3 x + \tan x) dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan x (1 + \tan^2 x) dx$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan x \sec^2 x dx, \text{ let } \tan x = u \text{ then } \frac{du}{dx} = \sec^2 x \text{ and } \tan \frac{\pi}{3} = \sqrt{3}, \tan \frac{\pi}{4} = 1.$$

$$\therefore \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan x \sec^2 x dx = \int_1^{\sqrt{3}} u du$$

Question 14

$\int_0^r \frac{\tan^2 2x}{\cos^2 2x} dx = \frac{1}{6}$, where $r \in (\frac{\pi}{2}, \pi)$. Hence, the value of r is

- A. $\frac{3\pi}{4}$
- B. $\frac{5\pi}{8}$
- C. $\frac{7\pi}{8}$
- D. $\frac{2\pi}{3}$
- E. $\frac{5\pi}{6}$

Answer is B.

Worked solution

Let $\tan 2x = u$, $\frac{du}{dx} = 2 \sec^2 2x$, $du = \frac{2}{\cos^2 x} dx$, $\frac{du}{2} = \frac{dx}{\cos^2 2x}$

$$\therefore \int_0^r \frac{\tan^2 2x}{\cos^2 2x} dx$$

$$\begin{aligned} \int_0^{\tan 2r} \frac{u^2}{2} du &= \left[\frac{u^3}{6} \right]_0^{\tan 2r} \\ &= \frac{\tan^3 2r}{6} = \frac{1}{6} \end{aligned}$$

$$\therefore \tan^3 2r = 1$$

$$\therefore \tan 2r = 1$$

$$2r = \frac{5\pi}{4} \left(r \in \left(\frac{\pi}{2}, \pi \right) \right)$$

$$r = \frac{5\pi}{8}$$

Question 15

When Euler's method, with a step size of 0.3, is used to solve the differential equation

$\frac{dy}{dx} = \frac{1}{3} \sin^{-1}\left(x + \frac{1}{2}\right)$ with $x_0 = 0$ and $y_0 = 2$, the value of $10y_2$ would be given by

- A. $20 + \frac{\pi}{6}$
- B. $20 + \frac{\pi}{6} + \sin^{-1}(0.8)$
- C. $20 + \sin^{-1}(0.8)$
- D. $2 + 6 \sin^{-1}(0.8) + 2 \sin^{-1}(1.1)$
- E. $20 - \pi + \sin^{-1}(0.8) + 20 \sin^{-1}(1.1)$

Answer is B.

Worked solution

$$y_1 = y_0 + h \left(\frac{1}{3} \sin^{-1} \left(x_0 + \frac{1}{2} \right) \right)$$

$$= 2 + 0.3 \left(\frac{1}{3} \sin^{-1} \left(\frac{1}{2} \right) \right)$$

$$= 2 + 0.3 \left(\frac{\pi}{18} \right) = 2 + \frac{\pi}{60}$$

$$x_1 = x_0 + h = 0 + 0.3 = 0.3$$

$$y_2 = y_1 + h \left(\frac{1}{3} \sin^{-1} \left(x_1 + \frac{1}{2} \right) \right)$$

$$= 2 + \frac{\pi}{60} + \frac{3}{10} \left(\frac{1}{3} \sin^{-1}(0.8) \right)$$

$$= 2 + \frac{\pi}{60} + \frac{1}{10} \sin^{-1}(0.8)$$

$$\therefore 10y_2 = 20 + \frac{\pi}{6} + \sin^{-1}(0.8)$$

Question 16

The acceleration of an object starting at rest from the origin is $\frac{dv}{dt} = 2 - 3v$, where v is the velocity at time t seconds, $t \geq 0$. The velocity after 2 seconds is

- A. $-\frac{1}{3}\log_e 4$
- B. $\frac{2}{3}\left(1 - \frac{1}{e^6}\right)$
- C. $\frac{2}{3} - e^{-6}$
- D. $\frac{3}{2}\left(1 + \frac{1}{e^6}\right)$
- E. $\frac{3}{2}\left(1 - \frac{1}{e^6}\right)$

Answer is B.

Worked solution

$$\frac{dv}{dt} = 2 - 3v, \quad \frac{dt}{dv} = \frac{1}{2 - 3v}$$

$$\text{So } t = \int \frac{1}{2 - 3v} dv = -\frac{1}{3} \log_e |(2 - 3v)A|.$$

When $t = 0$ and $v = 0$:

$$0 = -\frac{1}{3} \log_e (2A)$$

$$\therefore A = \frac{1}{2}, \quad t = -\frac{1}{3} \log_e \left| \frac{2 - 3v}{2} \right|$$

To find the velocity after 2 seconds, $t = 2$:

$$2 = -\frac{1}{3} \log_e \left| \frac{2 - 3v}{2} \right|$$

$$\text{So } e^{-6} = \frac{2 - 3v}{2}, \quad 2e^{-6} = 2 - 3v.$$

$$\therefore v = \frac{2}{3} - \frac{2}{3}e^{-6} = \frac{2}{3}\left(1 - \frac{1}{e^6}\right)$$

Question 17

The position vector of a particle at time t seconds, $t \geq 0$, is given by

$\underline{r}(t) = (t \cos t)\underline{i} + (t \sin t)\underline{j} - 6t\underline{k}$. The direction of motion of the particle when $t = \pi$ is

- A. $-\pi\underline{i} + \underline{j}$
- B. $\pi\underline{i} + \underline{j}$
- C. $\pi\underline{i} - \underline{j}$
- D. $-\underline{i} + \pi\underline{j}$
- E. $-\underline{i} - \pi\underline{j}$

Answer is E.

Worked solution

To find the direction of the motion, we should find the velocity

$$\underline{v}(t) = (\cos t - t \sin t)\underline{i} + (\sin t + t \cos t)\underline{j}$$

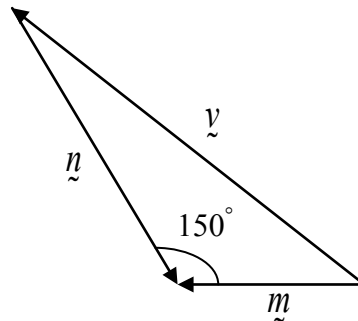
$$\begin{aligned}\therefore \underline{v}(\pi) &= (\cos \pi - \pi \sin \pi)\underline{i} + (\sin \pi + \pi \cos \pi)\underline{j} \\ &= (-1 - 0)\underline{i} + (0 + \pi(-1))\underline{j} = -\underline{i} - \pi\underline{j}\end{aligned}$$

Question 18

Vectors $\underline{m}$, $\underline{n}$ and $\underline{v}$ are as shown on the right.

From the diagram, it follows that

- A. $|\underline{v}|^2 = |\underline{m}|^2 + |\underline{n}|^2 + \sqrt{3}|\underline{m}||\underline{n}|$
- B. $|\underline{v}|^2 = |\underline{m}|^2 + |\underline{n}|^2$
- C. $|\underline{v}|^2 = |\underline{m}|^2 + |\underline{n}|^2 - \sqrt{3}|\underline{m}||\underline{n}|$
- D. $|\underline{v}|^2 = |\underline{m}|^2 + |\underline{n}|^2 - \sqrt{2}|\underline{m}||\underline{n}|$
- E. $|\underline{v}|^2 = |\underline{m}|^2 + |\underline{n}|^2 + |\underline{m}||\underline{n}|$



Answer is A.

Worked solution

$$\begin{aligned}|\underline{v}|^2 &= |\underline{m}|^2 + |\underline{n}|^2 - 2|\underline{m}||\underline{n}|\cos 150^\circ \\ &= |\underline{m}|^2 + |\underline{n}|^2 - 2|\underline{m}||\underline{n}|\frac{-\sqrt{3}}{2} \\ &= |\underline{m}|^2 + |\underline{n}|^2 + \sqrt{3}|\underline{m}||\underline{n}|\end{aligned}$$

Question 19

An object moves with a position vector $\underline{r}(t)$, expressed in metres, at time t seconds given by $\underline{r}(t) = (25 + 3t^2)\underline{i} + (14t - \frac{1}{4}t^4)\underline{j}$, $t \geq 0$. The angle (to the nearest degree) between the velocity vector and the acceleration vector of the body at a time $t = 2$ seconds is

- A. 90°
- B. 45°
- C. 60°
- D. 30°
- E. 0°

Answer is A.

Worked solution

$$\underline{v}(t) = (6t)\underline{i} + (14 - t^3)\underline{j} \quad \text{and} \quad \underline{a}(t) = 6\underline{i} - 3t^2\underline{j}$$

$$\text{So } \underline{v}(2) = 12\underline{i} + (14 - 8)\underline{j} = 12\underline{i} + 6\underline{j} \quad \text{and} \quad \underline{a}(2) = 6\underline{i} - 12\underline{j}.$$

$$\begin{aligned} \cos \theta &= \frac{\underline{a}(2) \cdot \underline{v}(2)}{|\underline{a}(2)| |\underline{v}(2)|} \\ &= \frac{72 - 72}{\sqrt{180} \sqrt{180}} = 0 \end{aligned}$$

$$\therefore \theta = 90^\circ$$

Alternatively

$$\underline{a}(2) \cdot \underline{v}(2) = 0$$

$$\therefore \theta = 90^\circ$$

Question 20

A 6 kg mass has an initial velocity of 10 ms^{-1} . After travelling 16 metres, the magnitude of the momentum of the mass is 84 kg ms^{-1} . The mass increases its speed by accelerating in a straight line at a constant rate of

- A. 2 ms^{-2}
- B. $\frac{5}{2} \text{ ms}^{-2}$
- C. 3 ms^{-2}
- D. $\frac{7}{2} \text{ ms}^{-2}$
- E. 4 ms^{-2}

Answer is C.

Worked solution

$$\text{Momentum} = 84 = 6 \times v$$

$$\therefore v = 14$$

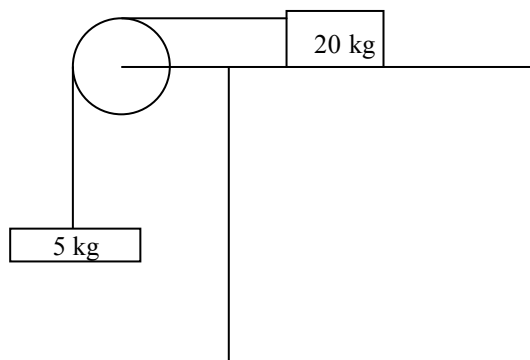
Hence, substitute $v = 14$, $u = 10$ and $s = 16$ into $v^2 = u^2 + 2as$, giving:

$$14^2 = 10^2 + 2 \times a \times 16$$

$$196 = 100 + 32a$$

$$96 = 32a$$

$$\therefore a = 3 \text{ ms}^{-2}$$

Question 21

A 20 kg mass on a rough horizontal table is connected to a 5 kg mass by a light inextensible string, which remains horizontal until it passes over a smooth pulley. The 20 kg mass moves along the table while the 5 kg mass falls toward the ground. Given that the acceleration of the 5 kg mass is $\frac{g}{25} \text{ ms}^{-2}$, the coefficient of friction between the 20 kg mass and the table is

- A. 0.10
- B. 0.15
- C. **0.20**
- D. 0.25
- E. 0.30

Answer is C.

Worked solution

For the 5 kg mass:

$$5g - T = 5 \times \frac{g}{25}$$

For the 20 kg mass:

$$T - \mu N = 20 \times \frac{g}{25}$$

Since $N = 20g$:

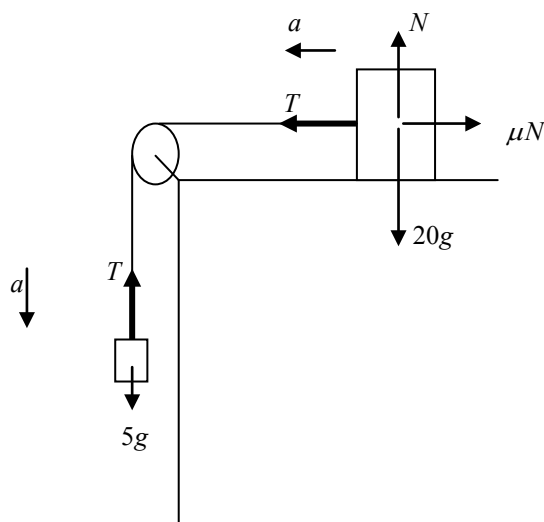
$$5g - T = \frac{5g}{25}$$

$$T - 20g\mu = \frac{20g}{25}$$

$$5g - 20g\mu = g$$

$$g(5 - 20\mu) = g$$

$$\mu = \frac{1}{5}$$



Question 22

A body of mass of 40 kg is placed on a rough surface which is inclined 60° to the horizontal. A force of $25g$ N is applied in an upwards direction to the body, which is parallel to the plane. Which of the following is closest to the magnitude of the acceleration of the body when $\mu = 0.4$, if it is moving down the plane?

- A. 0.04
- B. 0.2
- C. **0.4**
- D. 0.5
- E. 2.0

Answer is C.

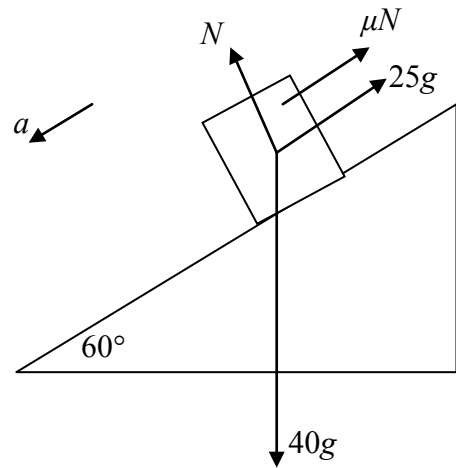
Worked solution

Upwards force is less than $40g \sin 60^\circ + \mu 40g \cos 60^\circ$.

So $40g \sin 60^\circ - (\mu 40g \cos 60^\circ + 25g) = 40a$.

$$40g \frac{\sqrt{3}}{2} - 33g = 40a$$

$$a = 0.402$$



SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.

A decimal approximation will not be accepted if an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Take the **acceleration due to gravity** to have magnitude $g \text{ m/s}^2$, where $g = 9.8$.

Question 1

- a. i. Express $w = 4\sqrt{2} - 4\sqrt{2}i$ in polar form.

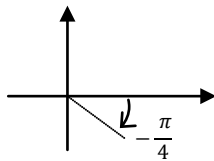
Worked solution

$$w = 4\sqrt{2} - 4\sqrt{2}i$$

$$r = \sqrt{(4\sqrt{2})^2 + (-4\sqrt{2})^2} = \sqrt{64} = 8$$

$$\tan \theta = \frac{-4\sqrt{2}}{4\sqrt{2}} = -1$$

$$\theta = -\frac{\pi}{4}$$



1 M

$$\therefore w = 8 \operatorname{cis} \left(-\frac{\pi}{4} \right)$$

1 M

2 marks

- ii. When $w = z^3$, show that one of the roots of the $z^3 = 4\sqrt{2} - 4\sqrt{2}i$ is $z_1 = -\sqrt{2} - \sqrt{2}i$.

Worked solution

$$z^3 = 8 \operatorname{cis}\left(-\frac{\pi}{4}\right)$$

$$z = 2 \operatorname{cis}\left(\frac{-\frac{\pi}{4} + 2k\pi}{3}\right) = 2 \operatorname{cis}\left(-\frac{\pi}{12} + \frac{2k\pi}{3}\right) \quad k \in \mathbb{Z} \quad 1 \text{ M}$$

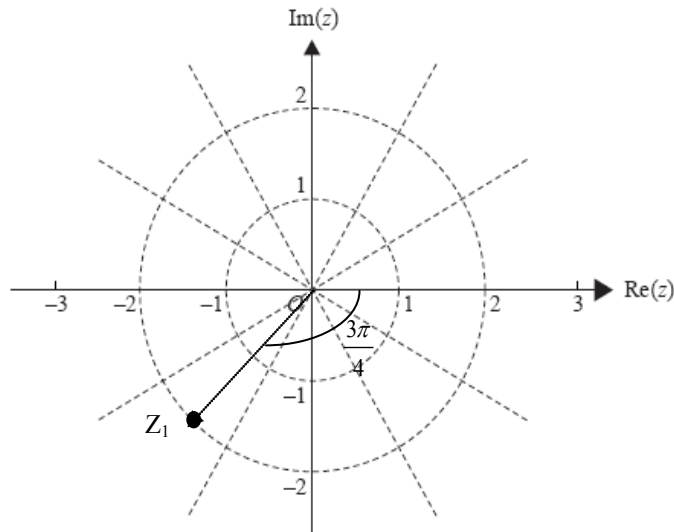
$$z = 2 \operatorname{cis}\left(-\frac{\pi}{12} + \frac{16\pi}{12}\right) \text{ when } k = 2. \quad 1 \text{ M}$$

$$\begin{aligned} z &= 2 \operatorname{cis}\left(\frac{15\pi}{12}\right) = 2 \operatorname{cis}\left(-\frac{9\pi}{12}\right) \\ &= 2\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = -\sqrt{2} - \sqrt{2}i \quad 1 \text{ M} \end{aligned}$$

3 marks

- iii. Plot and label z_1 on the Argand diagram below

Worked solution



1 mark

2+3+1=6 marks

- b. i.** By solving $z^2 - 2\sqrt{2}z + 4 = 0$ algebraically, show that the roots of this equation are $z_2 = \sqrt{2} - \sqrt{2}i$ and $z_3 = \sqrt{2} + \sqrt{2}i$.

Worked solution

$$\begin{aligned}
 z^2 - 2\sqrt{2}z + 4 &= 0 \\
 z^2 - 2\sqrt{2}z + 2 + 2 &= (z - \sqrt{2})^2 + 2 \\
 &= (z - \sqrt{2})^2 - 2i^2 && 1 \text{ M} \\
 &= (z - \sqrt{2} - \sqrt{2}i)(z - \sqrt{2} + \sqrt{2}i) \\
 &= 0 \\
 \therefore z &= \sqrt{2} + \sqrt{2}i \text{ and } z = \sqrt{2}i - \sqrt{2}i && 1 \text{ M}
 \end{aligned}$$

2 marks

- ii** Express the roots of $z^2 - 2\sqrt{2}z + 4 = 0$ in terms of z_1 , where $z_1 = -\sqrt{2} - \sqrt{2}i$.

Worked solution

$$\begin{aligned}
 z_3 &= \sqrt{2} + \sqrt{2}i = -z_1 && 1 \text{ M} \\
 z_2 &= \sqrt{2} - \sqrt{2}i = -\bar{z}_1 && 1 \text{ M}
 \end{aligned}$$

2 marks

2+2=4 marks

- c. i.** Show that the Cartesian equation for the relation $|z - z_1| = |z + z_1|$ is given by $y = -x$.

Worked solution

$$\begin{aligned}
 |z - z_1| &= |z + z_1|, \quad |x + yi + \sqrt{2} + \sqrt{2}i| = |x + yi - \sqrt{2} - \sqrt{2}i| && 1 \text{ M} \\
 \sqrt{(x + \sqrt{2})^2 + (y + \sqrt{2})^2} &= \sqrt{(x - \sqrt{2})^2 + (y - \sqrt{2})^2} \\
 x^2 + 2x\sqrt{2} + 2 + y^2 + 2y\sqrt{2} + 2 &= x^2 - 2x\sqrt{2} + 2 + y^2 - 2y\sqrt{2} + 2 \\
 4y\sqrt{2} &= -4x\sqrt{2}, \quad y = -x && 1 \text{ M}
 \end{aligned}$$

2 marks

- ii. Show that $\bar{z}_1$ satisfies the relation $|z - z_1| = |z + z_1|$.

Worked solution

$$\bar{z}_1 = -\sqrt{2} + \sqrt{2}i$$

$$x = -\sqrt{2}, \quad y = \sqrt{2}$$

$$y = -x \quad (\text{from part c i})$$

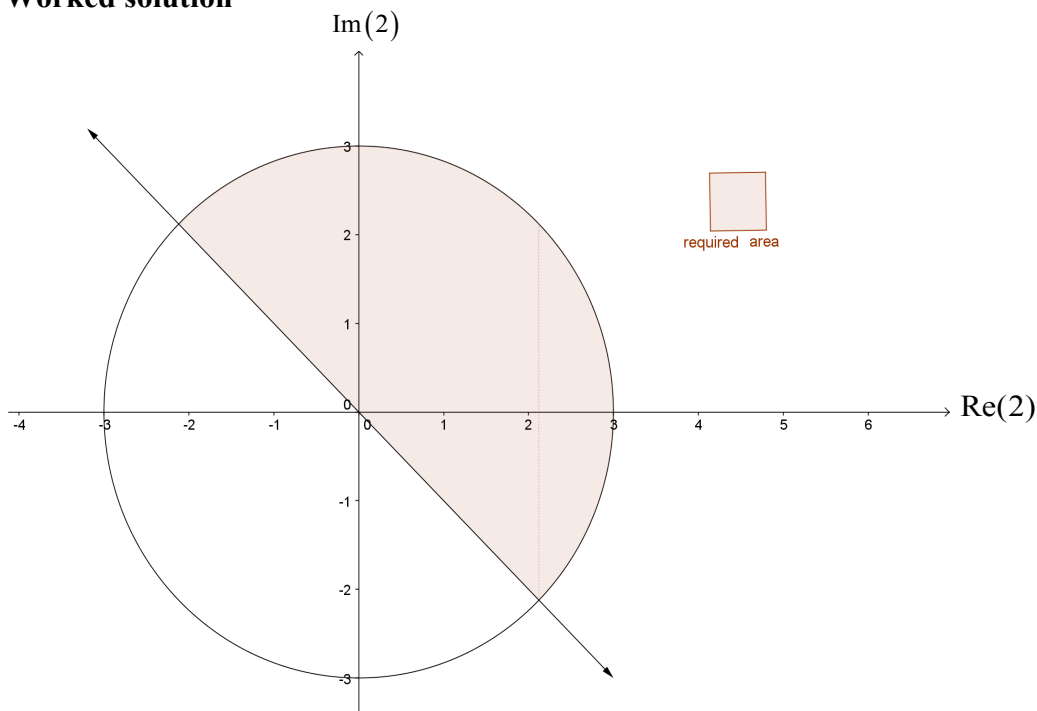
$$\sqrt{2} = -(-\sqrt{2})$$

1 mark

2+1=3 marks

- d. Sketch the region specified by $\{z : |z| \leq 3\} \cap \{z : |z - z_1| \geq |z + z_1|\}$.

Worked solution

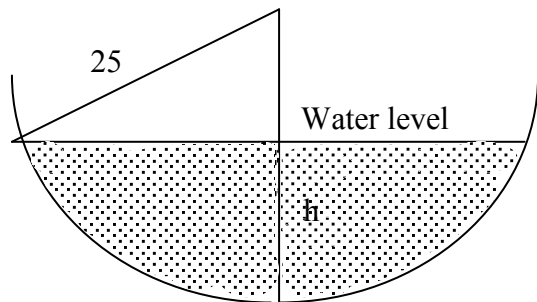


2 marks

Total 15 marks

Question 2

Water is flowing out at the rate of $4 \text{ cm}^3/\text{min}$ from a hemispherical bowl of radius 25 cm.



- a. i. At what rate is the water level changing when the water is 10 cm deep if the volume of water in a hemispherical bowl of radius r is $V = \left(\frac{\pi}{3}\right)h^2(3r - h)$ when the water is h cm deep?

Worked solution

$$\frac{dh}{dt} = ?$$

$$\frac{dV}{dt} = -4 \text{ cm}^3/\text{min}$$

$$V = \frac{\pi}{3}h^2 3r - \frac{\pi}{3}h^3$$

$$\frac{dV}{dh} = 2\pi hr - \pi h^2$$

1 M

$$\frac{dh}{dv} = \frac{1}{500\pi - 100\pi} = \frac{1}{400\pi}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \frac{dV}{dt} = \frac{1}{400\pi} \times (-4)$$

$$\frac{dh}{dt} = -\frac{1}{100\pi} \text{ cm/min}$$

1 M

2 marks

- ii. What is the radius r of the water's surface, in terms of h ?

Worked solution

$$\begin{aligned}(25-h)^2 + r^2 &= 25^2 \\ 625 - 50h + h^2 + r^2 &= 625 & 1 \text{ M} \\ r^2 &= 50h - h^2 \\ r &= \pm \sqrt{50h - h^2} \\ r &= \sqrt{50h - h^2} \quad (r > 0) & 1 \text{ M}\end{aligned}$$

2 marks

- iii. At what rate is the radius r changing when the water is 10 cm deep?

Worked solution

$$\begin{aligned}r &= \sqrt{50h - h^2} \\ r &= (50h - h^2)^{\frac{1}{2}} \\ \frac{dr}{dh} &= \frac{1}{2} \frac{50 - 2h}{\sqrt{50h - h^2}} & 1 \text{ M}\end{aligned}$$

$$\frac{dr}{dt} = \frac{dr}{dh} \frac{dh}{dt} & 1 \text{ M}$$

$$\frac{dr}{dt} = \frac{25-h}{\sqrt{50h-h^2}} \times \left(-\frac{1}{100\pi} \right)$$

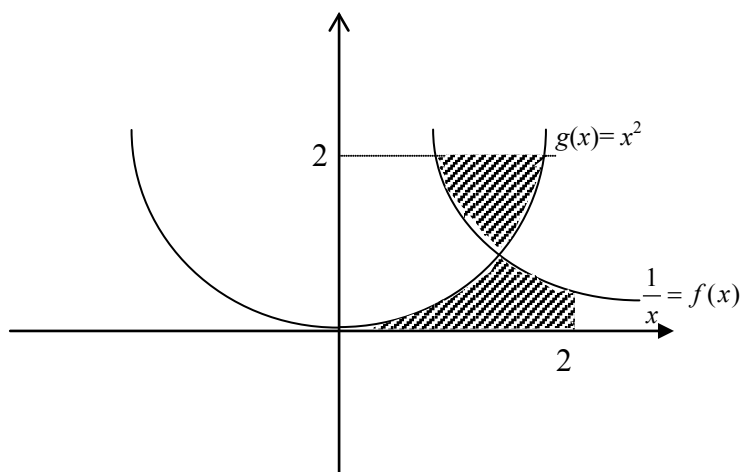
When $h = 10$ cm:

$$\begin{aligned}\frac{dr}{dt} &= \frac{15}{20} \times \frac{-1}{100\pi} \\ \frac{dr}{dt} &= -\frac{3}{400\pi} \text{ cm/min} & 1 \text{ M}\end{aligned}$$

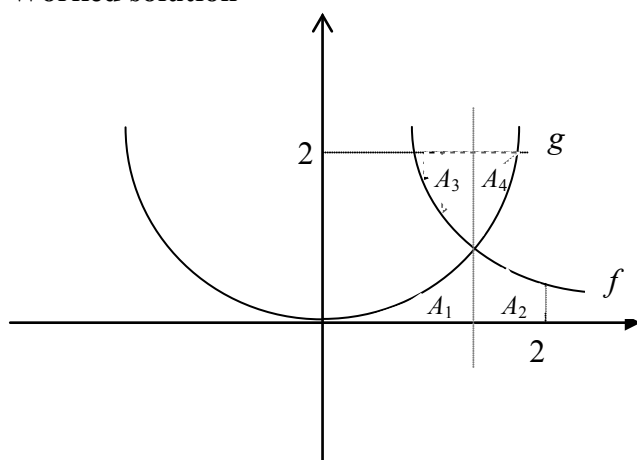
3 marks

2+2+3=7 marks

- b. i. Let M be the set of points inside the shaded area. Express M as a sum of integrals.



Worked solution



$$A_1 = \int_0^1 x^2 dx$$

$$A_3 = \int_{\frac{1}{2}}^1 \left(2 - \frac{1}{x} \right) dx$$

$$A_2 = \int_1^2 \frac{1}{x} dx$$

$$A_4 = \int_1^{\sqrt{2}} (2 - x^2) dx$$

Alternatively

$$M = \{g(x) \geq x^2 \cap f(x) \geq \frac{1}{x} \cap 1 \leq y \leq 2\} \cup \{g(x) < x^2 \cap f(x) < \frac{1}{x} \cap 0 \leq x \leq 2 \cap 1 \geq y \geq 0\}$$

2 marks

- ii. Using calculus, find the total shaded area in exact form and represent the total shaded area in the form of $\frac{a\sqrt{2}-b}{c}$.

Worked solution

Total shaded area can be found as the sum of three smaller shaded areas, which are: (1) the lower part before the intersection point (A_2); (2) the lower part after the intersection (A_3); and (3) the upper part (A_1).

$$A = A_1 + A_2 + A_3$$

$$g(x) = x^2$$

$$\text{So } x = \sqrt{y}$$

$$\frac{1}{x} = f(x)$$

$$\text{So } x = \frac{1}{y}$$

1 M

The x -coordinate of the intersection point can be found by the equation

$$x^2 = \frac{1}{x} \text{ and } x^3 = 1, \text{ so } x = 1.$$

Then:

$$A_1 = \int_1^2 \left(\sqrt{y} - \frac{1}{y} \right) dy = \left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1}{y} \right]_1^2 = \frac{2}{3} 2\sqrt{2} - \ln 2 - \frac{2}{3}$$

1 M

$$A_2 = \int_0^1 x^2 dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

1 M

$$A_3 = \int_1^2 \frac{1}{x^2} dx = [\ln x]_1^2 = \ln 2 - \ln 1 = \ln 2$$

1 M

$$\begin{aligned} A &= \frac{4}{3}\sqrt{2} - \ln 2 - \frac{2}{3} + \frac{1}{3} + \ln 2 \\ &= \frac{4\sqrt{2}}{3} - \frac{1}{3} \end{aligned}$$

1 M

5 marks

2+5=7 marks

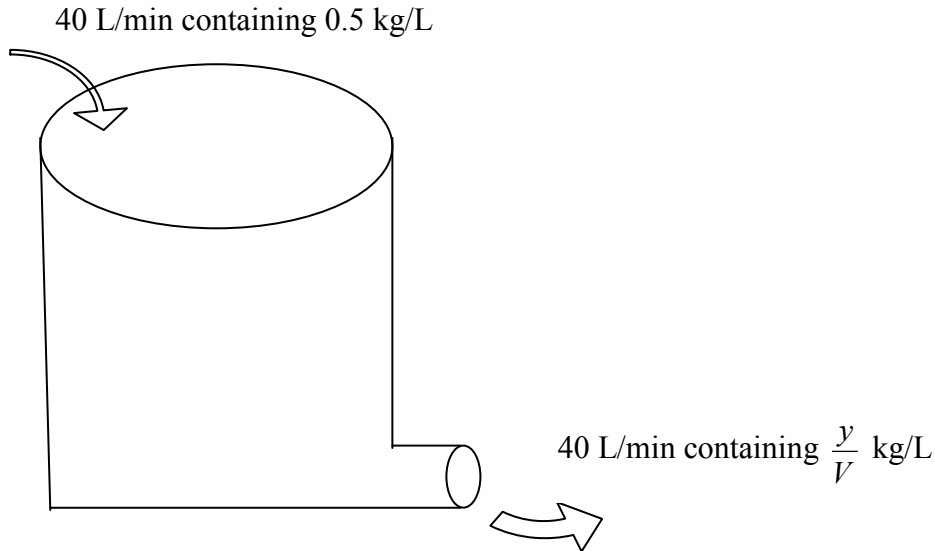
Note

The total area can be found using four smaller areas as well (whereby each area will be 1 mark, and the final result will be 1 mark).

Total 14 marks

Question 3

In an oil refinery, a storage tank contains 8000 L of gasoline that initially has 100 kg of an additive dissolved in it. In preparation for winter weather, gasoline containing 0.5 kg of additive per litre is pumped in at a rate of 40 L/min. The well-mixed solution is pumped out at a rate of 40 L/min.



- a. Let y kg be the amount of solution in the tank. Find the number of litres of gasoline and additive in solution, $V(t)$, in the tank at any time t minutes.

Worked solution

If $y = 100$ when $t = 0$, then

$$V(t) - 8000 + (40 \text{ L/min} - 10 \text{ L/min})(t \text{ min}) = 8000 \text{ L}$$

1 mark

- b. i. Find the inflow rate and outflow rate and represent $\frac{dy}{dt}$ as a differential equation.

Worked solution

$$\text{Rate out} = \left(\frac{y}{8000} \right) 40 = \frac{y}{200} \text{ kg/min}$$

1 M

$$\text{Rate in} = (0.5 \text{ kg/L})(40 \text{ L/min}) = 20 \text{ kg/min}$$

1 M

$$\frac{dy}{dt} = 20 - \frac{y}{200}$$

1 M

3 marks

- ii. Using calculus, solve for t as a function of y .

Worked solution

$$\frac{dy}{dt} = 20 - \frac{y}{200}$$

$$\frac{dy}{dt} = \frac{4000 - y}{200}$$

$$\frac{dt}{dy} = \frac{200}{4000 - y}$$

$$t = \int \frac{200}{4000 - y} dy$$

1 M

$$t = -200 \log_e |4000 - y| + c$$

Let $c = -200 \log_e A$

(Or you can find c by substituting $y = 100$ when $t = 0$, then $c = -200 \log_e \frac{1}{3900}$.)

$$t = -200 (\log_e |4000 - y| + \log_e A) = -200 \log_e ((4000 - y)A)$$

1 M

When $t = 0$, $y = 100$, so:

$$0 = -200 \log_e (3900A)$$

$$A = \frac{1}{3900}$$

$$c = -200 \log_e \frac{1}{3900}$$

$$t = -200 \log_e \left(\frac{4000 - y}{3900} \right)$$

$$= 200 \log_e \left(\frac{3900}{4000 - y} \right)$$

1 M

3 marks

3+3=6 marks

- c. i. Represent $y(t)$ for any time, t minutes.

Worked solution

$$t = -200 \log_e \left(\frac{4000 - y}{3900} \right)$$

$$-\frac{t}{200} = \log_e \left(\frac{4000 - y}{3900} \right)$$

$$e^{-\frac{t}{200}} = \frac{4000 - y}{3900} \quad 1 \text{ M}$$

$$3900 e^{-\frac{t}{200}} = 4000 - y$$

$$y = 4000 - 3900 e^{-\frac{t}{200}} \quad 1 \text{ M}$$

2 marks

- ii. How much additive solution is in the tank 20 minutes after the pumping process begins? Express your answer correct to two decimal places.

Worked solution

$$y(20) = 4000 - 3900 e^{-\frac{20}{200}} = 4000 - \frac{3900}{e^{0.1}}$$

$$y(20) \approx 471.13 \text{ kg} \quad 1 \text{ mark}$$

2+1=3 marks

Total 10 marks

Question 4

An object is moving along the curve $y = \frac{1}{4}\sin^2 x + \frac{1}{2}$ and point P is on the curve.

- a. Show that point $\frac{1}{x} = f(x)P$ is in the first quadrant if the x value of point P is $\frac{\pi}{4}$.

Worked solution

$$y\left(\frac{\pi}{4}\right) = \frac{1}{4}\sin^2\left(\frac{\pi}{4}\right) + \frac{1}{2} = \frac{1}{4} \times \frac{1}{2} + \frac{1}{2} = \frac{5}{8}$$

$$P\left(\frac{\pi}{4}, \frac{5}{8}\right)$$

1 mark

- b. i. Find unit vectors $\hat{u}$ and $-\hat{u}$ that are tangent to the curve at the point $x = \frac{\pi}{4}$.

(Hint: Use any vector that has the same slope.)

Worked solution

$$P\left(\frac{\pi}{4}, \frac{5}{8}\right)$$

$$\frac{dy}{dx} = \frac{1}{4} 2 \sin x \cos x = \frac{1}{2} \sin x \cos x$$

1 M

$$\frac{dy}{dx} = \frac{1}{2} \sin x \cos x \Big|_{x=\frac{\pi}{4}} = \frac{1}{2} \frac{\sqrt{2}}{2} \frac{\sqrt{2}}{2} = \frac{1}{4} = m_T$$

1 M

We look for a unit vector with this slope, $m_T = \frac{1}{4}$.

The vector $\underline{u} = 4\mathbf{i} + \mathbf{j}$ has slope $\frac{1}{4}$ (as does every non-zero multiple of $\underline{u}$).

$$\text{So } |\underline{u}| = \sqrt{1+16} = \sqrt{17}$$

$$\hat{u} = \frac{\underline{u}}{|\underline{u}|} = \frac{4}{\sqrt{17}}\mathbf{i} + \frac{1}{\sqrt{17}}\mathbf{j}$$

1 M

The vector $\hat{u}$ is tangent to the curve at $\left(\frac{\pi}{4}, \frac{5}{8}\right)$ because it has the same direction as $\underline{u}$ and

$-\hat{u} = -\frac{4}{\sqrt{17}}\mathbf{i} - \frac{1}{\sqrt{17}}\mathbf{j}$, which points in the opposite direction, is also tangent to the curve

at $\left(\frac{\pi}{4}, \frac{5}{8}\right)$.

1 M

4 marks

- ii. Hence, find unit vectors normal ($\hat{n}$ and $-\hat{n}$) to the curve at $x = \frac{\pi}{4}$.

Worked solution

$$\hat{n} = -\frac{1}{\sqrt{17}}i + \frac{4}{\sqrt{17}}j \text{ and } -\hat{n} = \frac{1}{\sqrt{17}}i - \frac{4}{\sqrt{17}}j$$

1 mark

4+1=5 marks

- c. If $\underline{a} = 3\hat{i} - 2\hat{j}$ is the acceleration of a particle moving along the path, express $\underline{a}$ as the sum of a vector parallel to $\underline{u} = 4\hat{i} + \hat{j}$ and a vector perpendicular to $\underline{u} = 4\hat{i} + \hat{j}$. (Hence, explain the reason.)

Worked solution

$$\underline{a} = 3\hat{i} - 2\hat{j}, \quad \underline{u} = 4\hat{i} + \hat{j}, \quad \hat{u} = \frac{1}{\sqrt{17}}(4\hat{i} + \hat{j})$$

$$\underline{a} = \underline{a}_{\parallel} + \underline{a}_{\perp} = (\hat{u} \cdot \underline{a})\hat{u} + \underline{a} - (\hat{u} \cdot \underline{a})\hat{u} \quad 1 \text{ M}$$

$$= \frac{12-2}{\sqrt{17}} \times \frac{1}{\sqrt{17}}(4\hat{i} + \hat{j}) + 3\hat{i} - 2\hat{j} - \frac{10}{17}(4\hat{i} + \hat{j})$$

$$= \frac{10}{17}(4\hat{i} + \hat{j}) + \frac{11}{17}\hat{i} - \frac{44}{17}\hat{j}$$

$$= \frac{10}{17}(4\hat{i} + \hat{j}) + \frac{11}{17}\hat{i} - \frac{44}{17}\hat{j}$$

$$= \frac{10}{17}(4\hat{i} + \hat{j}) + \frac{11}{17}(\hat{i} - 4\hat{j}) \quad 1 \text{ M}$$

Parallel component Perpendicular component

The first vector in the sum is parallel to $\underline{u}$ because it is $\frac{10}{17}\underline{u}$ (i.e. a multiple of $\underline{u}$).

The second vector is perpendicular to $\underline{u}$ because $(\frac{11}{17}\hat{i} - \frac{44}{17}\hat{j}) \cdot (4\hat{i} + \hat{j}) = \frac{44}{17} - \frac{44}{17} = 0$.

1 M

3 marks

Total 9 marks

Question 5

A cricket ball is hit when it is 1 metre above the ground. It leaves the bat with an initial speed of 30 m/s, making an angle of 20° with the horizontal. At the instant the ball is hit, an instantaneous gust of wind blows in the horizontal direction directly opposite the direction the ball is taking toward the outfield, adding a component $-3\hat{i}$ (m/s) to the ball's initial velocity.

- a. Show that the initial velocity and the initial position are
 $\mathbf{v}_0 = (30 \cos 20^\circ - 3)\hat{i} + 30(\sin 20^\circ)\hat{j}$ and $\mathbf{r}_0 = \hat{j}$, respectively.

Worked solution

$$\mathbf{v}_0 = v_0(\cos 20^\circ)\hat{i} + v_0(\sin 20^\circ)\hat{j} - 3\hat{i} = (30 \cos 20^\circ - 3)\hat{i} + 30(\sin 20^\circ)\hat{j} \quad 1 \text{ M}$$

$$\mathbf{r}_0 = 0\hat{i} + 1\hat{j} \text{ (1 metre above the ground), so } \mathbf{r}_0 = \hat{j}. \quad 1 \text{ M}$$

2 marks

- b. In terms of $\mathbf{v}_0$ and $\mathbf{r}_0$, define a vector equation (position vector) for the path of the cricket ball and, hence, find it.

Worked solution

$$\frac{d^2\mathbf{r}}{dt^2} = -g\hat{j} \quad \text{or} \quad \mathbf{a}(t) = -g\hat{j} \quad 1 \text{ M}$$

$$\begin{aligned} \frac{d\mathbf{r}}{dt} &= (-gt)\hat{j} + \mathbf{v}_0 & \text{or} & & \mathbf{v}(t) &= (-gt)\hat{j} + \mathbf{c}, \\ \mathbf{v}(0) &= \mathbf{c} = (30 \cos 20^\circ - 3)\hat{i} + 30(\sin 20^\circ)\hat{j} & & & & 1 \text{ M} \\ \mathbf{v}(t) &= (30 \cos 20^\circ - 3)\hat{i} + (30(\sin 20^\circ) - gt)\hat{j} \end{aligned}$$

$$\begin{aligned} \mathbf{r}(t) &= -\frac{1}{2}gt^2\hat{j} + \mathbf{v}_0t + \mathbf{r}_0 & \text{or} & & \mathbf{r}(t) &= (30t \cos 20^\circ - 3t)\hat{i} + \left(30(\sin 20^\circ)t - \frac{1}{2}gt^2\right)\hat{j} + \mathbf{d} \\ & & & & \mathbf{d} &= \mathbf{r}(0) = \hat{j} \end{aligned}$$

$$\begin{aligned} \therefore \mathbf{r}(t) &= -\frac{1}{2}gt^2\hat{j} + (30 \cos 20^\circ - 3)t\hat{i} + 30(\sin 20^\circ)t\hat{j} + \hat{j} \\ &= (30 \cos 20^\circ - 3)t\hat{i} + (30(\sin 20^\circ)t + 1 - \frac{1}{2}gt^2)\hat{j} \end{aligned}$$

1 M

3 marks

- c. How high does the cricket ball go, and when does it reach its maximum height? Give your answer correct to two decimal places.

Worked solution

The ball reaches its highest point when the vertical component of velocity is zero.

$$\text{So } \frac{dy}{dt} = 30 \sin 20^\circ - 9.8t = 0 \quad 1 \text{ M}$$

$$t = \frac{30 \sin(20^\circ)}{9.8} = 1.05 \text{ s} \quad 1 \text{ M}$$

Substituting this time value into the vertical component for z gives the maximum height.

$$Y_{\max} = 1 + 30 \sin(20^\circ)(1.05) - 4.9(1.05)^2 = 6.37 \text{ m} \quad 1 \text{ M}$$

That is, the maximum height of the cricket ball is about 6.37 m, reached at about 1.05 second after leaving the bat.

3 marks

- d. Assuming that the ball is not caught, find its range and flight time, giving your answers correct to three decimal places.

Worked solution

On Cas calculator type:

$$\text{solve } (1 + 30 \sin(20^\circ)t - 4.9t^2 = 0, t)$$

$$t = 2.187 \text{ s and } t = -0.093 \text{ s}$$

$$\therefore t = 2.187 \text{ s} \quad 1 \text{ M}$$

Substituting the positive time value into the horizontal component for z , we find the range.

$$R = (30 \cos 20^\circ - 3)(2.187) = 55.092 \text{ m}$$

$$R : [0, 55.092] \quad 1 \text{ M}$$

2 marks

Total 10 marks

END OF SOLUTIONS BOOK