INSIGHT

Trial Exam Paper

2010

SPECIALIST MATHEMATICS

Written examination 1

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions.
- mark allocations
- tips on how to approach the questions.

Question 1

The position of a particle is given by $\underset{\sim}{r} = \sec \frac{t}{2}\, \underset{\sim}{i} + 2 \tan \frac{t}{2}\, \underset{\sim}{j}$, where $t \in [0, \pi)$.

a. Show that the Cartesian equation of the particle is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Find the value of a and b and state the domain and range.

Worked solution

$\underset{\sim}{r} = \sec \frac{t}{2}\, \underset{\sim}{i} + 2 \tan \frac{t}{2}\, \underset{\sim}{j}$

$x = \sec \frac{t}{2}$ $\quad y = 2 \tan \frac{t}{2}$, $x \geq 1$ and $y \geq 0$ for $t \in [0, \pi)$.

$x = \sec \frac{t}{2}$ $\quad \frac{y}{2} = \tan \frac{t}{2}$

$\sec^2 \frac{t}{2} - \tan^2 \frac{t}{2} = 1$

$x^2 - \left(\frac{y}{2}\right)^2 = 1$

$x^2 - \frac{y^2}{4} = 1$

This is a part of the hyperbola with centre (0, 0) and $a = 1$ and $b = 2$.

So, vertex is (1, 0).

Asymptotes are $y = \pm 2x$.

Domain is $\{x: x \geq 1\}$.

Range is $\{y: y \geq 0\}$.

3 marks

Mark allocation

- 1 mark for using the correct trigonometric identity.
- 1 mark for finding the correct values of a and b.
- 1 mark for the correct domain and range.

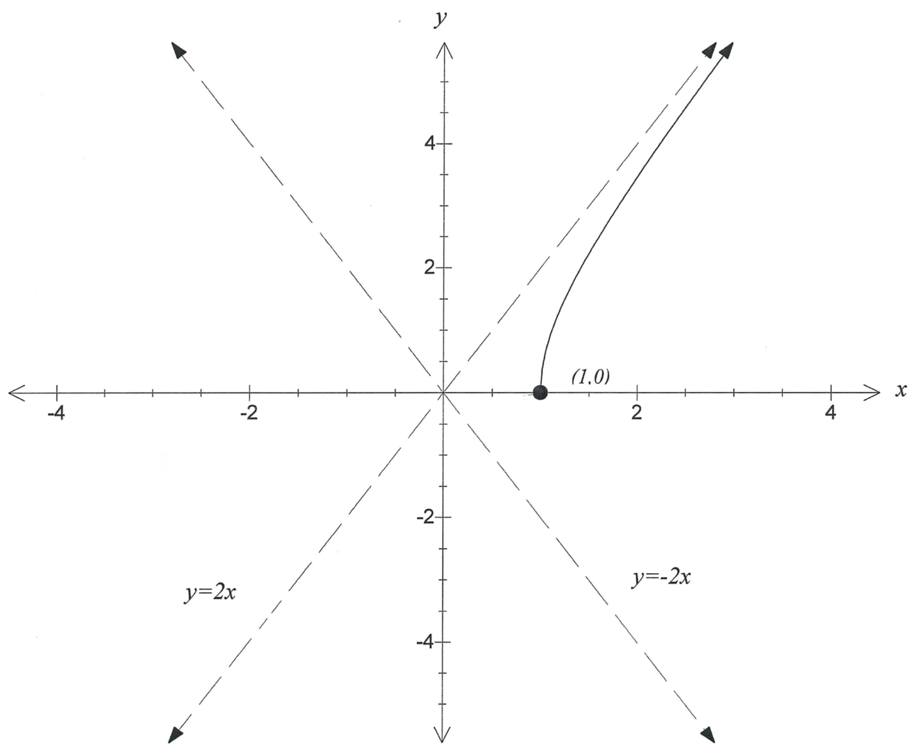
Tip

- *Use the trigonometric identity $1 + \tan^2 \theta = \sec^2 \theta$ or $\sec^2 \theta - \tan^2 \theta = 1$ to establish the Cartesian relationship.*

Question 1 – continued
TURN OVER

b. Sketch the graph of its path on the axes below.

Worked solution



1 mark

Mark allocation

- 1 mark for showing the correct part of the hyperbola and its asymptotes.

Tip

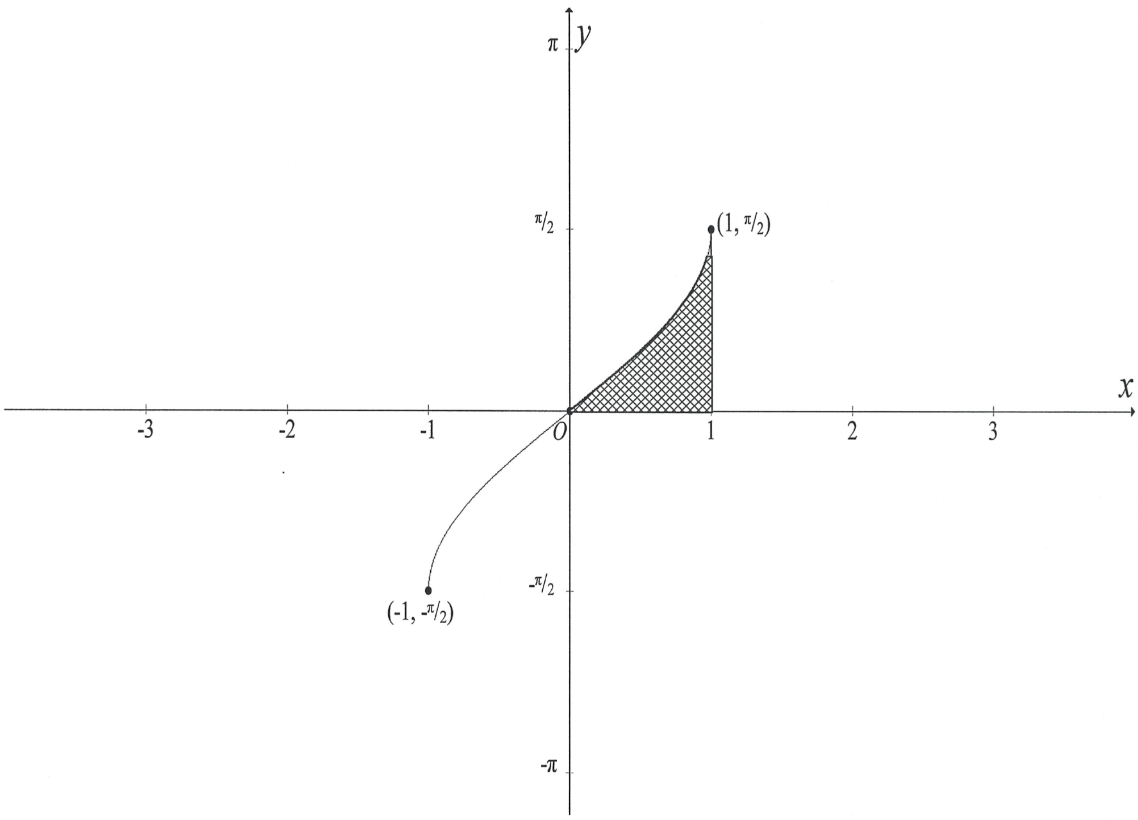
The domain of $\frac{t}{2}$ *are* $[0, \frac{\pi}{2})$, *so the respective ranges of* $\sec\theta$ *and* $\tan\theta$ *are* $[1, \infty)$ *and* $[0, \infty)$.

End of Question 1

Question 2

a. On the set of axes below, shade the region enclosed by the graph of $y = \sin^{-1} x$ and the lines $y = 0$ and $x = 1$.

Worked solution



1 mark

Mark allocation

- 1 mark for shading the correct region.

b. Find the exact area enclosed by the graph of $y = \sin^{-1} x$ and the lines $y = 0$ and $x = 1$.

Worked solution

Area required = Area of rectangle – Area between graph of $y = \sin^{-1} x$ and the y-axis from $y = 0$ to $y = \frac{\pi}{2}$.

$y = \sin^{-1} x$

$x = \sin y = f(y)$

$$A = \left(\frac{\pi}{2} \times 1\right) - \int_0^{\frac{\pi}{2}} f(y).dy$$

$$A = \left(\frac{\pi}{2} \times 1\right) - \int_0^{\frac{\pi}{2}} \sin y.dy$$

$$= \frac{\pi}{2} - [-\cos y]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} + [\cos y]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} + \cos\frac{\pi}{2} - \cos 0$$

$$= \frac{\pi}{2} - 1$$

The area required is $\frac{\pi}{2} - 1$ square units.

2 marks

Mark allocation

- 1 mark for setting up the correct area.
- 1 mark for finding the correct answer for the area.

Tip

- *Since* $\int \sin^{-1} x.dx$ *cannot be done, y = sin*$^{-1}$ *x must be expressed as x = sin y and then the area between the curve and the y-axis is found.*

Question 2 – continued

c. Find the exact volume generated by rotating this area about the y-axis.

Worked solution

Volume required = volume generated by rotating the area between the graphs of $x = 1$ and $x = \sin y$ and the lines $y = 0$ to $y = \frac{\pi}{2}$ about the y-axis.

$$V = \pi\int_0^{\frac{\pi}{2}} \left(1^2 - \sin^2 y\right).dy$$

$$= \pi\int_0^{\frac{\pi}{2}} \left(1 - \sin^2 y\right).dy$$

$$= \pi\int_0^{\frac{\pi}{2}} \left(1 - \left(\frac{1-\cos 2y}{2}\right)\right).dy$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \left(1 + \cos 2y\right).dy$$

$$= \frac{\pi}{2}\left[y + \frac{1}{2}\sin 2y\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}\left[\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi\right) - \left(0 + \frac{1}{2}\sin 0\right)\right]$$

$$= \frac{\pi}{2} \times \frac{\pi}{2}$$

$$= \frac{\pi^2}{4}$$

The volume is $\frac{\pi^2}{4}$ cubic units.

3 marks

Mark allocation

- 1 mark for setting up the integral for the volume correctly.
- 1 mark for using the correct identity to simplify the integral.
- 1 mark for correctly evaluating the integral.

Tip

- *The volume could also be found by calculating* $\pi(1)^2\left(\frac{\pi}{2}\right) - \pi\int_0^{\frac{\pi}{2}} \sin^2 y.dy$.

End of Question 2
TURN OVER

Question 3

Given that $z_1 = 2$ is a solution to the equation $z^2 + (i\sqrt{3} - 3)z + 2 - 2\sqrt{3}\,i = 0,\ z \in C$

a. Show that the other solution is $z_2 = 1 - \sqrt{3}\,i$.

Worked solution

$$(z-2)(z-1+i\sqrt{3})$$
$$= z^2 - z + i\sqrt{3}z - 2z + 2 - 2\sqrt{3}\,i$$
$$= z^2 + (i\sqrt{3} - 3)z + 2 - 2\sqrt{3}\,i$$

$\therefore$ The other solution is $z_2 = 1 - \sqrt{3}\,i$.

1 mark

Mark allocation

- 1 mark for verifying that $z_2 = 1 - \sqrt{3}\,i$ is the other solution.

Tip

- *z_2 could also be found by dividing $z - 2$ into $z^2 + (i\sqrt{3} - 3)z + 2 - 2\sqrt{3}\,i$.*

b. Find the square root of z_2 in the form $a + bi$.

Worked solution

$$z_2 = 1 - \sqrt{3}\,i$$

$$|z_2| = \sqrt{1^2 + \sqrt{3}^2} \qquad \text{and Arg } z = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right)$$
$$= \sqrt{4} \qquad\qquad = \frac{-\pi}{3}$$
$$= 2$$

$$z_2 = 2\,\text{cis}\left(\frac{-\pi}{3}\right)$$

$$\sqrt{z_2} = \pm\sqrt{2}\,\text{cis}\left(\frac{-\pi}{6}\right)$$
$$= \pm\sqrt{2}\left(\cos\frac{-\pi}{6} + \sin\frac{-\pi}{6}i\right)$$
$$= \pm\sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$
$$\sqrt{z_2} = \pm\left(\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i\right)$$

3 marks

Mark allocation

- 1 mark for the correctly converting z_2 to polar form.
- 1 mark for correctly taking the square root.
- 1 mark for expressing the answer correctly in Cartesian form.

End of Question 3

Question 4

Find the equation of the tangent to the function $x\log_e y + 2x^2 = 3$ at the point where $x = 1$.

Worked solution

$x\log_e y + 2x^2 = 3$

When $x = 1$:

$\log_e y + 2 = 3$

$\log_e y = 1$

$y = e$

$$\frac{d}{dx}\left(x\log_e y + 2x^2 = 3\right)$$

$$1.\log_e y + x\frac{d}{dy}(\log_e y)\frac{dy}{dx} + 4x = 0$$

$$\log_e y + \frac{x}{y}.\frac{dy}{dx} + 4x = 0$$

$$\frac{x}{y}.\frac{dy}{dx} = -4x - \log_e y$$

$$\frac{dy}{dx} = \frac{y}{x}(-4x - \log_e y)$$

Substitute $x = 1$ and $y = e$:

$$\frac{dy}{dx} = \frac{e}{1}(-4 - \log_e e)$$

$= e(-4-1)$

$= -5e =$ gradient of tangent

Using $y - y_1 = m(x - x_1)$

$y - e = -5e(x - 1)$

$y - e = -5ex + 5e$

$y = -5ex + 6e$

The equation of the tangent is $y = -5ex + 6e$.

4 marks

Mark allocation

- 1 mark for correctly evaluating y when $x = 1$.
- 1 mark for correctly differentiating the equation.
- 1 mark for correctly evaluating the gradient of the tangent.
- 1 mark for the correct answer.

Tip

- *Implicit differentiation is used to find $\frac{dy}{dx}$, otherwise it is possible to express y as a function of x and then find $\frac{dy}{dx}$.*

End of Question 4

TURN OVER

Question 5

a. Show that $\dfrac{2x}{4-x^2} = \dfrac{1}{2-x} - \dfrac{1}{2+x}$.

Worked solution

$$\frac{2x}{4-x^2} = \frac{2x}{(2-x)(2+x)} = \frac{a}{2-x} + \frac{b}{2+x}$$

$$\frac{2x}{(2-x)(2+x)} \equiv \frac{a(2+x)+b(2-x)}{(2-x)(2+x)}$$

$$2x \equiv a(2+x) + b(2-x)$$

Let $x = 2$, $4a = 4$

$a = 1$

Let $x = -2$, $4b = -4$

$b = -1$

$$\frac{2x}{4-x^2} = \frac{1}{2-x} - \frac{1}{2+x}$$

2 marks

Mark allocation

- 1 mark for correctly setting up the partial fractions.
- 1 mark for correct workings to find a and b.

b. Hence, or otherwise, find $\displaystyle\int_{-1}^{1} \frac{2x}{4-x^2}.dx$.

Worked solution

$$\int_{-1}^{1} \frac{2x}{4-x^2}.dx = \int_{-1}^{1}\left(\frac{1}{2-x} - \frac{1}{2+x}\right).dx$$

$$= \left[-\log_e|2-x| - \log_e|2+x|\right]_{-1}^{1}$$

$$= [-\log_e|1| - \log_e|3|] - [-\log_e|3| - \log_e|1|]$$

$$= -\log_e 3 + \log_e 3$$

$$= 0$$

2 marks

Mark allocation

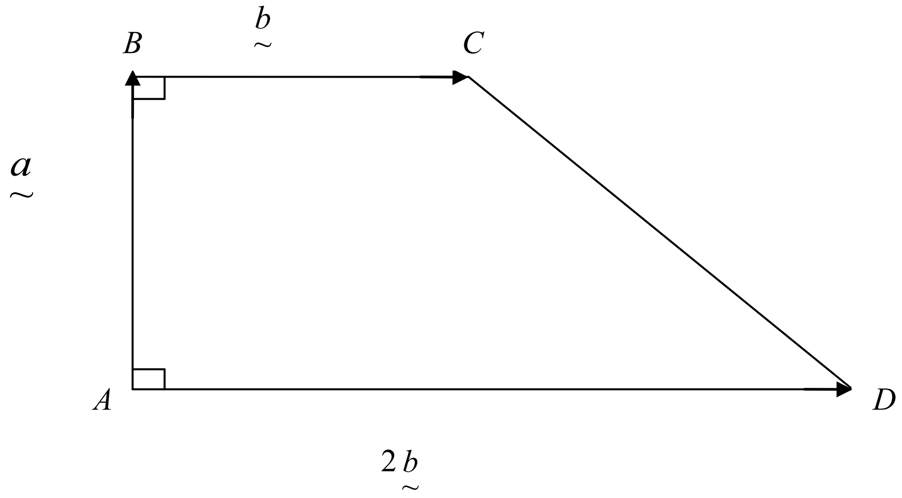
- 1 mark for correctly integrating.
- 1 mark for the correct answer.

Tip

- *The substitution $u = 4 - x^2$ could also have been used to set up an antidifferentiable integrand.*

End of Question 5

Question 6



Given that $|\underset{\sim}{a}| = |\underset{\sim}{b}|$ in the trapezium shown above, use vectors to show that $\triangle ACD$ is right angled.

Worked solution

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$
$$= \underset{\sim}{a} + \underset{\sim}{b}$$

$$\overrightarrow{CD} = -\overrightarrow{BC} - \overrightarrow{AB} + \overrightarrow{AD}$$
$$= -\underset{\sim}{b} - \underset{\sim}{a} + 2\underset{\sim}{b}$$
$$= \underset{\sim}{b} - \underset{\sim}{a}$$

$$\overrightarrow{AC}.\overrightarrow{CD} = (\underset{\sim}{b} + \underset{\sim}{a}).(\underset{\sim}{b} - \underset{\sim}{a})$$
$$= \underset{\sim}{b}.\underset{\sim}{b} - \underset{\sim}{b}.\underset{\sim}{a} + \underset{\sim}{a}.\underset{\sim}{b} - \underset{\sim}{a}.\underset{\sim}{a}$$
$$= b^2 - a^2$$
$$= 0 \text{ since } |\underset{\sim}{a}| = |\underset{\sim}{b}|$$

Therefore, $\triangle ABC$ is right angled at C because $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{CD}$.

3 marks

Mark allocation

- 1 mark for correctly expressing $\overrightarrow{AC}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.
- 1 mark for correctly expressing $\overrightarrow{CD}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.
- 1 mark for correctly showing the dot product of $\overrightarrow{AC}$ and $\overrightarrow{CD}$ is 0.

End of Question 6
TURN OVER

Question 7

A parachutist free-falls from rest and reaches a speed of $2g$ m/s when his parachute is activated.
The acceleration of the parachutist from this point is $(g - 2v)$ m/s^2, where v is the speed of the parachutist t seconds after the parachute is activated.

Five seconds after the parachute is activated, the speed of the parachutist is $\frac{g}{2}(a + be^{-10})$.

Find the values of a and b.
The acceleration due to gravity is $g = 9.8$ m/s^2.

Worked solution

$$a = \frac{dv}{dt} = g - 2v$$

$$\frac{dt}{dv} = \frac{1}{g - 2v}$$

$$t = \int \frac{1}{g - 2v}.dv$$

$t = -\frac{1}{2}\log_e\left|k(g - 2v)\right|$, where k is the constant of integration

When $t = 0$, $v = 2g$:

$$0 = -\frac{1}{2}\log_e\left|k(g - 4g)\right|$$

$$0 = -\frac{1}{2}\log_e\left|k(-3g)\right|$$

$-3kg = 1$ since $\ln 1 = 0$.

$$k = \frac{-1}{3g}$$

$$t = \frac{-1}{2}\log_e\left|\frac{-(g - 2v)}{3g}\right|$$

$$t = \frac{-1}{2}\log_e\left|\frac{2v - g}{3g}\right|$$

$$\log_e\left|\frac{2v - g}{3g}\right| = -2t$$

$$\left|\frac{2v - g}{3g}\right| = e^{-2t}$$

$$\frac{2v - g}{3g} = \pm e^{-2t}$$

Question 7 – continued

When $t = 0$, $v = 2g$

and $\dfrac{2v-g}{g} = \dfrac{3g}{3g} = 1 = +e^{0}$

So $\dfrac{2v-g}{3g} = e^{-2t}$

$2v - g = 3ge^{-2t}$

$2v = g + 3ge^{-2t}$

$v = \dfrac{1}{2}\left(g + 3ge^{-2t}\right)$

$v(5) = \dfrac{1}{2}\left(g + 3ge^{-10}\right)$

$v(5) = \dfrac{g}{2}\left(1 + 3e^{-10}\right)$

Hence, $a = 1$ and $b = 3$.

The speed of the parachutist after 5 seconds is $\dfrac{g}{2}\left(1 + 3e^{-10}\right)$ m/s.

4 marks

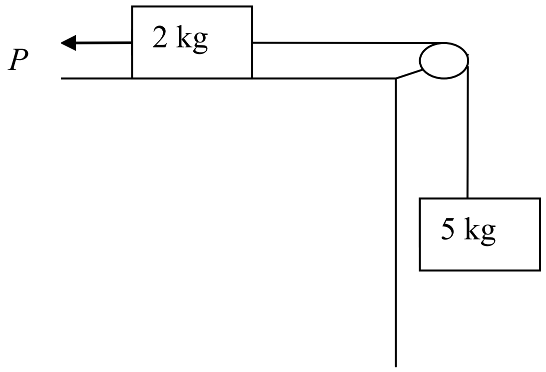
Mark allocation

- 1 mark for setting up the correct integral for t.
- 1 mark for the correct integration and evaluation of the constant.
- 1 mark for correctly expressing v as a function of t.
- 1 mark for correctly finding the values of a and b.

End of Question 7
TURN OVER

Question 8

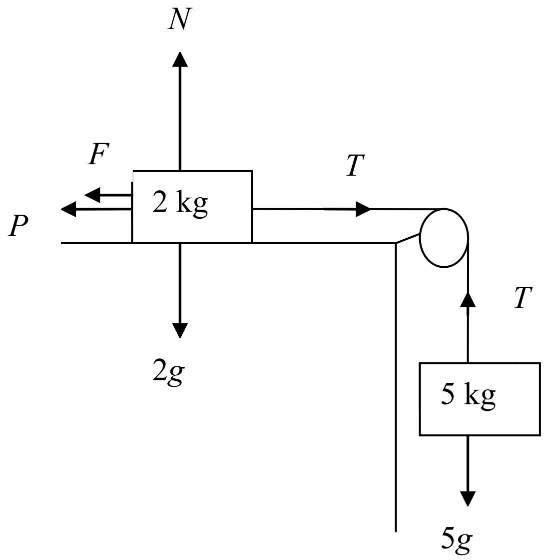
A 2 kg mass on a horizontal bench is connected by an inextensible string via a smooth pulley to a 5 kg mass, as shown below. The force of *P* newtons is acting on the 2 kg mass, as shown. The coefficient of friction between the 2 kg mass and the bench is 0.25.



a. Label all forces on the diagram above.

Worked solution

a.



1 mark

Mark allocation

- 1 mark for correctly labelling all the forces.

Tip

- *The minimum value of the force P required occurs when the 2 kg mass is on the verge of moving to the right.*

Question 8 – continued

b. Show that the minimum value of the force P required to maintain equilibrium is $4.5g$ newtons.

Worked solution

The minimum value of P required to maintain equilibrium occurs when the friction, F, is maximum.

Resolving the forces:

<u>For the 2 kg mass</u>

$N = 2g$

$F = N\mu$

$= 0.25 \times 2g$

$= 0.5g$

<u>For the combined system</u>

$R = ma = 0$

$R = 5g - T + T - P - F = 0$

$5g - P - 0.5g = 0$

$P = 4.5g$

2 marks

Mark allocation

- 1 mark for correctly finding the maximum friction.
- 1 mark for the correct equation of motion for the combined system.

c. If $P = 2.5g$, find the exact acceleration of the system.

Worked solution

$R = 5g - T + T - F - P = ma$

For motion $F_{max} = N\mu = 0.5g$

$R = 5g - T + T - 0.5g - 2.5g = (5 + 2)a$

$7a = 2g$

$$a = \frac{2g}{7}$$

2 marks

Mark allocation

- 1 mark for determining the correct equation of motion.
- 1 mark for the correct answer.

End of Question 8
TURN OVER

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Question 9

Evaluate $\int_{e}^{e^2} \frac{1}{x \ln x} .dx$.

Worked solution

$$\int_{e}^{e^2} \frac{1}{x \log_e x} .dx = \int_{e}^{e^2} \frac{1}{x} . \frac{1}{\log_e x} .dx$$

Let $u = \log_e x$

$$\frac{du}{dx} = \frac{1}{x}$$

$x = e^2$, $u = \log_e e^2 = 2$

$x = e$, $u = \log_e e = 1$

$$\begin{aligned}
\int_{e}^{e^2} \frac{1}{x} . \frac{1}{\log_e x} .dx &= \int_{1}^{2} \frac{du}{dx} . \frac{1}{u} .dx \\
&= \int_{1}^{2} \frac{1}{u} .du \\
&= \left[\log_e |u|\right]_1^2 \\
&= \log_e 2 - \log_e 1 \\
&= \log_e 2
\end{aligned}$$

3 marks

Mark allocation

- 1 mark for using the correct substitution for u.
- 1 mark for setting up the correct integral in terms of u.
- 1 mark for the correct answer.

End of Question 9
TURN OVER

Question 10

$S = \{z: (z+1)(\bar{z}+1) = 4,\ z \in C\}$ and $T = \{z: \text{Arg}\,(z+1) = \frac{-3\pi}{4},\ z \in C\}$.

a. Find the Cartesian equations of S and T.

Worked solution

Cartesian equation for S:

$((x+1)+yi)((x+1)-yi) = 4$

$(x+1)^2 + y^2 = 4$

S is represented by a circle of centre $(-1, 0)$ and radius = 2.

Cartesian equation for T:

$$\tan^{-1}\left(\frac{y}{x+1}\right) = \frac{-3\pi}{4}$$

$$\frac{y}{x+1} = \tan\frac{-3\pi}{4}$$

$$\frac{y}{x+1} = 1,\ x < -1$$

$y = x + 1$, $x < -1$, which is a ray in the third quadrant.

T is represented by a the straight line $y = x + 1$, $x < -1$.

2 marks

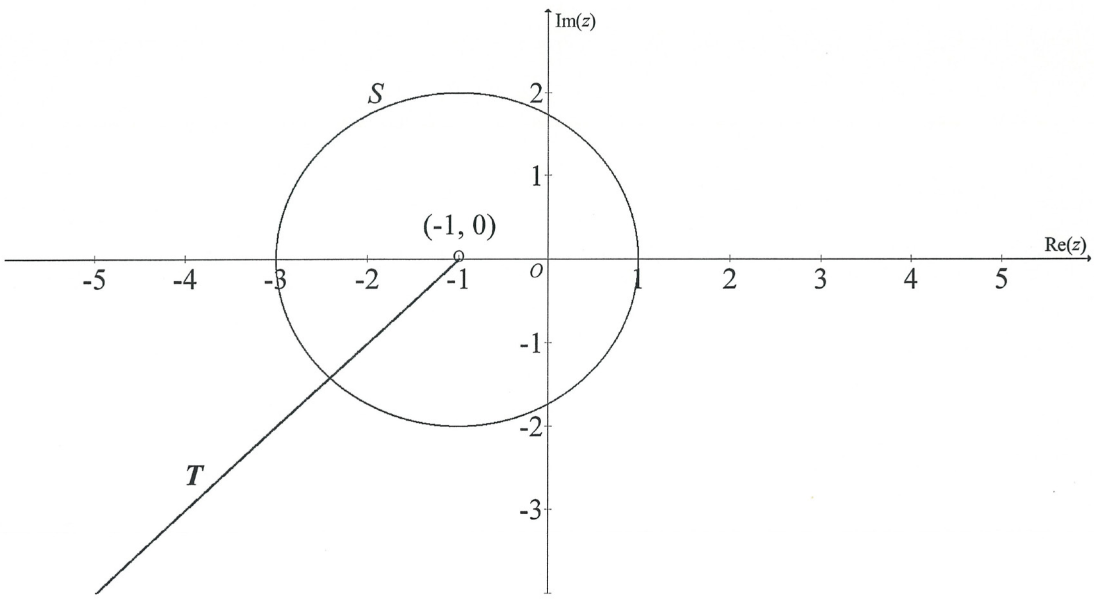
Mark allocation

- 1 mark for the correct Cartesian equation for S.
- 1 mark for the correct Cartesian equation for T.

Question 10 – continued

b. Show the subsets S and T on the complex plane below.

Worked solution



1 mark

Mark allocation

- 1 mark for the correct graph of S and T.

END OF SOLUTIONS BOOK