

**THE
HEFFERNAN
GROUP**

P.O. Box 1180
Surrey Hills North VIC 3127
Phone 03 9836 5021
Fax 03 9836 5025
info@theheffernangroup.com.au
www.theheffernangroup.com.au

**SPECIALIST MATHS
TRIAL EXAMINATION 1
SOLUTIONS
2010**

Question 1

$z = 1 - 2i$ is a solution so $z = 1 + 2i$ is also a solution (conjugate root theorem).

(1 mark)

Now, $(z - 1 + 2i)(z - 1 - 2i)$
 $= z^2 - z - 2iz - z + 1 + 2i + 2iz - 2i + 4$
 $= z^2 - 2z + 5$

Method 1

$$\begin{array}{r} z + 4 \\ z^2 - 2z + 5 \overline{) z^3 + 2z^2 - 3z + 20} \\ \underline{z^3 - 2z^2 + 5z} \\ 4z^2 - 8z + 20 \\ \underline{4z^2 - 8z + 20} \\ 0 \end{array}$$

So $z^3 + 2z^2 - 3z + 20 = (z^2 - 2z + 5)(z + 4) = 0$.
The other solution is $z = -4$.

(1 mark)

Method 2

$$\begin{aligned} z^3 + 2z^2 - 3z + 20 &= (z^2 - 2z + 5)(z + 4) \\ &= z(z^2 - 2z + 5) + 4(z^2 - 2z + 5) \\ &= z(z^2 - 2z + 5) + 4z^2 - 8z + 20 \\ &= (z + 4)(z^2 - 2z + 5) \end{aligned}$$

So $z^3 + 2z^2 - 3z + 20 = (z^2 - 2z + 5)(z + 4) = 0$.
The other solution is $z = -4$.

(1 mark)

Question 2

- a. $A(2,1,\sqrt{15}), B(2,-4,0), O(0,0,0)$

$$\begin{aligned}\vec{AB} &= \vec{AO} + \vec{OB} \\ &= -(\vec{OA}) + (\vec{OB}) \\ &= -2\vec{i} - \vec{j} - \sqrt{15}\vec{k} + 2\vec{i} - 4\vec{j} \\ &= -5\vec{j} - \sqrt{15}\vec{k}\end{aligned}$$

(1 mark)

- b. To prove: $\triangle ABO$ contains a right angle and has two sides of equal length.

$$\begin{aligned}\vec{AB} \cdot \vec{AO} &= (-5\vec{j} - \sqrt{15}\vec{k}) \cdot (-2\vec{i} - \vec{j} - \sqrt{15}\vec{k}) \\ &= 0 + 5 + 15 \neq 0 \\ \vec{AB} \cdot \vec{BO} &= (-5\vec{j} - \sqrt{15}\vec{k}) \cdot (-2\vec{i} + 4\vec{j}) \\ &= 0 - 20 + 0 \neq 0 \\ \vec{OA} \cdot \vec{OB} &= (2\vec{i} + \vec{j} + \sqrt{15}\vec{k}) \cdot (2\vec{i} - 4\vec{j}) \\ &= 4 - 4 = 0\end{aligned}$$

So $\vec{OA}$ is at right angles to $\vec{OB}$ so $\triangle ABO$ contains a right angle.

(1 mark)

$$\begin{aligned}|\vec{AB}| &= \sqrt{25 + 15} = \sqrt{40} \\ |\vec{AO}| &= \sqrt{4 + 1 + 15} = \sqrt{20} \\ |\vec{BO}| &= \sqrt{4 + 16} = \sqrt{20}\end{aligned}$$

Since $|\vec{AO}| = |\vec{BO}|$, $\triangle ABO$ contains two sides of equal length.

So $\triangle ABO$ is a right-angled, isosceles triangle.

(1 mark)

Question 3

a.

$$3y^2 + 4x - 2x^2y = 5$$

$$6y \frac{dy}{dx} + 4 - 2x^2 \frac{dy}{dx} - 4xy = 0$$

(1 mark)

$$\frac{dy}{dx}(6y - 2x^2) = 4xy - 4$$

$$\frac{dy}{dx} = \frac{4xy - 4}{6y - 2x^2}$$

$$\frac{dy}{dx} = \frac{2xy - 2}{3y - x^2}$$

(1 mark)

b.

When $x=1$,

$$3y^2 + 4x - 2x^2y = 5$$

becomes $3y^2 + 4 - 2y = 5$

$$3y^2 - 2y - 1 = 0$$

$$(3y+1)(y-1) = 0$$

$$y = -\frac{1}{3} \text{ or } y = 1$$

The first quadrant point is (1,1).

(1 mark)

$$\frac{dy}{dx} = \frac{2xy - 2}{3y - x^2}$$

$$= \frac{2 - 2}{3 - 1}$$

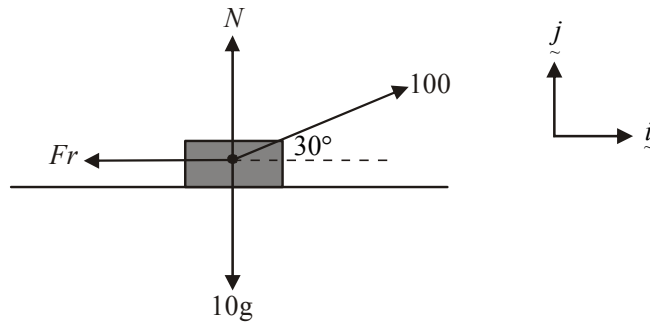
$$= 0$$

The gradient at the point where $x=1$ is zero.

(1 mark)

Question 4

a.



(1 mark)

b.

$$\vec{R} = m \vec{a}$$

$$(100 \cos 30^\circ - Fr) \vec{i} + (N + 100 \sin 30^\circ - 10g) \vec{j} = m \vec{a}$$

Resolving horizontally:

$$100 \cos(30^\circ) - Fr = 10 \times 8$$

$$\frac{100\sqrt{3}}{2} - \mu N = 80$$

$$\mu N = 50\sqrt{3} - 80$$

$$\mu = \frac{50\sqrt{3} - 80}{N} \quad \text{---(1)}$$

(1 mark)

Resolving vertically:

$$N + 100 \sin(30^\circ) = 10g$$

$$N = 98 - 50$$

$$= 48$$

$$\begin{aligned} \text{In (1)} \quad \mu &= \frac{50\sqrt{3} - 80}{48} \\ &= \frac{25\sqrt{3} - 40}{24} \end{aligned}$$

(1 mark)

c.

At the point at which the crate begins to move $Fr = \mu N$.

Resolving vertically:

$$N + 8 \sin(30^\circ) = 10g$$

$$N = 98 - 4$$

$$= 94$$

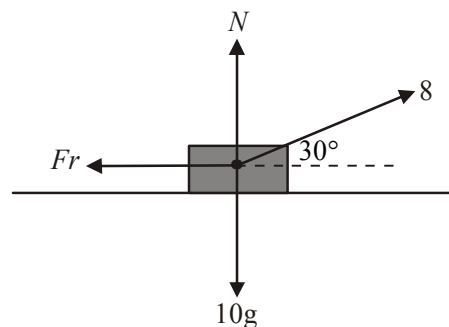
$$\begin{aligned} \text{So at the point of moving } Fr &= \frac{1}{10} \times 94 \\ &= 9.4 \end{aligned}$$

Resolving horizontally:

$$Fr = 8 \cos(30^\circ)$$

$$= \frac{8\sqrt{3}}{2}$$

$$= 4\sqrt{3}$$



(1 mark)

Since $4\sqrt{3} < 9.4$, the crate does not move.

(1 mark)

Question 5

$$\frac{dy}{dx} = 3\sqrt{4-y^2}$$

$$\frac{dx}{dy} = \frac{1}{3\sqrt{4-y^2}}$$

$$x = \frac{1}{3} \int \frac{1}{\sqrt{4-y^2}} dy$$

$$x = \frac{1}{3} \arcsin\left(\frac{y}{2}\right) + c$$

(1 mark)Given $y(0) = 2$

$$0 = \frac{1}{3} \arcsin(1) + c$$

$$c = -\frac{1}{3} \times \frac{\pi}{2}$$

$$c = -\frac{\pi}{6}$$

$$x = \frac{1}{3} \arcsin\left(\frac{y}{2}\right) - \frac{\pi}{6}$$

(1 mark)

$$3\left(x + \frac{\pi}{6}\right) = \arcsin\left(\frac{y}{2}\right)$$

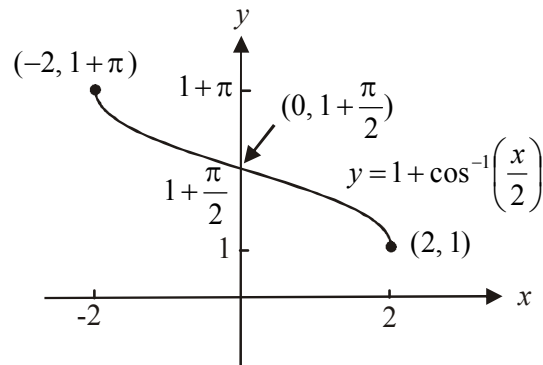
$$\sin\left(3\left(x + \frac{\pi}{6}\right)\right) = \frac{y}{2}$$

$$y = 2 \sin\left(3\left(x + \frac{\pi}{6}\right)\right)$$

(1 mark)

Question 6

a.



(1 mark) – correct endpoints

(1 mark) – correct y-intercept and shape

b. i. $d_f = [-2, 2]$

(1 mark)

ii. $r_f = [1, 1 + \pi]$

(1 mark)

c. $f(x) = 1 + \cos^{-1}\left(\frac{x}{2}\right)$

$$f'(x) = \frac{-1}{\sqrt{4-x^2}}$$

When $x = \sqrt{3}$

$$f'(x) = \frac{-1}{1} = -1$$

The gradient of the normal is therefore 1.

(1 mark)

$$f(\sqrt{3}) = 1 + \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$= 1 + \frac{\pi}{6}$$

At the point $\left(\sqrt{3}, 1 + \frac{\pi}{6}\right)$,

$y - y_1 = m(x - x_1)$ becomes

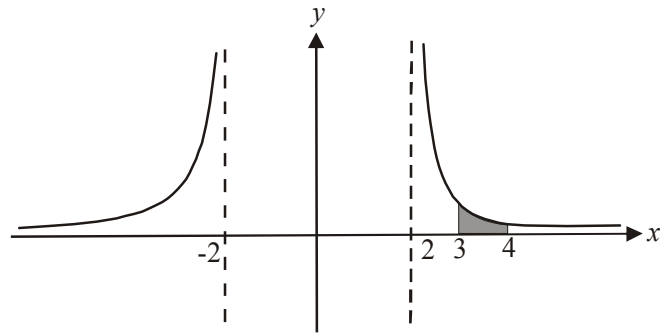
$$y - \left(1 + \frac{\pi}{6}\right) = 1(x - \sqrt{3})$$

$$y = x - \sqrt{3} + 1 + \frac{\pi}{6} \text{ is the equation of the normal.}$$

(1 mark)

Question 7

$$\begin{aligned}
 y &= \frac{2}{\sqrt{x^2 - 4}} \\
 \text{Volume} &= \pi \int_3^4 y^2 dx \\
 &= \pi \int_3^4 \frac{4}{x^2 - 4} dx \\
 &= 4\pi \int_3^4 \frac{1}{x^2 - 4} dx
 \end{aligned}$$



(1 mark)

$$\begin{aligned}
 \text{Let } \frac{1}{x^2 - 4} &\equiv \frac{A}{x - 2} + \frac{B}{x + 2} \\
 &\equiv \frac{A(x + 2) + B(x - 2)}{(x - 2)(x + 2)}
 \end{aligned}$$

$$\text{True iff } 1 \equiv A(x + 2) + B(x - 2)$$

$$\text{Put } x = -2, \quad 1 = -4B \quad B = -\frac{1}{4}$$

$$\text{Put } x = 2, \quad 1 = 4A \quad A = \frac{1}{4}$$

$$\text{So } \frac{1}{x^2 - 4} = \frac{1}{4(x - 2)} - \frac{1}{4(x + 2)}$$

(1 mark)

$$\text{Volume} = 4\pi \int_3^4 \left(\frac{1}{4(x - 2)} - \frac{1}{4(x + 2)} \right) dx$$

$$= \pi \int_3^4 \left(\frac{1}{x - 2} - \frac{1}{x + 2} \right) dx$$

(1 mark)

$$= \pi [\log_e |x - 2| - \log_e |x + 2|]_3^4$$

$$= \pi \{ (\log_e(2) - \log_e(6)) - (\log_e(1) - \log_e(5)) \}$$

$$= \pi (\log_e(2) - \log_e(6) - 0 + \log_e(5))$$

$$= \pi \left\{ \log_e \left(\frac{2 \times 5}{6} \right) \right\}$$

$$= \pi \log_e \left(\frac{5}{3} \right) \text{ cubic units}$$

(1 mark)

Question 8

$$a = v - 5$$

$$v \frac{dv}{dx} = v - 5$$

(1 mark)

$$\frac{dv}{dx} = \frac{v-5}{v}$$

$$\frac{dx}{dv} = \frac{v}{v-5}$$

Method 1

$$x = \int \frac{v}{v-5} dv$$

$$\text{let } u = v - 5$$

$$= \int \frac{1}{u} \times (u+5) \frac{du}{dv} dv$$

$$\text{so } \frac{du}{dv} = 1 \quad \text{and} \quad v = u + 5$$

$$= \int \left(1 + \frac{5}{u} \right) du$$

$$x = u + 5 \log_e |u| + c$$

$$x = v - 5 + 5 \log_e |v - 5| + c$$

(1 mark)When $x = 2$, $v = 6$

$$2 = 6 - 5 + 5 \log_e (1) + c$$

$$c = 1$$

$$\text{So } x = v - 4 + 5 \log_e |v - 5|$$

(1 mark)Method 2Since $\frac{v}{v-5}$ is an improper fraction we divide.

$$\begin{array}{r} 1 \\ v-5 \overline{) v} \\ \underline{v-5} \\ 5 \end{array}$$

$$\text{So } \frac{v}{v-5} = 1 + \frac{5}{v-5}$$

$$\text{So } \frac{dx}{dv} = 1 + \frac{5}{v-5}$$

$$x = \int \left(1 + \frac{5}{v-5} \right) dv$$

$$= v + 5 \log_e |v - 5| + c$$

(1 mark)When $x = 2$, $v = 6$

$$2 = 6 + 5 \log_e (1) + c$$

$$c = -4$$

$$\text{So } x = v - 4 + 5 \log_e |v - 5|$$

(1 mark)

Question 9

a. Let $z = 2 + 2\sqrt{3}i$

$$r = \sqrt{4 + 4 \times 3}$$

$$= \sqrt{16}$$

$$= 4$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$= \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right)$$

$$= \tan^{-1}(\sqrt{3})$$

$$= \frac{\pi}{3} \text{ since } z \text{ is a first quadrant angle.}$$

$$\text{So } 2 + 2\sqrt{3}i = 4\text{cis}\left(\frac{\pi}{3}\right)$$

(1 mark)

b. $\sqrt{3}z^2 + \sqrt{2}z - \frac{i}{2} = 0$

This is a quadratic equation in z .

$$\begin{aligned} z &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-\sqrt{2} \pm \sqrt{2 - 4 \times \sqrt{3} \times -\frac{i}{2}}}{2\sqrt{3}} \\ &= \frac{-\sqrt{2} \pm \sqrt{2 + 2\sqrt{3}i}}{2\sqrt{3}} \end{aligned}$$

(1 mark)

$$\text{Now } \sqrt{2 + 2\sqrt{3}i} = \sqrt{4\text{cis}\left(\frac{\pi}{3}\right)} \text{ from part a.}$$

$$= \sqrt{4}\text{cis}\left(\frac{\pi}{3} \times \frac{1}{2}\right) \text{ De Moivre}$$

$$= 2\text{cis}\left(\frac{\pi}{6}\right)$$

$$= 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$$

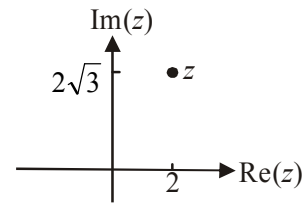
$$= 2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$$

$$= \sqrt{3} + i$$

(1 mark)

$$\text{So } z = \frac{-\sqrt{2} \pm (\sqrt{3} + i)}{2\sqrt{3}}$$

$$z = \frac{-\sqrt{2} + \sqrt{3} + i}{2\sqrt{3}} \text{ or } z = \frac{-\sqrt{2} - \sqrt{3} - i}{2\sqrt{3}}$$

(1 mark) – correct answers

Question 10

$$\begin{aligned}
\text{Area} &= \int_0^1 (f(x) - g(x)) dx \\
&= \int_0^1 \left(\frac{2}{4+x^2} - \frac{1-x}{\sqrt{4-x^2}} \right) dx && \text{(1 mark)} \\
&= \int_0^1 \frac{2}{4+x^2} dx - \int_0^1 \frac{1}{\sqrt{4-x^2}} dx + \int_0^1 \frac{x}{\sqrt{4-x^2}} dx \\
&= \left[\tan^{-1}\left(\frac{x}{2}\right) - \sin^{-1}\left(\frac{x}{2}\right) \right]_0^1 + \int_4^3 -\frac{1}{2} \frac{du}{dx} u^{-\frac{1}{2}} dx \\
&= \left\{ \left(\tan^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right) \right) - (\tan^{-1}(0) - \sin^{-1}(0)) \right\} - \frac{1}{2} \int_4^3 u^{-\frac{1}{2}} du \\
&= \tan^{-1}\left(\frac{1}{2}\right) - \frac{\pi}{6} - 0 - 0 - \frac{1}{2} \left[2u^{\frac{1}{2}} \right]_4^3 \\
&= \tan^{-1}\left(\frac{1}{2}\right) - \frac{\pi}{6} - \frac{1}{2} \{2\sqrt{3} - 2\sqrt{4}\} \\
&= \tan^{-1}\left(\frac{1}{2}\right) - \frac{\pi}{6} - \sqrt{3} + 2 \text{ square units}
\end{aligned}$$

$$\text{let } u = 4 - x^2$$

$$\frac{du}{dx} = -2x$$

$$x=1, \quad u=3$$

$$x=0, \quad u=4$$

$$\text{(1 mark) - } \tan^{-1}\left(\frac{x}{2}\right)$$

$$\text{(1 mark) - } \sin^{-1}\left(\frac{x}{2}\right)$$

(1 mark) for $2u^{\frac{1}{2}}$ and correct terminals

(1 mark) for correct substitution of terminals

(1 mark) for correct answer