

SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 2



2009 Trial Examination

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: C

Explanation:

The asymptotes of the hyperbola of the form $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$ are of the form

$$y - k = \pm \frac{b}{a}(x - h), \quad a = 2, b = 4$$

$$y - 3 = \pm 2(x + 1)$$

$$2x - y + 5 = 0$$

$$2x + y - 1 = 0$$

The asymptote $2x + y - 1 = 0$ is offered as alternative C.

Question 2

Answer: D

Explanation:

$$f(x) = \frac{x+a}{x^2 - 2ax - 3a^2} = \frac{x+a}{(x+a)(x-3a)}$$

$f(x)$ has a vertical asymptote $x = 3a$ and a hole for $x = -a$. Therefore the first three options are incorrect.

D is a correct option because: $\lim_{x \rightarrow -a} f(x) = \lim_{x \rightarrow -a} \frac{x+a}{(x+a)(x-3a)} = \lim_{x \rightarrow -a} \frac{1}{x-3a} = -\frac{1}{4a}$.

Question 3*Answer: A**Explanation:*

$y = \cos(\sin^{-1} 2x)$ is a composite function $y = f \circ g$. The range of g must be a subset of the domain of f .

The range of $\sin^{-1} 2x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, while the domain of $\cos x$ is $[0, \pi]$. Therefore, $\sin^{-1} 2x$ must be restricted so that its range is $\left[0, \frac{\pi}{2}\right]$.

Domain of y

$$0 \leq \sin^{-1} 2x \leq \frac{\pi}{2}$$

$$0 \leq 2x \leq 1$$

$$0 \leq x \leq \frac{1}{2}$$

$$\text{Domain is } \left[0, \frac{1}{2}\right]$$

Range of y

$$\cos(\sin^{-1} 0) = \cos 0 = 1$$

$$\cos\left(\sin^{-1} 2 \times \frac{1}{2}\right) = \cos \frac{\pi}{2} = 0$$

$$\text{Range is } [0, 1]$$

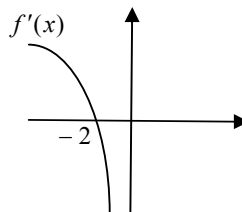
Question 4*Answer: D**Explanation:*

Option A can be correct: $f''(x)$ is discontinuous for $x = 0$, so $f(x)$ is also discontinuous for $x = 0$.

Option B can be correct: $f'(-2)$ can equal 0.

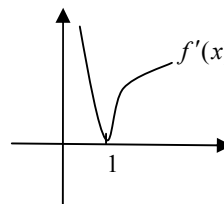
For example, the graph on the right shows $f'(x)$.

The derivative of this function ($f''(x)$) is negative for every $x < 0$.



Option C can be correct: $f'(x)$ can be positive for every $x > 0$.

The function ($f'(x)$) on the right has a negative gradient ($f''(x)$) for all $x < 1$, a gradient of zero for $x = 1$ and a positive gradient for $x > 1$.



Option D is incorrect: For $f(x)$ to have a local maximum at $x = 1$, according to second derivative test, $f''(1)$ would have to be less than zero, while in the given graph $f''(1) = 0$.

Option E can be correct: $f''(1) = 0$, so $f(x)$ can have a stationary point of inflexion for $x = 1$.

Question 5*Answer:* B*Explanation:*

Let $z = a + bi$, $a, b \geq 0$ and $a = 4b$. Then $z = 4b + bi$.

$$\begin{aligned} z^2 &= b^2(4+i)^2 \\ &= b^2(15+8i) \end{aligned}$$

$$\operatorname{Re} z^2 = 15b^2$$

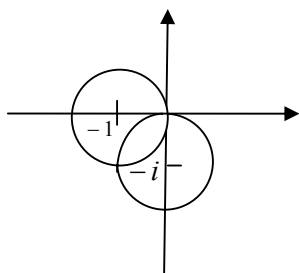
$$\operatorname{Im} z^2 = 8b^2$$

$$\operatorname{Re} z^2 : \operatorname{Im} z^2 = 15 : 8$$

Question 6*Answer:* C*Explanation:*

Complex number z which satisfies $|z+1| = |z+i| = 1$ is represented by the point of intersection of two circles: $|z+1| = 1$ and $|z+i| = 1$. From the diagram below, as z is a non-zero number,

we have $z = -1 - i$ and $|z| = \sqrt{2}$, $\operatorname{Arg} z = -\frac{3\pi}{4}$.

**Question 7***Answer:* E*Explanation:*

If $z = 2 - i\sqrt{3}$ is a solution of $z^4 + az^3 + bz^2 - 32z + 56 = 0$, $a, b \in R$, then $z = 2 + i\sqrt{3}$ is also a solution (conjugate root theorem). It follows that one quadratic factor is

$$z^2 - (2 - i\sqrt{3} + 2 + i\sqrt{3})z + (2 - i\sqrt{3})(2 + i\sqrt{3}) = 0$$

$$z^2 - 4z + 7 = 0$$

$$z^4 + az^3 + bz^2 - 32z + 56 = (z^2 - 4z + 7)(z^2 + kz + 8)$$

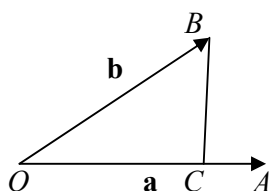
By equating coefficients of z : $-32 + 7k = -32 \Rightarrow k = 0$, $z^2 + 8$ is the other quadratic factor.

$$(z^2 - 4z + 7)(z^2 + 8) = z^4 - 4z^3 + 15z^2 - 32z + 56 \Rightarrow a = -4, b = 15$$

Question 8*Answer:* A*Explanation:*

The shortest distance of point B from the line OA is the magnitude of vector $\vec{BC}$, which is the perpendicular component of vector $\vec{OB}$ in the direction of $\vec{OA}$. Let $\vec{OA} = \mathbf{a}$ and $\vec{OB} = \mathbf{b}$.

Then $\vec{OC} = (\mathbf{b} \cdot \hat{\mathbf{a}})\hat{\mathbf{a}}$, $\hat{\mathbf{a}} = \frac{1}{\sqrt{10}}(3\mathbf{i} + \mathbf{j})$



$$\vec{OC} = \frac{1}{10} \times 15(3\mathbf{i} + \mathbf{j}) = \frac{3}{2}(3\mathbf{i} + \mathbf{j})$$

$$|\vec{OC}| = \sqrt{\frac{90}{4}}, \quad |\vec{OB}| = 5$$

$$|\vec{BC}| = \sqrt{25 - \frac{90}{4}} = \sqrt{\frac{10}{4}} = \frac{\sqrt{10}}{2}$$

Question 9*Answer:* C*Explanation:*

$$|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$$

$$= |\mathbf{a}|^2 + 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2$$

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\frac{2\pi}{3} = 4 \times 3 \times \frac{-1}{2} = -6$$

$$|\mathbf{a} + \mathbf{b}|^2 = 16 - 12 + 9 = 13.$$

$$\text{Thus, } |\mathbf{a} + \mathbf{b}| = \sqrt{13}$$

Question 10*Answer:* E*Explanation*

The equation $a \tan^2 x + b \cot^2 x = a$, $a, b \in R^+$ can be written as

$$a \tan^2 x + \frac{b}{\tan^2 x} = a \quad (1)$$

$$a \tan^4 x - a \tan^2 x + b = 0$$

Let $t = \tan^2 x$. The equation (1) becomes $at^2 - at + b = 0$. It has a solution if the discriminant is greater than or equal to zero.

Discriminant $= a^2 - 4ab \geq 0$ for $a \in (-\infty, 0] \cup [4b, \infty)$. Because $a \in R^+$, the smallest positive value of a from the interval $(-\infty, 0] \cup [4b, \infty)$ is $4b$. Therefore, $a = 4b$ is the required value.

Question 11*Answer:* E*Explanation:*

The gradient of the tangent to the curve $y = \arcsin \frac{k}{x}$ at the point $\left(2k, \frac{\pi}{6}\right)$ is $y'(2k)$. First,

finding the derivative: $y' = \frac{1}{\sqrt{1 - \frac{k^2}{x^2}}} \times \left(-\frac{k}{x^2}\right) = -\frac{k}{x\sqrt{x^2 - k^2}}$. The gradient when $x = 2k$ is

$$y'(2k) = -\frac{k}{2k\sqrt{3k^2}} = -\frac{1}{2k\sqrt{3}} = -\frac{\sqrt{3}}{6k}. \quad \text{The angle with the } x\text{-axis} = \arctan\left(-\frac{\sqrt{3}}{6k}\right).$$

Question 12*Answer:* B*Explanation:*

$$\text{Let } \cos^{-1} \frac{2}{\sqrt{m^2 + 4}} = x. \quad \text{Then } \cos x = \frac{2}{\sqrt{m^2 + 4}}.$$

$$\text{Using the identity } \sec^2 x = 1 + \tan^2 x, \quad \tan^2 x = \frac{m^2 + 4}{4} - 1 = \frac{m^2}{4}.$$

As inverse cosine is restricted to $[0, \pi]$, x must be from the first quadrant. Thus, $\tan x = \frac{m}{2}$.

$$\tan\left(2 \cos^{-1} \frac{2}{\sqrt{m^2 + 4}}\right) = \tan 2x = \frac{2 \times \frac{m}{2}}{1 - \frac{m^2}{4}} = \frac{4m}{4 - m^2}$$

Question 13*Answer:* E*Explanation:*

Option A is correct as $\frac{\sin 2x}{1 + \cos 2x} = \frac{2 \sin x \cos x}{1 + \cos^2 x - \sin^2 x} = \frac{2 \sin x \cos x}{2 \cos^2 x} = \tan x$

Option B: If $u = 1 + \cos 2x$, then $du = -2 \sin 2x dx$ and $\sin x dx = -\frac{1}{2} du$. Substitution into the integral gives option B.

Option C is correct (similar to option B)

Option D: $\frac{\sin 2x}{1 + \cos 2x} = \frac{2 \sin x \cos x}{1 + 1 - 2 \sin^2 x} = \frac{2 \sin x \cos x}{2(1 - \sin^2 x)} = \frac{\sin x \cos x}{(1 - \sin^2 x)}$

Let $u = \sin x$. Then $du = \cos x dx$ and $I = \int \frac{u}{1 - u^2} du = \frac{1}{2} \int \frac{1}{1 - u} - \frac{1}{1 + u} du$

Therefore, option D is correct.

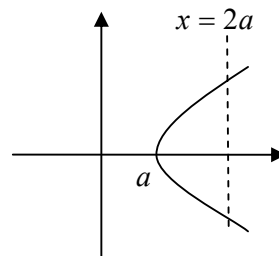
Option E is incorrect: $\frac{(\sin x + \cos x)^2}{1 + \sin 2x} = \frac{\sin^2 x + 2 \sin x \cos x + \cos^2 x}{1 + 2 \sin x \cos x} = \frac{1 + 2 \sin x \cos x}{1 + 2 \sin x \cos x} = 1$

Question 14*Answer:* D*Explanation:*

$$V = \pi \int_a^{2a} y^2 dx, \text{ where } y^2 = \frac{b^2}{a^2} (x^2 - a^2)$$

$$V = \frac{b^2 \pi}{a^2} \int_a^{2a} (x^2 - a^2) dx = \frac{b^2 \pi}{a^2} \left[\frac{x^3}{3} - a^2 x \right]_a^{2a}$$

$$V = \frac{b^2 \pi}{a^2} \left[\frac{8a^3}{3} - 2a^3 - \frac{a^3}{3} + a^3 \right] = \frac{b^2 \pi}{a^2} \times \frac{4a^3}{3} = \frac{4ab^2 \pi}{3}$$

**Question 15***Answer:* A*Explanation:*

The differential equation for the melting block is $\frac{dm}{dt} = \frac{k}{m}$. Solving: $\frac{dt}{dm} = \frac{m}{k} \Rightarrow t = \frac{m^2}{2k} + c$

When $t = 0$, $m = 100$ and $\frac{10000}{2k} + c = 0 \Rightarrow c = -\frac{5000}{k}$. When $t = 2$, $m = 50$ and after

substituting c , we have: $\frac{2500}{2k} - \frac{5000}{k} = 2 \Rightarrow k = -1875$ and thus $c = \frac{8}{3} = 2.67$.

For $m = 0$, $t = c$, therefore the block will melt in 2.67 days.

Question 16*Answer:* B*Explanation:*

From the slope field given, it can be concluded that $\frac{dy}{dx} = 0$ when $x = -y$. Also $\frac{dy}{dx}$ is undefined when $x = y$.

$\frac{dy}{dx} = \frac{x+y}{x-y}$ is the only differential equation among offered options that satisfies both conditions.

Question 17*Answer:* D*Explanation:*

$$x_{n+1} = x_n + h, \quad y_{n+1} = y_n + hf(x_n, y_n), \quad x_0 = 0, y_0 = 1, h = 0.2, \quad f(x, y) = y + e^{-2x} - 1$$

$$x_0 = 0 \quad y_1 = 1 + 0.2(1 + e^0 - 1) = 1.2$$

$$\begin{aligned} x_1 = 0.2, \quad y_2 &= 1.2 + 0.2(1.2 + e^{-0.4} - 1) \\ &= 1.2 + 0.2(0.2 + e^{-0.4}) \\ &= 1.2 + 0.04 + 0.2e^{-0.4} \\ &= 1.24 + 0.2e^{-0.4} \end{aligned}$$

Question 18*Answer:* C*Explanation:*

$$x_{up} = 5gt - \frac{1}{2}gt^2, \quad x_{down} = \frac{1}{2}gt^2$$

$$v_{up} = 5g - gt, \quad v_{down} = gt$$

When the balls meet: $v_{down} = 4v_{up}$

$$gt = 4(5g - gt) \Rightarrow t = 4$$

$$\text{Distance traveled by the falling ball } \frac{1}{2}g \times 4^2 = 8g = 8 \times 9.6 = 78.4m$$

Question 19*Answer:* C*Explanation:*

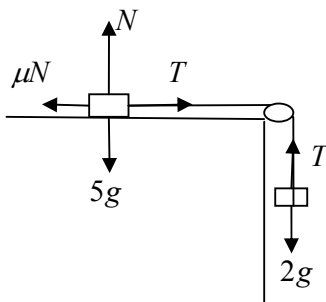
$$\dot{x}(t) = 30 + \frac{10}{(1+10t)^2}, \quad \ddot{x}(t) = -\frac{200}{(1+10t)^3}$$

When $t = 2$, $\ddot{x}(t) = -\frac{200}{(1+20)^3} = -0.0215959\dots$ and when $t \rightarrow \infty$, $\dot{x} \rightarrow 30$ as $\frac{10}{(1+10t)^2} \rightarrow 0$

Question 20*Answer:* A*Explanation:*

$$\dot{r}(t) = 2t \mathbf{i} - 2e^{-2t} \mathbf{j}$$

$$|\dot{r}(t)| = \sqrt{4t^2 + 4e^{-4t}} = 4.2$$

Using calculator: $\Rightarrow t = 2.0999\dots$ **Question 21***Answer:* D*Explanation:*

$$s = ut + \frac{1}{2}at^2, \quad s = 1.5, t = 2, u = 0$$

$$1.5 = \frac{1}{2}a \times 4 \Rightarrow a = \frac{3}{4}$$

$$2g - T = 2 \times \frac{3}{4} \Rightarrow T = 2g - \frac{3}{2}$$

$$N = 5g$$

$$T - \mu N = \frac{3}{4} \times 5$$

$$5g\mu = T - \frac{15}{4} = 2g - \frac{3}{2} - \frac{15}{4} = 14.35$$

$$\mu = \frac{14.35}{5 \times 9.8} = 0.29$$

Question 22*Answer:* B*Explanation:*

$$a = \frac{1+v^2}{v}, \quad v \frac{dv}{dx} = \frac{1+v^2}{v}$$

$$\frac{dv}{dx} = \frac{1+v^2}{v^2} \Rightarrow \frac{dx}{dv} = \frac{v^2}{1+v^2}$$

$$x = \int \frac{v^2}{1+v^2} dv = \int \left(1 - \frac{1}{1+v^2} \right) dv = v - \tan^{-1} v + c$$

When $x = 0, v = 1$.

$$\tan^{-1} 1 - 1 = c$$

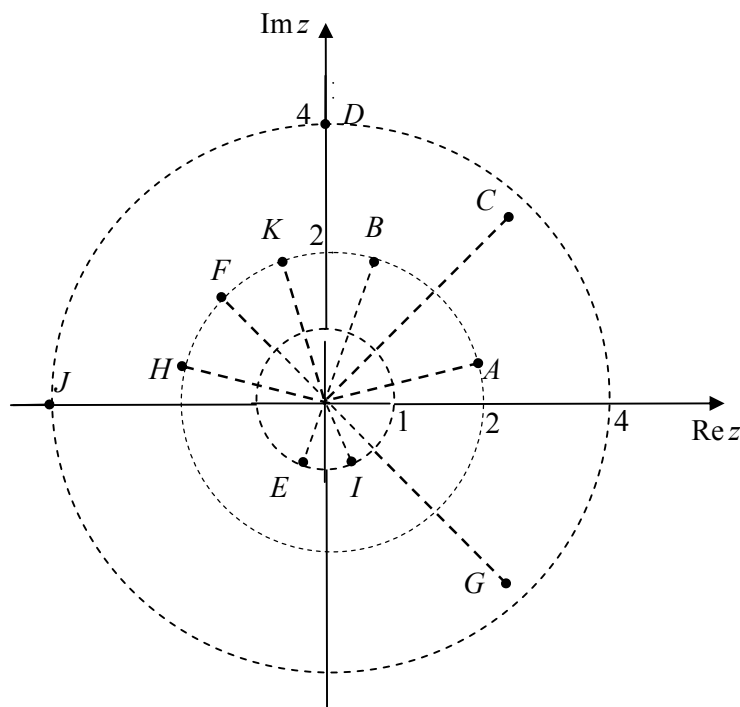
$$c = \frac{\pi}{4} - 1$$

$$\text{When } v = 3, x = 3 - \tan^{-1} 3 + \frac{\pi}{4} - 1 = 1.53635....$$

SECTION 2

Question 1

a.



$$\frac{u}{v} = \frac{2cis\frac{\pi}{12}}{2cis\frac{5\pi}{12}} = cis\left(\frac{\pi}{12} - \frac{5\pi}{12}\right) = cis\left(-\frac{\pi}{3}\right), \text{ represented by point } I. \quad A1$$

$$uv = 4cis\left(\frac{\pi}{12} + \frac{5\pi}{12}\right) = 4cis\frac{\pi}{2}, \text{ represented by point } D. \quad A1$$

The sum $u + v$ is represented by point C. Because $\bar{u} + \bar{v} = \overline{(u + v)}$, the required point is G.

$$\frac{16}{v^3} = \frac{16}{8cis\frac{5\pi}{4}} = 2cis\frac{-5\pi}{4} = 2cis\frac{3\pi}{4}, \text{ represented by point } F. \quad A1$$

b. i Rearrangement of $x + \frac{1}{x} = 2\cos\alpha$ gives $x^2 - 2\cos\alpha x + 1 = 0$.

By using the quadratic formula or by completing the square

$$x = \frac{2\cos\alpha \pm \sqrt{4\cos^2\alpha - 4}}{2} = \frac{2\cos\alpha \pm \sqrt{-4\sin^2\alpha}}{2} = \cos\alpha \pm i\sin\alpha. \quad A1$$

As $\alpha \in [0, \pi]$, $\sin\alpha > 0$, thus $x = \cos\alpha + i\sin\alpha$ A1

$$\text{ii} \quad x^n = cis(n\alpha), \text{ and } \frac{1}{x^n} = cis(-n\alpha). \quad M1$$

$$x^n - \frac{1}{x^n} = (\cos(n\alpha) + i\sin(n\alpha)) - (\cos(n\alpha) - i\sin(n\alpha)) = 2i\sin(n\alpha) \quad A1$$

iii.

$$\begin{aligned}
 (\sqrt{3} + i)^7 - (\sqrt{3} - i)^7 &= 2^7 \operatorname{cis} \frac{7\pi}{6} - 2^7 \operatorname{cis} \left(-\frac{7\pi}{6} \right) \\
 &= 2^7 \left(\operatorname{cis} \frac{7\pi}{6} - \operatorname{cis} \left(-\frac{7\pi}{6} \right) \right)
 \end{aligned}$$

M 1

Using the result from ii, $(\sqrt{3} + i)^7 - (\sqrt{3} - i)^7 = 2^7 \times 2i \sin \left(\frac{7\pi}{6} \right)$

M1

$$= 2^7 \times 2i \left(-\frac{1}{2} \right) = -2^7 i$$

A1

Question 2a. $\mathbf{a} = 2p\mathbf{i} + \mathbf{j} + (1-p)\mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 3\mathbf{j}$, $\mathbf{c} = 5\mathbf{i} - \mathbf{j} + 8\mathbf{k}$

$$\angle(\mathbf{a}, \mathbf{b}) = \angle(\mathbf{a}, \mathbf{c})$$

$$\frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{\mathbf{a} \bullet \mathbf{c}}{|\mathbf{a}| |\mathbf{c}|},$$

$$|\mathbf{b}| = \sqrt{10}, \quad |\mathbf{c}| = \sqrt{90}, \quad (\mathbf{a} \bullet \mathbf{b}) = -2p + 3, \quad (\mathbf{a} \bullet \mathbf{c}) = 10p - 1 + 8(1 - p)$$

M1

$$\frac{-2p + 3}{\sqrt{10}} = \frac{10p - 1 + 8(1 - p)}{3\sqrt{10}}$$

$$3(-2p + 3) = 10p - 1 + 8p - 8p$$

M1

$$-6p + 9 = 2p + 7$$

$$p = \frac{1}{4}$$

A1

b. Vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly dependent if one of them can be written as a linear combination of the other two.

$$\mathbf{a} = m\mathbf{b} + n\mathbf{c}$$

$$2p\mathbf{i} + \mathbf{j} + (1-p)\mathbf{k} = m(-\mathbf{i} + 3\mathbf{j}) + n(5\mathbf{i} - \mathbf{j} + 8\mathbf{k}).$$

M1

By equating coefficients of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$:

$$-m + 5n = 2p \quad (1)$$

$$3m - n = 1$$

$$8n = 1 - p \Rightarrow n = \frac{1-p}{8} \quad (2)$$

$$3m = n + 1 = \frac{1-p}{8} + 1 \Rightarrow m = \frac{9-p}{24} \quad (3)$$

M1

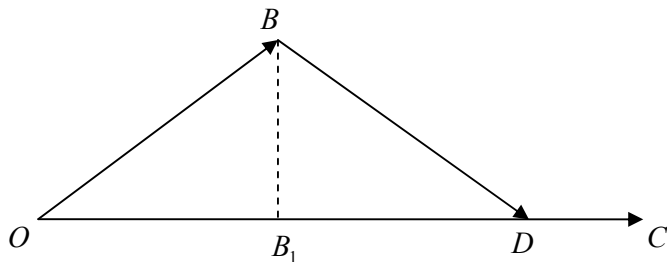
Substituting (2) and (3) into the equation (1) gives

$$-\frac{9-p}{24} + \frac{5-5p}{8} = 2p, \quad -9 + p + 15 - 15p = 48p \Rightarrow p = \frac{3}{31}$$

A1

c. Method 1

In the diagram below, $|\overrightarrow{BD}| = |\overrightarrow{OB}|$ and B_1 is the midpoint of OD .



M1

Vector $\overrightarrow{OB_1}$ is a vector resolute of $\overrightarrow{OB}$ in the direction of $\overrightarrow{OC}$.

$$\overrightarrow{OB_1} = (\mathbf{b} \cdot \hat{\mathbf{c}})\hat{\mathbf{c}} = \frac{1}{90}(-5-3)(5\mathbf{i} - \mathbf{j} + 8\mathbf{k})$$

$$= \frac{-4}{45}(5\mathbf{i} - \mathbf{j} + 8\mathbf{k})$$

A1

$$= -\frac{4}{9}\mathbf{i} + \frac{4}{45}\mathbf{j} - \frac{32}{45}\mathbf{k}$$

From $\overrightarrow{OD} = 2\overrightarrow{OB_1}$, the position vector of point D is $\overrightarrow{OD} = -\frac{8}{9}\mathbf{i} + \frac{8}{45}\mathbf{j} - \frac{64}{45}\mathbf{k}$

A1

Method 2

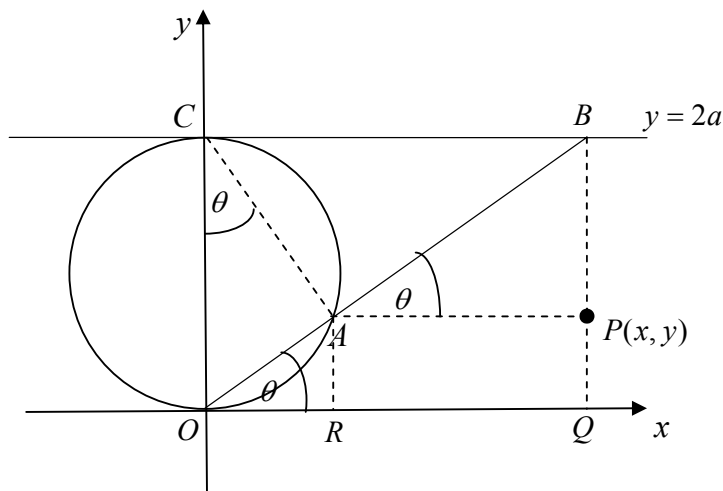
Let $\overrightarrow{OD} = m\overrightarrow{OC}$. Then $\overrightarrow{OD} = 5m\mathbf{i} - m\mathbf{j} + 8m\mathbf{k}$

$$\overrightarrow{BD} = \overrightarrow{BO} + \overrightarrow{OD} = (1+5m)\mathbf{i} + (-3-m)\mathbf{j} + 8m\mathbf{k}$$

$$|\overrightarrow{OB}| = \sqrt{10}, \quad |\overrightarrow{BD}| = \sqrt{(1+5m)^2 + (3+m)^2 + 64m^2} = \sqrt{90m^2 + 16m + 10}$$

$$90m^2 + 16m + 10 = 10 \Rightarrow m = -\frac{8}{45}$$

$$\text{Therefore } \overrightarrow{OD} = -\frac{8}{9}\mathbf{i} + \frac{8}{45}\mathbf{j} - \frac{64}{45}\mathbf{k}$$

Question 3**a.**

From the triangle OBQ , $\frac{BQ}{OQ} = \tan \theta$, $OQ = \frac{BQ}{\tan \theta}$ and $BQ = 2a$, $OQ = x$.

Thus $x = \frac{2a}{\tan \theta} = 2a \cot \theta$.

A1

To find y , note that $\angle OAC = \frac{\pi}{2}$ (angle subtended by diameter).

From the triangle OAC , $OA = 2a \sin \theta$ and from the triangle OAR , $AR = y = OA \sin \theta$.

After substituting $OA = 2a \sin \theta$ into $AR = y = OA \sin \theta$, it follows $y = 2a \sin^2 \theta$.

A1

b. From $\cot \theta = \frac{x}{2a}$, using the identity $\cot^2 \theta + 1 = \frac{1}{\sin^2 \theta}$, it follows $\sin^2 \theta = \frac{4a^2}{x^2 + 4a^2}$

$$y = 2a \sin^2 \theta$$

$$= 2a \times \frac{4a^2}{x^2 + 4a^2}$$

M2

$$= \frac{8a^3}{x^2 + 4a^2}$$

c. $y' = \frac{-16a^3x}{(x^2 + 4a^2)^2}$, $y'' = \frac{16a^3(3x^2 - 4a^2)}{(x^2 + 4a^2)^3}$

M1

$$y'' = 0 \text{ when } 3x^2 - 4a^2 = 0 \Rightarrow x = \pm \frac{2a}{\sqrt{3}}.$$

A1

After substituting into $y = \frac{8a^3}{x^2 + 4a^2}$, $y = \frac{8a^3}{\left(\frac{2a}{\sqrt{3}}\right)^2 + 4a^2} \Rightarrow y = \frac{3a}{2}$

Thus, the points of inflexion are $\left(-\frac{2a}{\sqrt{3}}, \frac{3a}{2}\right)$ and $\left(\frac{2a}{\sqrt{3}}, \frac{3a}{2}\right)$

A1

d. i

$$\begin{aligned}
 A &= 2 \int_0^{\frac{2a}{\sqrt{3}}} \frac{8a^3}{x^2 + 4a^2} dx = 8a^2 \left(\tan^{-1} \frac{x}{2a} \right)_0^{\frac{2a}{\sqrt{3}}} \\
 &= 8a^2 \tan^{-1} \frac{1}{\sqrt{3}} \\
 &= \frac{4a^2 \pi}{3}
 \end{aligned}$$

M1, A1

ii

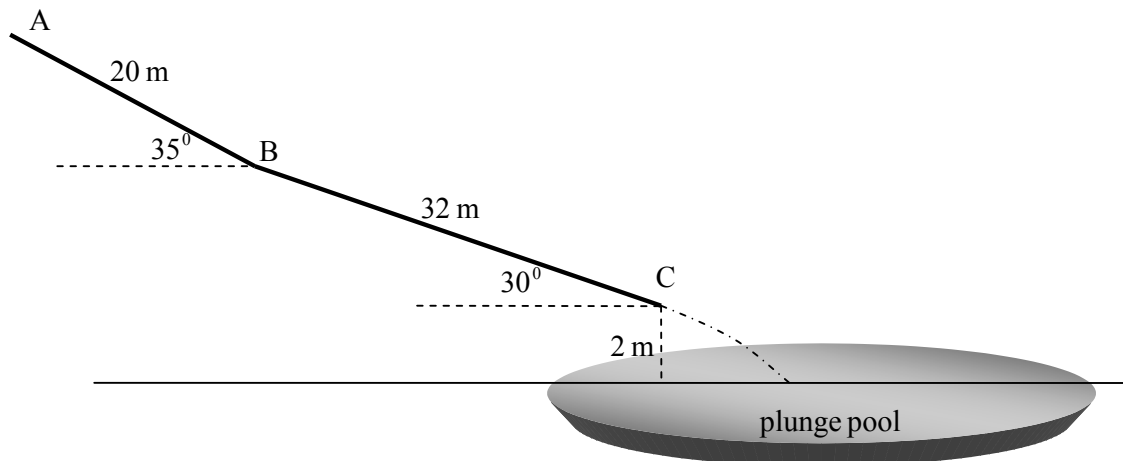
$$\begin{aligned}
 V &= \pi \int_a^{2a} x^2 dy = \pi \int_a^{20} \left(\frac{8a^3}{y} - 4a^2 \right) dy \\
 &= \pi \left[8a^3 \log_e y - 4a^2 y \right]_a^{2a} \\
 &= \pi (8a^3 \log_e 2 - 4a^3) \\
 &= 4a^3 \pi (2 \log_e 2 - 1)
 \end{aligned}$$

M1, A1

iii By using a graphics calculator, the point of intersection of the area and volume functions occurs when $a = 0.863$

A1

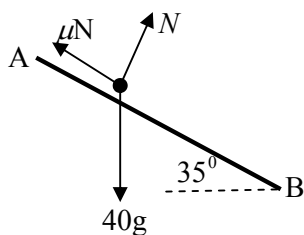
Question 4



a. i Given that $u = 0$, $v = 6$, $s = 20$, using constant acceleration formula $v^2 = u^2 + 2as$,
 $36 = 40a \Rightarrow a = \frac{9}{10} \text{ ms}^{-2}$.

A1

ii



From the diagram: $40g \sin 35^\circ - \mu N = 40 \times \frac{9}{10}$ and $N - 40g \cos 35^\circ = 0$. It follows

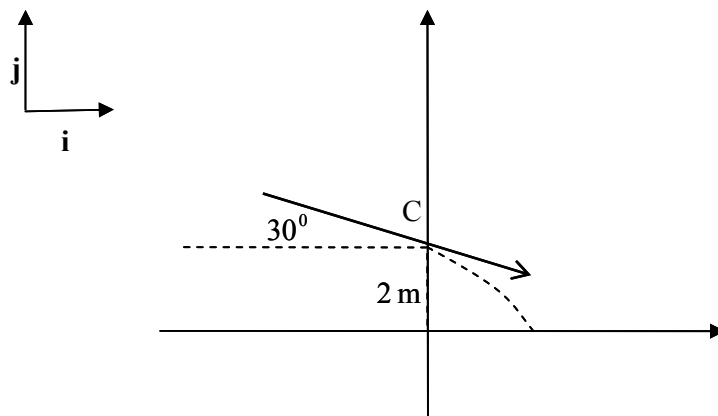
$$40g \sin 35^\circ - \mu \times 40g \cos 35^\circ = 40 \times \frac{9}{10} \quad \text{M1}$$

$$\mu = \frac{g \sin 35^\circ - 0.9}{g \cos 35^\circ} = 0.575 \quad \text{A1}$$

b. It is known that $u = 6$, $s = 32$ and $a = -0.5$. Therefore $v^2 = 36 - 32 \Rightarrow v = 2$.
M1

$$\text{From } s = \frac{u+v}{2}t, \quad t = \frac{2s}{u+v} = 8 \text{ seconds} \quad \text{A1}$$

c.



i By choosing the directions $\mathbf{i}$ and $\mathbf{j}$ as in the diagram above and using $v = 2$ at point C, we have $\mathbf{r}(0) = 2 \cos 30^\circ \mathbf{i} - 2 \sin 30^\circ \mathbf{j}$, thus $\dot{\mathbf{r}}(0) = \sqrt{3}\mathbf{i} - \mathbf{j}$. A1

$$\begin{aligned} \text{ii From } \ddot{\mathbf{r}}(t) &= -g\mathbf{j}, \quad \dot{\mathbf{r}}(t) = -gt\mathbf{j} + \sqrt{3}\mathbf{i} - \mathbf{j} \\ \dot{\mathbf{r}}(t) &= \sqrt{3}\mathbf{i} - (gt + 1)\mathbf{j} \end{aligned} \quad \text{A1}$$

After integrating the velocity vector: $\mathbf{r}(t) = \sqrt{3}t\mathbf{i} - \left(\frac{gt^2}{2} + t\right)\mathbf{j} + 2\mathbf{j}$

$$\mathbf{r}(t) = \sqrt{3}t\mathbf{i} - \left(\frac{gt^2}{2} + t - 2\right)\mathbf{j} \quad \text{A1}$$

d. i Before the speed can be found, we need to know the time when Adam hits the water. This is when the **j** component of the position vector is zero.

Solving $\frac{gt^2}{2} + t - 2 = 0$ gives $t = 0.5449$. A1

The velocity vector at $t = 0.5449$ is $\dot{\mathbf{r}}(t) = \sqrt{3}\mathbf{i} - (9.8 \times 0.5449 + 1)\mathbf{j} = 1.732\mathbf{i} - 6.34\mathbf{j}$. M1

The speed is the magnitude of the velocity vector.

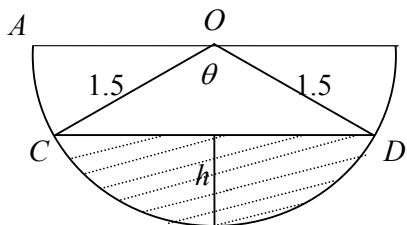
Speed = $\sqrt{3 + 6.34^2} = 6.57 \text{ ms}^{-1}$ A1

ii The horizontal distance from the bottom of the slide is the **i** component of the position vector at $t = 0.5449$, which is $\sqrt{3} \times 0.5449 = 0.9438$.

The required distance is 0.94 m. A1

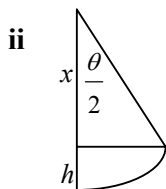
Question 5

AB is the diameter of the semi-circle, $\angle COD = \theta$ and h is the depth



a. i The shaded area (segment) = $\frac{1}{2} \times 1.5^2 (\theta - \sin \theta) = \frac{9}{8} (\theta - \sin \theta)$.

Therefore $V = 8 \times \frac{9}{8} (\theta - \sin \theta) = 9(\theta - \sin \theta)$. M1



$$\frac{x}{1.5} = \cos \frac{\theta}{2} \Rightarrow x = \frac{3}{2} \cos \frac{\theta}{2}$$

$$h = \frac{3}{2} - \frac{3}{2} \cos \frac{\theta}{2} = \frac{3}{2} \left(1 - \cos \frac{\theta}{2} \right) \quad \text{M1}$$

b. $\frac{dh}{dt} = \frac{dh}{d\theta} \frac{d\theta}{dV} \frac{dV}{dt}$ (1)

From $h = \frac{3}{2} \left(1 - \cos \frac{\theta}{2}\right)$, $\frac{dh}{d\theta} = \frac{3}{4} \sin \frac{\theta}{2}$.

Also, from $V = 9(\theta - \sin \theta)$, $\frac{dV}{d\theta} = 9(1 - \cos \theta)$ and $\frac{dV}{dt} = 0.2$ (given).

By substituting into (1), we have $\frac{dh}{dt} = \frac{3}{4} \sin \frac{\theta}{2} \times \frac{1}{9(1 - \cos \theta)} \times 0.2$ M1

$$= \frac{\sin \frac{\theta}{2}}{60(1 - \cos \theta)} \quad (2) \quad \text{A1}$$

For $h = 0.8$, $\frac{3}{2} \left(1 - \cos \frac{\theta}{2}\right) = 0.8$

Solving this equation for θ gives $\theta = 2.17^\circ$ or $\theta = 124.36^\circ$ M1

After substitution into equation (2): $\frac{dh}{dt} = 0.009 \text{ ms}^{-1}$ (3 decimal places)

A1

c. i $\frac{dx}{dt} = \text{Rate in} - \text{Rate out}$,

$$\frac{dx}{dt} = 0.05 \times 100 + 0.04 \times 50 - \frac{150x}{20000} \quad \text{M1}$$

After simplifying, $\frac{dx}{dt} = 7 - \frac{3x}{400}$

ii $\frac{dx}{dt} = \frac{2800 - 3x}{400}$

$$\frac{dx}{2800 - 3x} = \frac{1}{400}$$

A1

$$t = \int \frac{400}{2800 - 3x} dx = -\frac{400}{3} \log_e (2800 - 3x) + c \quad 0 \leq x < \frac{2800}{3}$$

$$\log_e (2800 - 3x) = -\frac{3t}{400} + \frac{3c}{400} \quad \text{M1}$$

$$2800 - 3x = Ae^{-\frac{3t}{400}}, \text{ where } A = e^{\frac{3c}{400}} \quad \text{M1}$$

When $x = 0$, $t = 0 \Rightarrow A = \frac{2800}{3}$ and therefore $x = \frac{2800}{3} (1 - e^{-\frac{3t}{400}})$.

d. i The concentration of 0.03 kg / L gives $x = 20000 \times 0.03 = 600 \text{ kg of salt}$. M1
After solving the equation (algebraically or using graphics calculator)

$$\frac{2800}{3} (1 - e^{-\frac{3t}{400}}) = 600 \Rightarrow t = 137.28 \text{ minutes} \quad \text{A1}$$

ii When $t \rightarrow \infty$, $e^{-\frac{3t}{400}} \rightarrow 0$ and $x \rightarrow \frac{2800}{3} = 933.33$ kg of salt. A1

The limiting concentration is $\frac{2800}{20000} = 0.047$ kg/L A1