

SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 1



2009 Trial Examination

SOLUTIONS

Question 1

a. Vectors $\mathbf{u} = \mathbf{a} + k\mathbf{b}$ and $\mathbf{v} = \mathbf{a} - \mathbf{b}$ are perpendicular, therefore $\mathbf{u} \cdot \mathbf{v} = 0$

$$(\mathbf{a} + k\mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = 0$$

$$|\mathbf{a}|^2 - \mathbf{a} \cdot \mathbf{b} + k\mathbf{a} \cdot \mathbf{b} - k|\mathbf{b}|^2 = 0 \quad (1)$$

M1

Also $\angle(\mathbf{a}, \mathbf{b}) = 120^\circ$, so $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos 120^\circ = -|\mathbf{a}|^2$.

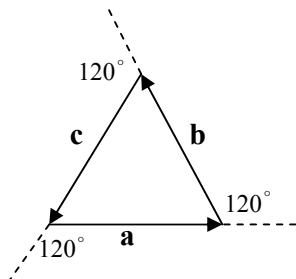
M1

Substituting $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}|^2$ and $|\mathbf{b}| = 2|\mathbf{a}|$ into the equation (1), gives $|\mathbf{a}|^2 + |\mathbf{a}|^2 - k|\mathbf{a}|^2 - 4k|\mathbf{a}|^2 = 0$

After simplifying we have $(2 - 5k)|\mathbf{a}|^2 = 0$ and $k = \frac{2}{5}$.

A1

b. As $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$ and $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1$, vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ form an equilateral triangle.



M1

The angle between each two vectors is 120° and $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c} = -\frac{1}{2}$.

It follows $\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a} = -\frac{1}{2} - \frac{1}{2} - \frac{1}{2} = -\frac{3}{2}$.

A1

Question 2**a.**

$$\begin{aligned}
 w &= \frac{1+i}{1-i} \times \frac{1+i}{1+i} \\
 &= \frac{1+2i+i^2}{1-i^2} \\
 &= \frac{2i}{2} \\
 &= i
 \end{aligned}$$

A1

b. Let $z = x + iy$. Then

$$|x - iy - i(x + iy)| = 2\sqrt{2}$$

M1

$$|x + y - i(x + y)| = 2\sqrt{2}$$

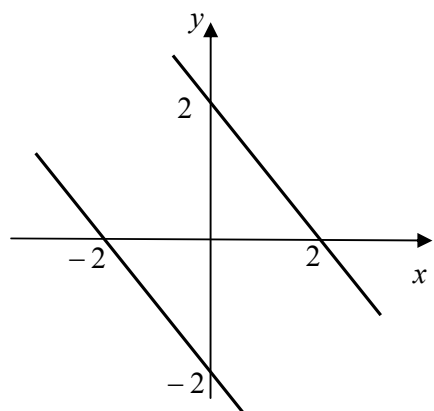
$$(x + y)^2 + (x + y)^2 = 8$$

$$(x + y)^2 = 4$$

$$x + y = \pm 2$$

This subset is represented by two parallel lines $x + y = 2$ and $x + y = -2$.

A1



A1

c. $z_1 w = z_1 i$ which represents a rotation by $\frac{\pi}{2}$, therefore the required numbers are

M1

$$z_1 = 2, -2, 2i, -2i$$

A1

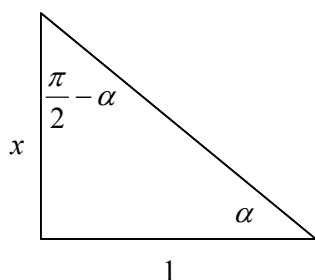
Question 3

$$\begin{aligned} \text{a. } \sin\left(\tan^{-1} \frac{1}{\sqrt{3}}\right) - \cos\left(\tan^{-1} \sqrt{3}\right) &= \sin \frac{\pi}{6} - \cos \frac{\pi}{3} \\ &= \frac{1}{2} - \frac{1}{2} \\ &= 0 \end{aligned}$$

A1

b. Given that $x = \tan \alpha$ and $\alpha \in \left(0, \frac{\pi}{2}\right)$ using the right-angle triangle below, we have

$$\tan\left(\frac{\pi}{2} - \alpha\right) = \frac{1}{x} \quad \text{and} \quad \left(\frac{\pi}{2} - \alpha\right) = \tan^{-1} \frac{1}{x}$$



M1

$$\begin{aligned} \sin(\tan^{-1} x) - \cos\left(\tan^{-1} \frac{1}{x}\right) &= \sin \alpha - \cos\left(\frac{\pi}{2} - \alpha\right) \\ &= 0 \quad \text{as } \sin \alpha = \cos\left(\frac{\pi}{2} - \alpha\right) \end{aligned}$$

A1

Question 4

To find the points of intersection, substitute $x = 1$ into $y^2 - 2x^2 + 2xy - 6 = 0$ which gives $y^2 + 2y - 8 = 0 \Rightarrow y = 2$ and $y = -4$.

A1

To find the gradient, differentiate implicitly as follows:

$$2y \frac{dy}{dx} - 4x + 2y + 2x \frac{dy}{dx} = 0 \quad . \quad \text{When rearranged, } \frac{dy}{dx} = \frac{2x - y}{x + y} \quad .$$

The gradient of the normal is given by the expression $\frac{dy}{dx} = \frac{x + y}{y - 2x}$.

M1

At the point (1, 2) the gradient of the normal is undefined, therefore its equation is $x = 1$.

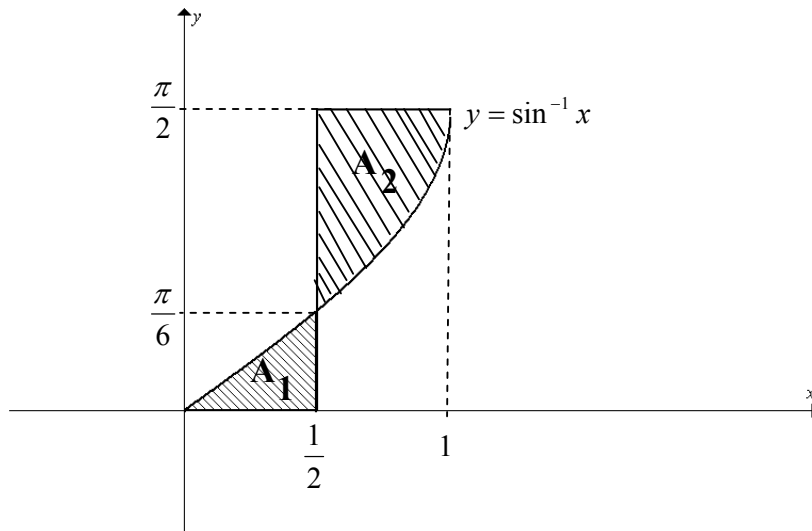
M1

At the point (1, -4), the gradient of the normal is $\frac{1}{2}$ and the equation of the normal is

$$y + 4 = \frac{1}{2}(x - 1). \quad \text{After simplifying, the equation is } x - 2y - 9 = 0 \quad .$$

A1

Question 5



The area can be found by integrating $x = \sin y$.

M1

$$A_1 = \frac{\pi}{6} \times \frac{1}{2} - \int_0^{\frac{\pi}{6}} \sin y dy$$

$$= \frac{\pi}{12} + (\cos y) \Big|_0^{\frac{\pi}{6}}$$

M1

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

$$A_2 = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin y dy - \left(\frac{\pi}{2} - \frac{\pi}{6} \right) \times \frac{1}{2}$$

$$= (-\cos y) \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} - \frac{\pi}{6}$$

M1

$$= \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

$$\text{Total area} = A_1 + A_2 = \sqrt{3} - \frac{\pi}{12} - 1$$

A1

Question 6

Let $\ln x = u$. Then $\frac{1}{x} dx = du$.

For $x = e^{-3}$, $u = \ln e^{-3} = -3$ and for $x = e^{-2}$, $u = \ln e^{-2} = -2$.

M1

$$\int_{e^{-3}}^{e^{-2}} \frac{1 + \ln x}{x \ln x (1 - \ln x)} dx = \int_{-3}^{-2} \frac{u + 1}{u(1 - u)} du$$

A1

$$\frac{u + 1}{u(1 - u)} = \frac{A}{u} + \frac{B}{1 - u}$$

M1

$$A(1 - u) + Bu = u + 1 \Rightarrow A = 1, B = 2$$

$$\begin{aligned} \int_{-3}^{-2} \frac{u + 1}{u(1 - u)} du &= \int_{-3}^{-2} \left(\frac{1}{u} + \frac{2}{1 - u} \right) du \\ &= \left(\ln|u| - 2 \ln|1 - u| \right) \Big|_{-3}^{-2} \\ &= \left(\ln \left| \frac{u}{(1 - u)^2} \right| \right) \Big|_{-3}^{-2} \\ &= \ln \frac{2}{9} - \ln \frac{3}{16} \\ &= \ln \frac{\frac{2}{9}}{\frac{3}{16}} = \ln \frac{32}{27} \end{aligned}$$

M1, A1

Question 7

a. $x = 2 \cos \frac{\pi}{10} t, \quad y = 3t, \quad t \geq 0$

$$t = \frac{10}{\pi} \cos^{-1} \frac{x}{2}, \text{ so the Cartesian equation is } y = \frac{30}{\pi} \cos^{-1} \frac{x}{2}. \quad \text{A1}$$

The domain of this function is $[-2, 2]$, the range is $[0, 30]$. A1

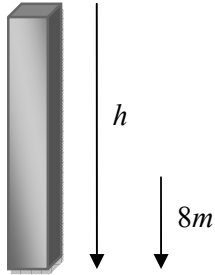
b. The speed of a particle can be found as the magnitude of its velocity vector. The position vector is $\mathbf{r}(t) = 2 \cos \frac{\pi}{10} t \mathbf{i} + 3t \mathbf{j}$, its velocity vector is $\dot{\mathbf{r}}(t) = -\frac{\pi}{5} \sin \frac{\pi}{10} t \mathbf{i} + 3 \mathbf{j}$.

The expression for the speed is $|\dot{\mathbf{r}}(t)| = \sqrt{\frac{\pi^2}{25} \sin^2 \frac{\pi}{10} t + 9}$. M1

The minimum speed is 3 and it occurs when $\sin^2 \frac{\pi}{10} t = 0$, which is for $\frac{\pi}{10} t = 0, \pi$
or $t = 0, 10$ A1

Question 8

At the time t when the object hits the ground: $a = g$, $v = gt$, $a = \frac{1}{2}gt^2$



When the object is 8 m above the ground, the time elapsed is $(t - 0.4)$ seconds and the velocity is $v_1 = g(t - 0.4)$. Therefore considering the last 0.4 seconds of travel M1

$$8 = g(t - 0.4) \times 0.4 + \frac{1}{2}g \times 0.4^2$$

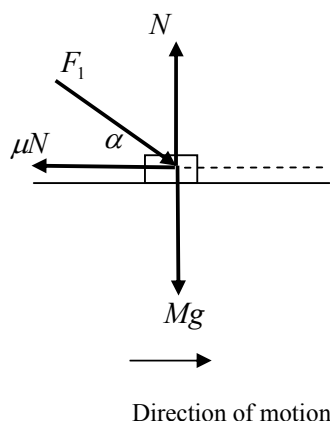
Solving for t :

$$g\left(t - \frac{2}{5}\right) \times \frac{2}{5} + \frac{1}{2}g \times \frac{4}{25} = 8$$

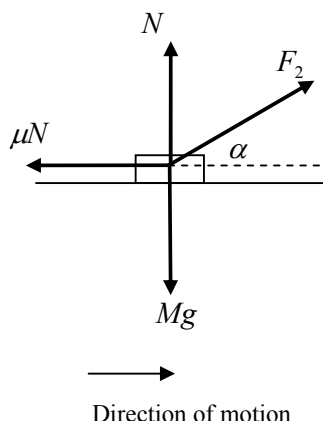
$$\frac{2g}{5}t - \frac{4g}{25} + \frac{2g}{25} = 8$$

$$\frac{2gt}{5} = 8 + \frac{2g}{25} \Rightarrow t = \frac{20}{g} + \frac{1}{5} = \frac{100 + g}{5g} \quad \text{A1}$$

$$\text{Substituting back into } h = \frac{1}{2}gt^2 \text{ and simplifying gives } h = \frac{(100 + g)^2}{50g}. \quad \text{A1}$$

Question 9**a.** Pushing force:

Pulling force:



The velocity of the object is constant, therefore the acceleration is zero. By resolving the forces vertically and horizontally we have:

$$N - Mg - F_1 \sin \alpha = 0, \quad N = Mg + F_1 \sin \alpha$$

$$-\mu N + F_1 \cos \alpha = 0$$

$$\mu Mg + \mu F_1 \sin \alpha - F_1 \cos \alpha = 0$$

$$F_1 = \frac{\mu Mg}{\cos \alpha - \mu \sin \alpha}$$

$$N - Mg + F_2 \sin \alpha = 0, \quad N = Mg - F_2 \sin \alpha$$

$$-\mu N + F_2 \cos \alpha = 0$$

$$-\mu Mg + \mu F_2 \sin \alpha + F_2 \cos \alpha = 0$$

$$F_2 = \frac{\mu Mg}{\cos \alpha + \mu \sin \alpha} \quad \text{M3}$$

$$\cos \alpha - \mu \sin \alpha < \cos \alpha + \mu \sin \alpha \quad \text{for every } \alpha \in \left(0, \frac{\pi}{2}\right) \quad \text{A1}$$

Therefore $F_1 > F_2$

b.

$$\begin{aligned} \frac{F_1}{F_2} &= \frac{\mu Mg}{\cos \alpha - \mu \sin \alpha} \times \frac{\cos \alpha + \mu \sin \alpha}{\mu Mg} \\ &= \frac{\cos \alpha + \mu \sin \alpha}{\cos \alpha - \mu \sin \alpha} \quad \text{M1} \end{aligned}$$

$$\text{Dividing each term by } \cos \alpha \text{ gives } \frac{F_1}{F_2} = \frac{1 + \mu \tan \alpha}{1 - \mu \tan \alpha} \quad \text{A1}$$