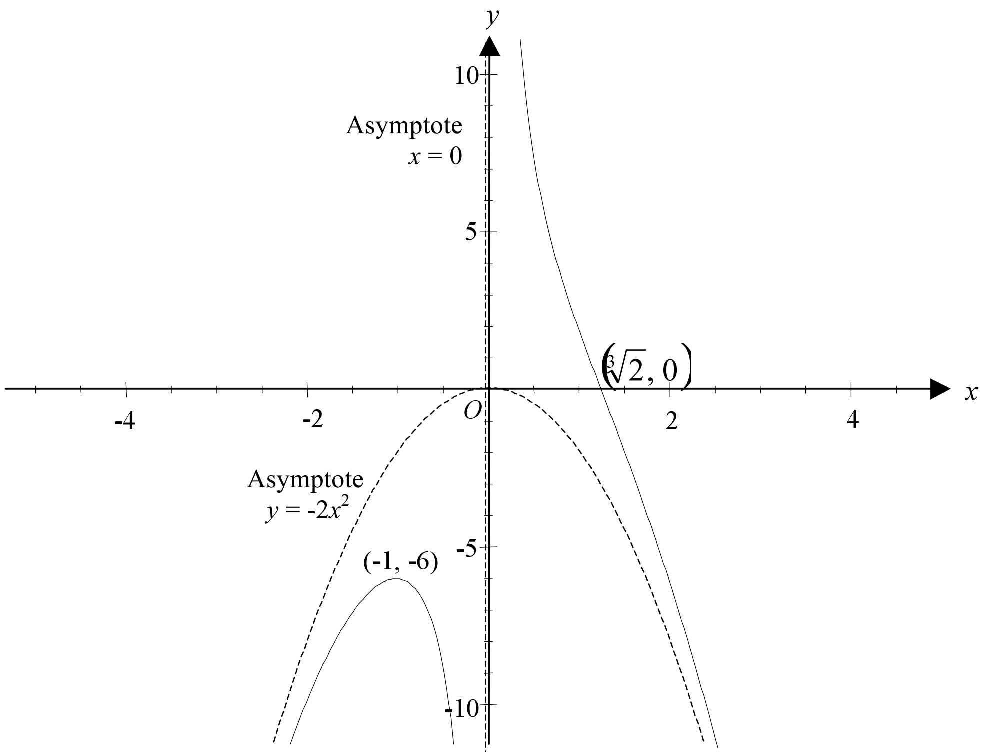
Question 1



correct shape **[A1]**

x-intercept: $y = 0$ $\quad \frac{4}{x} - 2x^2 = 0$

$$4 - 2x^3 = 0$$

$$x^3 = 2$$

$$x = \sqrt[3]{2}$$ **[A1]**

Turning point: $\frac{dy}{dx} = 0$ $\quad y = 4x^{-1} - 2x^2$

$$\frac{dy}{dx} = -4x^{-2} - 4x$$

$$-\frac{4}{x^2} - 4x = 0$$

$$4 + 4x^3 = 0$$

$$x^3 = -1$$

$$x = -1$$

When, $y = \dfrac{4}{-1} - 2(-1)^2$

Maximum at $(-1, -6)$ **[A1]**

Asymptotes: $x = 0$, $y = -2x^2$ **[A2]**

Total 5 marks

Question 2

a.

$P(z) = z^2 + (1 + i\sqrt{2})z + 4i\sqrt{2} - 12$

$P(-4) = 16 - 4 - 4i\sqrt{2} + 4i\sqrt{2} - 12$ **[M1]**

$= 0$

b.

$z^2 + (1 + i\sqrt{2})z + 4i\sqrt{2} - 12 = 0$

$z^2 + (1 + i\sqrt{2})z + 4(i\sqrt{2} - 3) = (z + 4)(z + a + bi)$ **[M1]**

$= z^2 + az + biz + 4z + 4a + 4bi$

$= z^2 + (a + 4 + bi)z + 4a + 4bi$ **[A1]**

Equating coefficients of z: **[M1]**

$1 + i\sqrt{2} = a + 4 + bi$ equating real components and equating imaginary components

$a + 4 = 1$ and $b = \sqrt{2}$

$a = -3$

Therefore the other factor is $(z - 3 + i\sqrt{2})$, hence the other solution is $3 - i\sqrt{2}$ **[A1]**

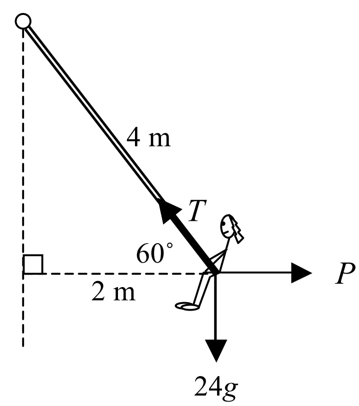
Total 5 marks

Question 3

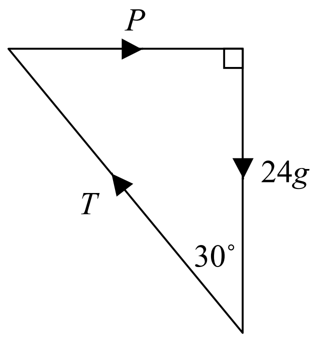
The three forces P, T, and $24g$ will be in equilibrium.

$\cos\theta = \dfrac{2}{4}$

$\theta = 60°$



Adding the forces head to tail.



[M1]

From the triangle of forces

$\sin(30^0) = \dfrac{P}{T}$ and $\cos(30^0) = \dfrac{24g}{T}$

$P = \dfrac{1}{2}T$ ... (1) $T = \dfrac{24g}{\cos(30^0)} = \dfrac{48g}{\sqrt{3}} = 16\sqrt{3}g$ … (2) **[A1]**

Substituting (2) into (1)

$P = \dfrac{1}{2} \times 16\sqrt{3}g$

$P = 8\sqrt{3}\,g$ newtons **[A1]**

Total 3 marks

Question 4

$\cot(2x) - \cot(x) = 2$

$\dfrac{\cos(2x)}{\sin(2x)} - \dfrac{\cos(x)}{\sin(x)} = 2$

$\dfrac{\sin(x)\cos(2x) - \cos(x)\sin(2x)}{\sin(2x)\sin(x)} = 2$ **[M1]**

$\dfrac{\sin(x - 2x)}{\sin(2x)\sin(x)} = 2$ **[M1]**

$\dfrac{\sin(-x)}{\sin(2x)\sin(x)} = 2$

$\dfrac{-\sin(x)}{\sin(2x)\sin(x)} = 2$

$\dfrac{-1}{\sin(2x)} = 2$

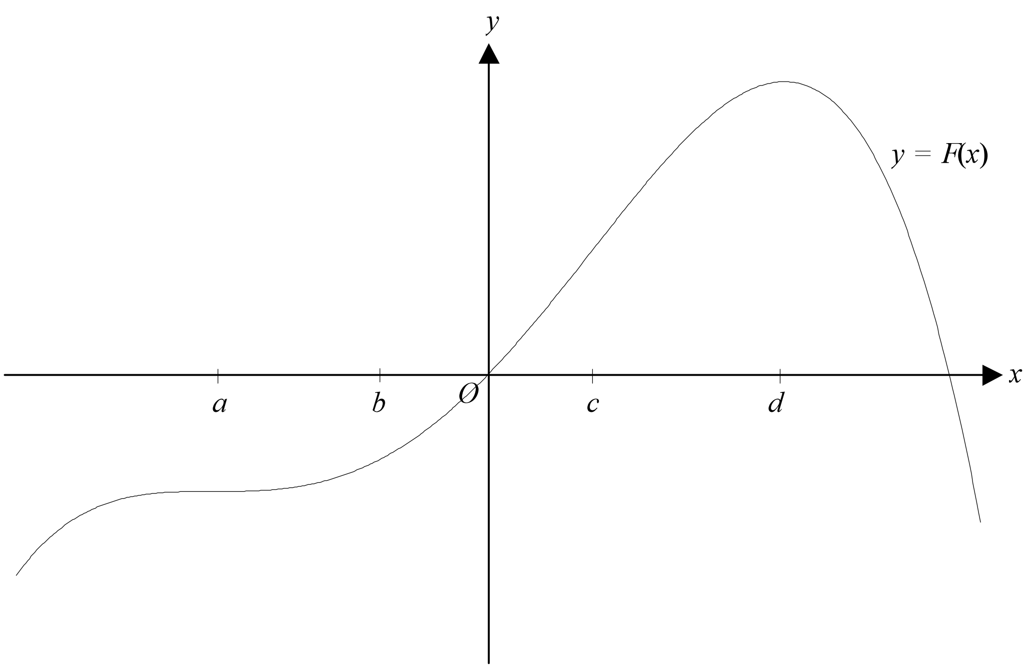
$\sin(2x) = -\frac{1}{2}$ **[A1]**

$2x = -\frac{5\pi}{6},\quad -\frac{\pi}{6},\quad \frac{7\pi}{6},\quad \frac{11\pi}{6}$

$x = -\frac{5\pi}{12},\quad -\frac{\pi}{12},\quad \frac{7\pi}{12},\quad \frac{11\pi}{12}$ **[A1]**

Total 4 marks

Question 5



There is a turning point on $y = f(x)$ at $x = a$ that touches the x-axis, so $y = F(x)$ will have a stationary point of inflexion at $x = a$. **[A1]**

There is a maximum turning point on $y = f(x)$ at $x = c$, so $y = F(x)$ will have a non-stationary point of inflexion at $x = c$. This is the point of steepest slope on $y = F(x)$ for $a \le x \le d$ **[A1]**

There is an x-intercept on $y = f(x)$ at $x = d$ cutting the x-axis from positive to negative, so $y = F(x)$ will have a local maximum at $x = d$. **[A1]**

Total 3 marks

Question 6

$y = \cot(x)$

$y = \dfrac{\cos(x)}{\sin(x)}$

$\dfrac{dy}{dx} = \dfrac{\sin(x)\times(-\sin(x)) - \cos(x)\times\cos(x)}{\sin^2(x)}$ **[M1]**

$= \dfrac{-\left(\sin^2(x) + \cos^2(x)\right)}{\sin^2(x)}$

$= \dfrac{-1}{\sin^2(x)}$

$= -\operatorname{cosec}^2(x)$ **[A1]**

Total 2 marks

Question 7

Use implicit differentiation and apply the product rule.

$5y = 2xy^2 - 3x$

$\dfrac{d}{dx}(5y) = \dfrac{d}{dx}\left(2xy^2\right) - \dfrac{d}{dx}(3x)$

$5\dfrac{dy}{dx} = \left(y^2\dfrac{d}{dx}(2x) + 2x\dfrac{d}{dx}\left(y^2\right)\right) - \dfrac{d}{dx}(3x)$

$5\dfrac{dy}{dx} = \left(2\times y^2 + 2x\times 2y\times\dfrac{dy}{dx}\right) - 3$ **[M1]**

$5\dfrac{dy}{dx} = 2y^2 + 4xy\dfrac{dy}{dx} - 3$

$(5 - 4xy)\dfrac{dy}{dx} = 2y^2 - 3$

$\dfrac{dy}{dx} = \dfrac{2y^2 - 3}{5 - 4xy}$ **[A1]**

Total 2 marks

Question 8

$\int \frac{\sqrt{x}}{x+1}dx$ Let $u=\sqrt{x}=x^{\frac{1}{2}}$ hence $x=u^2$

$$\frac{du}{dx}=\frac{1}{2}x^{-\frac{1}{2}}$$
$$=\frac{1}{2\sqrt{x}}$$
$$=\frac{1}{2u}$$ **[M1]**

$$\int \frac{u}{u+1}2u\,\frac{du}{dx}dx$$
$$=\int \frac{2u^2}{u^2+1}du$$

$$\frac{2u^2}{u^2+1}=\frac{(u^2+1)+(u^2+1)-2}{u^2+1}$$ Alternatively, perform long division.
$$=2-\frac{2}{u^2+1}$$ **[M1, A1]**

$$=\int\left(2-\frac{2}{u^2+1}\right)du$$
$$=2u-2\tan^{-1}(u)+c$$
$$=2\sqrt{x}-2\tan^{-1}\left(\sqrt{x}\right)+c$$ **[A1]**

Total 4 marks

Question 9

$$\int \frac{1}{1+\cos(x)}dx$$ $\cos^2(x)=\frac{1}{2}(1+\cos(2x))$

$$=\int \frac{1}{2\cos^2\left(\frac{x}{2}\right)}dx$$ $2\cos^2\left(\frac{x}{2}\right)=1+\cos(x)$ **[M1]**

$$=\int \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx$$ **[A1]**

$$=\frac{1}{2}\times 2\tan\left(\frac{x}{2}\right)+c$$
$$=\tan\left(\frac{x}{2}\right)+c$$ **[A1]**

Total 3 marks

Question 10

a.

Finding a relationship between the radius and height of the solution in the filter:

$\tan(60°) = \dfrac{r}{h} \quad \Rightarrow \quad r = \sqrt{3}\,h$ … (1)

Volume of cone: $V = \dfrac{1}{3}\pi r^2 h$

Substitute (1) $V = \dfrac{1}{3}\pi \left(\sqrt{3}\,h\right)^2 h$

$V = \pi h^3$ … (2) **[A1]**

Related rate equation:

$\dfrac{dV}{dt} = \dfrac{dV}{dh} \times \dfrac{dh}{dt}$ Given $\dfrac{dV}{dt} = -8\sqrt{h}$ and from (2) $\dfrac{dV}{dh} = 3\pi h^2$

$-8\sqrt{h} = 3\pi h^2 \times \dfrac{dh}{dt}$

$\dfrac{dh}{dt} = -\dfrac{8}{3\pi h^{\frac{3}{2}}}$ **[A1]**

$\dfrac{dh}{dt} = -\dfrac{8}{3\pi \sqrt{h^3}}$ as required.

b.

$\dfrac{dt}{dh} = -\dfrac{3\pi \sqrt{h^3}}{8}$

$t = -\dfrac{3\pi}{8}\int h^{\frac{3}{2}}\,dt$

$t = -\dfrac{3\pi}{8} \times \dfrac{2}{5} h^{\frac{5}{2}} + c$

$t = -\dfrac{3\pi}{20} h^{\frac{5}{2}} + c$ **[M1]**

When $t = 0$, $h = 4$ $\Rightarrow \quad 0 = -\dfrac{3\pi}{20} \times (4)^{\frac{5}{2}} + c$

$\Rightarrow \quad c = \dfrac{3\pi}{20} \times 2^5$

$\Rightarrow \quad c = \dfrac{24\pi}{5}$ **[A1]**

Hence $t = -\dfrac{3\pi}{20} h^{\frac{5}{2}} + \dfrac{24\pi}{5}$

Solution will have passed through filter when $h = 0$

$t = -\frac{3\pi}{20}(0)^{\frac{5}{2}} + \frac{24\pi}{5}$

$t = \frac{24\pi}{5}$ minutes

The solution will have completely passed through the filter after $\frac{24\pi}{5}$ minutes. **[A1]**

Total 5 marks

Question 11

a.

Finding an expression for v in terms of x.

$a = -\frac{3\sqrt{x}}{8}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{3\sqrt{x}}{8}$

$\frac{1}{2}v^2 = \int -\frac{3\sqrt{x}}{8}\, dx$ **[M1]**

$\frac{1}{2}v^2 = -\frac{3}{8}\int x^{\frac{1}{2}}\, dx$

$\frac{1}{2}v^2 = -\frac{3}{8} \times \frac{2}{3} x^{\frac{3}{2}} + c$

Initially $x = 0$, $v = -2$

$\frac{1}{2}(-2)^2 = -\frac{1}{4}(0)^{\frac{3}{2}} + c$

$\therefore \quad c = 2$

$\frac{1}{2}v^2 = 2 - \frac{1}{4}x^{\frac{3}{2}}$ **[A1]**

$v^2 = 4 - \frac{1}{2}x^{\frac{3}{2}}$

$v = -\sqrt{4 - \frac{1}{2}x^{\frac{3}{2}}}$ Negative root since $v = -2$ m/s when $x = 0$ **[A1]**

b.

Acceleration valid for $4 - \frac{1}{2}x^{\frac{3}{2}} \geq 0$

$8 - \left(\sqrt{x}\right)^3 \geq 0$

$x \geq 0$ and $\sqrt{x} \leq 2$

$\therefore \quad 0 \leq x \leq 4$ **[A1]**

Total 4 marks

