

Year 2009

VCE

Specialist Mathematics

Solutions

Trial Examination 1



KILBAHA MULTIMEDIA PUBLISHING
PO BOX 2227
KEW VIC 3101
AUSTRALIA

TEL: (03) 9817 5374
FAX: (03) 9817 4334
kilbaha@gmail.com
<http://kilbaha.googlepages.com>

IMPORTANT COPYRIGHT NOTICE

This material is copyright. Subject to statutory exception and to the provisions of the relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Kilbaha Pty Ltd.

- The contents of this work are copyrighted. Unauthorised copying of any part of this work is illegal and detrimental to the interests of the author.
 - For authorised copying within Australia please check that your institution has a licence from Copyright Agency Limited. This permits the copying of small parts of the material, in limited quantities, within the conditions set out in the licence.
 - Teachers and students are reminded that for the purposes of school requirements and external assessments, students must submit work that is clearly their own.
 - Schools which purchase a licence to use this material may distribute this electronic file to the students at the school for their exclusive use. This distribution can be done either on an Intranet Server or on media for the use on stand-alone computers.
 - Schools which purchase a licence to use this material may distribute this printed file to the students at the school for their exclusive use.
-
- **The Word file (if supplied) is for use ONLY within the school.**
 - **It may be modified to suit the school syllabus and for teaching purposes.**
 - **All modified versions of the file must carry this copyright notice.**
 - **Commercial use of this material is expressly prohibited.**

Question 1

$$(x^2 + y^2)^2 = x^3 + y^3 \text{ expanding gives } x^4 + 2x^2y^2 + y^4 = x^3 + y^3$$

taking $\frac{d}{dx}$ of each term (implicit differentiation)

$$\frac{d}{dx}(x^4) + \frac{d}{dx}(2x^2y^2) + \frac{d}{dx}(y^4) = \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3)$$

product rule in the second term

$$4x^3 + 4xy^2 + 4x^2y \frac{dy}{dx} + 4y^3 \frac{dy}{dx} = 3x^2 + 3y^2 \frac{dy}{dx} \quad \text{M1}$$

$$4x^3 + 4xy^2 - 3x^2 = (3y^2 - 4x^2y - 4y^3) \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{4x^3 + 4xy^2 - 3x^2}{3y^2 - 4x^2y - 4y^3} \quad \text{A1}$$

Question 2

$$y = 4 \sin(3x) \quad V = \pi \int_a^b y^2 dx$$

$$V = \pi \int_0^{\frac{\pi}{6}} 16 \sin^2(3x) dx \quad \text{A1}$$

$$V = 8\pi \int_0^{\frac{\pi}{6}} (1 - \cos(6x)) dx \quad \text{M1}$$

$$V = 8\pi \left[x - \frac{1}{6} \sin(6x) \right]_0^{\frac{\pi}{6}}$$

$$V = 8\pi \left[\left(\frac{\pi}{6} - \frac{1}{6} \sin(\pi) \right) - \left(0 - \frac{1}{6} \sin(0) \right) \right]$$

$$V = \frac{4\pi^2}{3} \quad \text{A1}$$

Question 3

Let $\alpha = 2 + \sqrt{3}i$, then by the conjugate root theorem, since a and b are real,

$\beta = 2 - \sqrt{3}i$ is also a root. Now $\alpha + \beta = 4$ and $\alpha\beta = 4 - 3i^2 = 7$, so that

$z^2 - 4z + 7$ is a factor. A1

$$P(z) = z^3 + az^2 + bz - 21 = 0$$

$$P(z) = (z^2 - 4z + 7)(z - 3) = 0 \quad \text{expanding gives}$$

$$z^2: \quad a = -3 - 4 = -7 \quad \text{A1}$$

$$z: \quad b = 7 + 12 = 19$$

all the roots are $z = 2 \pm \sqrt{3}i$ and $z = 3$. A1

Question 4

$$\underline{c} = \alpha \underline{a} + \beta \underline{b}$$

$$\underline{c} = 3\underline{i} + y\underline{j} + 7\underline{k} = \alpha(2\underline{i} - 3\underline{j} + 4\underline{k}) + \beta(\underline{i} - \underline{j} + \underline{k}) \quad \text{M1}$$

$$\underline{i} \quad (1) \quad 3 = 2\alpha + \beta$$

$$\underline{j} \quad (2) \quad y = -3\alpha - \beta \quad \text{A1}$$

$$\underline{k} \quad (3) \quad 7 = 4\alpha + \beta$$

$$(3) - (1) \Rightarrow 2\alpha = 4$$

$$\alpha = 2 \quad \text{and} \quad \beta = -1, \text{ substituting gives } y = -5 \quad \text{A1}$$

Question 5

$$\text{a.} \quad v = \sqrt{9 - 4x^2} \quad \Rightarrow \quad \frac{dv}{dx} = -8x \times \frac{1}{2} x (9 - 4x^2)^{-\frac{1}{2}} = \frac{-4x}{\sqrt{9 - 4x^2}}$$

$$a = v \frac{dv}{dx} = -4x \quad \text{A1}$$

$$\text{b.} \quad v = \frac{dx}{dt} = \sqrt{9 - 4x^2}$$

$$t = \int \frac{1}{\sqrt{9 - 4x^2}} dx \quad \text{A1}$$

$$t = \frac{1}{2} \sin^{-1} \left(\frac{2x}{3} \right) + C \quad \text{now when } x = 0 \quad t = 0 \Rightarrow C = 0 \quad \text{A1}$$

$$2t = \sin^{-1} \left(\frac{2x}{3} \right) \quad \Rightarrow \quad \sin(2t) = \frac{2x}{3}$$

$$x = \frac{3}{2} \sin(2t) \quad \text{A1}$$

Question 6

a. $\vec{r}(t) = \left(t + \frac{1}{t}\right)\vec{i} + \left(t^2 + \frac{1}{t^2}\right)\vec{j}$ for $t > 0$ vector equation,

the parametric equations are (1) $x = t + \frac{1}{t}$ (2) $y = t^2 + \frac{1}{t^2}$ M1

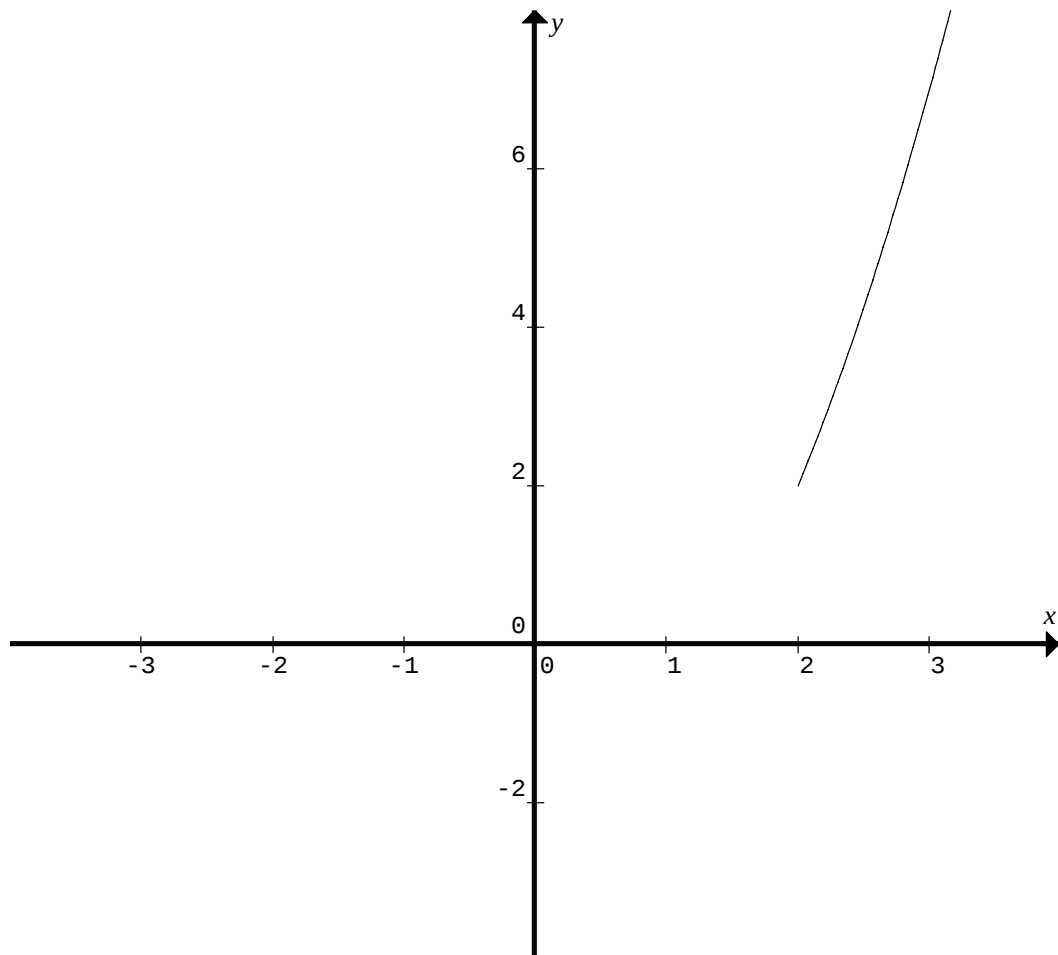
squaring (1) $x^2 = t^2 + 2 + \frac{1}{t^2} = \left(t^2 + \frac{1}{t^2}\right) + 2 = y + 2$

$y = x^2 - 2$ is the Cartesian equation of the path. A1

b. since $t > 0$, the minimum value of x , occurs when $\frac{dx}{dt} = 1 - \frac{1}{t^2} = 0 \Rightarrow t = 1$

$\Rightarrow x \geq 2$ and $y \geq 2$ A1

graph starts from the point (2,2) G1



Question 7

$$y = \frac{x^4 - 16}{2x^2} = \frac{x^2}{2} - \frac{8}{x^2}$$

$y = \frac{x^2}{2}$ is an asymptote, and $x = 0$ is a vertical asymptote A1

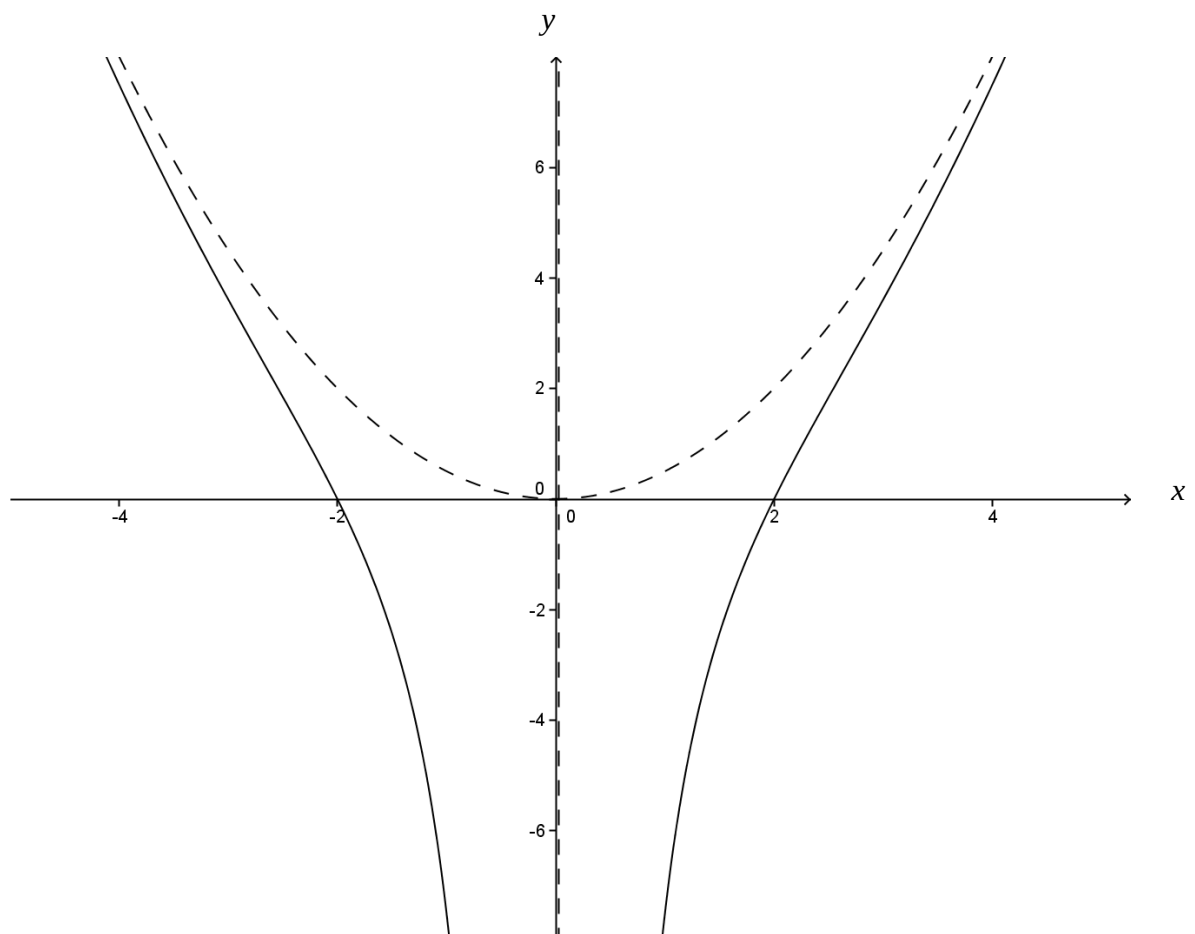
the graph does not cross the y-axis,

crosses the x-axis when $y = 0 \Rightarrow x^4 = 16 \Rightarrow x = \pm 2$ at $(2, 0)$ $(-2, 0)$ A1

for turning points, $\frac{dy}{dx} = x + \frac{16}{x^3} = 0 \Rightarrow x^4 = -16$ this has no real solution,

so there are no turning points. A1

correct graph G1



Question 8

$$\frac{dy}{dx} = \frac{x}{\sqrt{2x+3}}$$

$$y = \int \frac{x}{\sqrt{2x+3}} dx \quad \text{let } u = 2x+3 \quad \frac{du}{dx} = 2 \quad x = \frac{1}{2}(u-3)$$

$$y = \frac{1}{4} \int \frac{u-3}{\sqrt{u}} du$$

$$y = \frac{1}{4} \int \left(u^{\frac{1}{2}} - 3u^{-\frac{1}{2}} \right) du$$

M1

$$y = \frac{1}{4} \left(\frac{2}{3} u^{\frac{3}{2}} - 6u^{\frac{1}{2}} \right) + C$$

$$y = \frac{u^{\frac{1}{2}}}{2} \left(\frac{u}{3} - 3 \right) + C = \frac{\sqrt{u}}{2} \left(\frac{u-9}{3} \right) + C$$

A1

$$y = \frac{1}{6} (2x-6) \sqrt{2x+3} + C = \frac{1}{3} (x-3) \sqrt{2x+3} + C$$

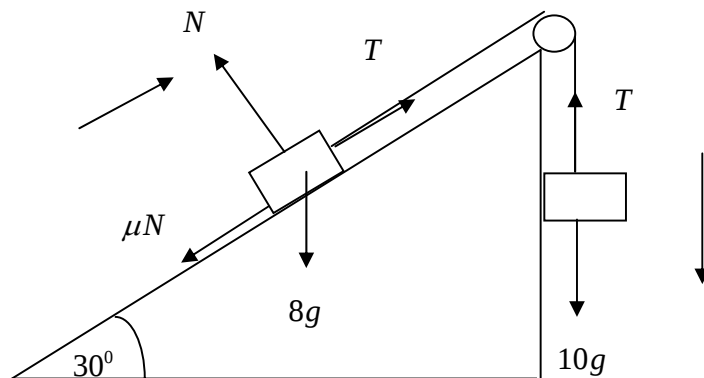
$$\text{now when } x=3 \quad y=0 \Rightarrow C=0$$

$$y = \left(\frac{x}{3} - 1 \right) \sqrt{2x+3} \quad a = \frac{1}{3} \quad b = -1$$

A1

Question 9

i.



correct forces on the diagram

A1

ii. resolving downwards for the 10 kg weight hanging vertically

$$(1) \quad 10g - T = 10a$$

for the crate on the incline plane, resolving upwards parallel to the plane

$$(2) \quad T - \mu N - 8g \sin(30^\circ) = 8a \quad \text{A1}$$

resolving perpendicular to the plane

$$(3) \quad N - 8g \cos(30^\circ) = 0 \quad N = 8g \cos(30^\circ)$$

$$(2) \text{ becomes } T - 8\mu g \cos(30^\circ) - 8g \sin(30^\circ) = 8a \quad \text{A1}$$

adding this to equation (1) to eliminate T ,

$$10g - 8g(\sin(30^\circ) + \mu \cos(30^\circ)) = 18a \text{ substituting} \quad \text{M1}$$

$$\mu = \frac{\sqrt{3}}{4} \quad \cos(30^\circ) = \frac{\sqrt{3}}{2} \quad \sin(30^\circ) = \frac{1}{2}$$

$$g \left[10 - 8 \left(\frac{1}{2} + \frac{3}{8} \right) \right] = 18a$$

$$a = \frac{g}{6} \text{ m/s}^2 \quad \text{A1}$$

Question 10

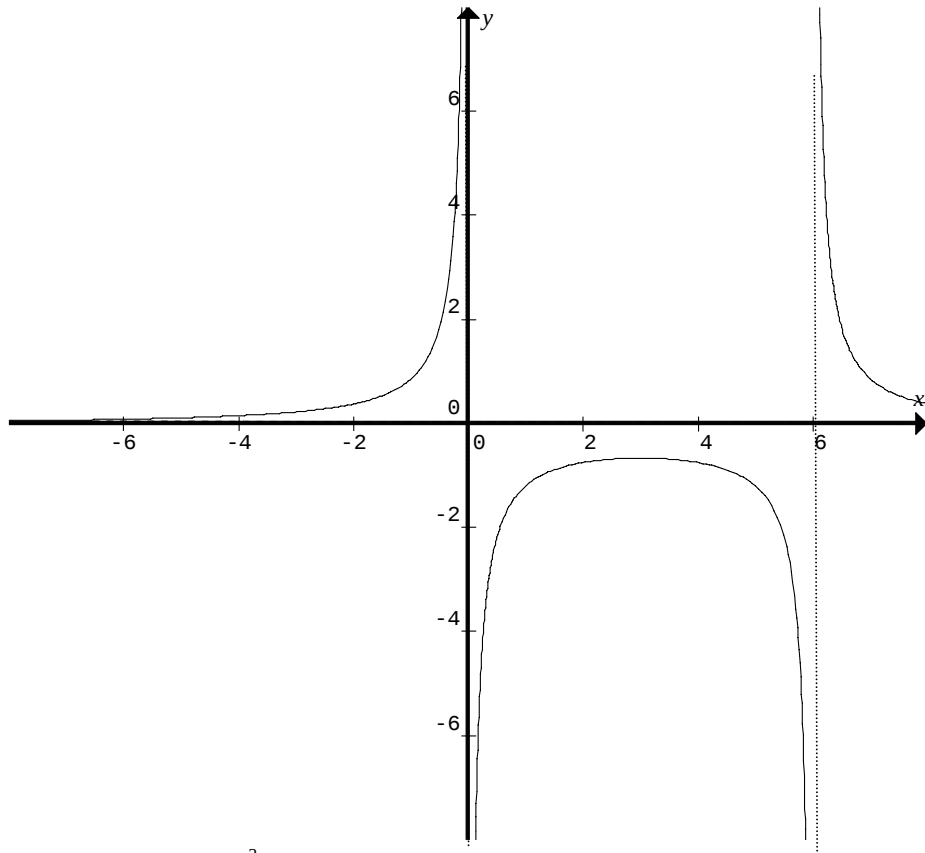
a. $y = \frac{6}{x^2 - 6x} = \frac{6}{x(x-6)}$

vertical asymptotes at $x = 0$ and $x = 6$

horizontal asymptotes at $y = 0$ (the x -axis) A1

the turning point occurs when $2x - 6 = 0 \Rightarrow x = 3$

the maximum turning point is $\left(3, -\frac{2}{3} \right)$ and correct graph A1



b. the area is $\int_1^3 \frac{6}{x^2 - 6x} dx$ but this is below the x -axis and negative,

so the area is $A = \int_1^3 \frac{6}{6x - x^2} dx$ A1

by partial fractions $\frac{6}{6x - x^2} = \frac{B}{x} + \frac{C}{6 - x}$ adding the partial fractions

$$= \frac{B(6 - x) + Cx}{x(6 - x)} = \frac{x(C - B) + 6B}{6x - x^2}$$

(1) $6B = 6$ and (2) $C - B = 0$ so that $B = C = 1$

$$A = \int_1^3 \frac{6}{6x - x^2} dx = \int_1^3 \left(\frac{1}{x} + \frac{1}{6 - x} \right) dx$$
 M1

$$A = \left[\log_e(x) - \log_e(6 - x) \right]_1^3 = \left[\log_e \left(\frac{x}{6 - x} \right) \right]_1^3$$

$$A = \left[\log_e(1) - \log_e \left(\frac{1}{5} \right) \right] = \log_e(5) \quad a = 5$$
 A1

Question 11

$$(\sqrt{3} + i)^m - (\sqrt{3} - i)^m = 0$$

$$\text{now } \sqrt{3} + i = 2\text{cis}\left(\frac{\pi}{6}\right) \quad \text{and} \quad \sqrt{3} - i = 2\text{cis}\left(-\frac{\pi}{6}\right) \quad \text{A1}$$

$$\left(2\text{cis}\left(\frac{\pi}{6}\right)\right)^m - \left(2\text{cis}\left(-\frac{\pi}{6}\right)\right)^m = 0, \text{ using DeMoivre's theorem}$$

$$2^m \text{cis}\left(\frac{m\pi}{6}\right) - 2^m \text{cis}\left(-\frac{m\pi}{6}\right) = 0 \quad \text{M1}$$

$$2^m \left(\left(\cos\left(\frac{m\pi}{6}\right) + i \sin\left(\frac{m\pi}{6}\right) \right) - \left(\cos\left(-\frac{m\pi}{6}\right) + i \sin\left(-\frac{m\pi}{6}\right) \right) \right) = 0$$

but $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$, so that

$$2^{m+1} i \sin\left(\frac{m\pi}{6}\right) = 0 \quad \text{A1}$$

$$\sin\left(\frac{m\pi}{6}\right) = 0$$

$$\frac{m\pi}{6} = 0, \pi, 2\pi, 3\pi, \dots = k\pi$$

$$m = 6k \quad \text{where} \quad k \in \mathbb{Z} \quad \text{A1}$$

END OF SUGGESTED SOLUTIONS