

2009 VCAA Specialist Math Exam 2 Solutions

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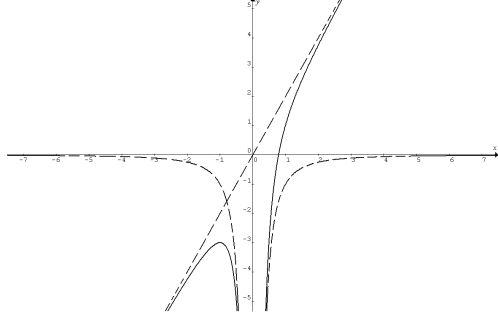
Section 1

1	2	3	4	5	6	7	8	9	10	11
E	D	D	A	C	A	C	E	B	B	C

12	13	14	15	16	17	18	19	20	21	22
E	A	B	C	D	D	B	A	E	D	B

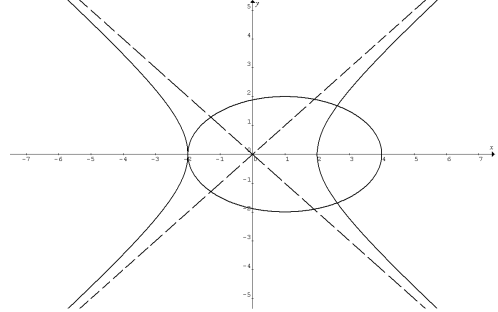
Q1 Sketch graph: addition of ordinates.

E



Q2 Sketch the ellipse and the hyperbola.

D



Q3 $-1 \leq x-b \leq 1$, $b-1 \leq x \leq b+1$, $b-1=2$, $b=3$.
 $a\pi = 6\pi$, $a=6$.

D

Q4 $\sec t = \frac{x+1}{2}$, $\tan t = \frac{y-1}{3}$, $\sec^2 t = 1 + \tan^2 t$,

$$\left(\frac{x+1}{2}\right)^2 = 1 + \left(\frac{y-1}{3}\right)^2, \therefore \frac{(x+1)^2}{4} - \frac{(y-1)^2}{9} = 1$$

A

Q5 $x^2 + 2ax + 2y^2 + 4by = -16$, $x^2 + 2ax + 2(y^2 + 2by) = -16$,

$$(x+a)^2 + \frac{(y+b)^2}{\frac{1}{2}} = \text{const.} \therefore a = -3 \text{ and } b = 2.$$

C

Q6 Distance between z and $-\bar{z}$ is $|z - (-\bar{z})| = |z + \bar{z}| = |2\text{Re}(z)|$
Best choice

A

Q7 The conjugate $z = -2 - i$ is also a root of $P(z) = 0$.

C

Q8 $(1+i)^n = ai$, $(1+i)^{2n+2} = ((1+i)^n)^2 (1+i)^2 = (ai)^2 (2i) = -2a^2 i$

E

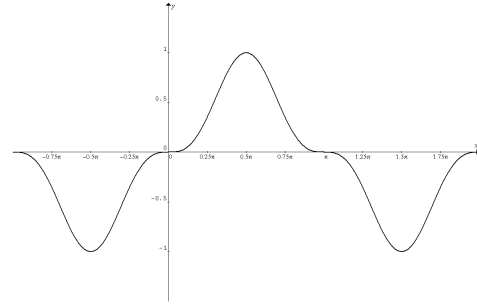
$$\text{Q9 } \frac{(x-6)^2}{a^2} + \frac{(y-3)^2}{b^2} = 1.$$

Implicit differentiation: $\frac{2(x-6)}{a^2} + \frac{2(y-3)}{b^2} \frac{dy}{dx} = 0$

$$\therefore \frac{dy}{dx} = -\frac{b^2(x-6)}{a^2(y-3)} = \frac{b^2(6-x)}{a^2(y-3)}.$$

B

Q10



$$\text{Area} = 3 \times \int_0^{\pi} (\sin x)^3 dx = 3 \times \int_0^{\pi} (\sin x)^2 \sin x dx$$

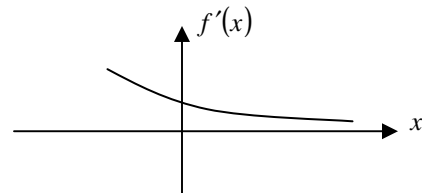
$$= 3 \times \int_0^{\pi} (1 - \cos^2 x) \sin x dx$$

$$\text{Let } u = \cos x, \quad -\frac{du}{dx} = \sin x$$

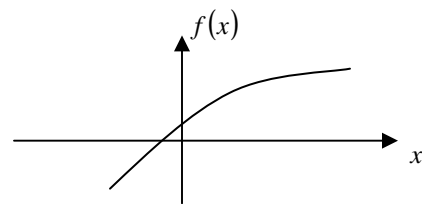
$$= -3 \times \int_1^{-1} (1 - u^2) du = 3 \times \int_{-1}^1 (1 - u^2) du$$

B

Q11 $f'(x) > 0$ and $f''(x) < 0$, the graph of $f'(x)$ would be



and a corresponding graph of $f(x)$ would be



C

Q12 $v = f(x)$, $a = v \frac{dv}{dx} = f(x)f'(x)$

E

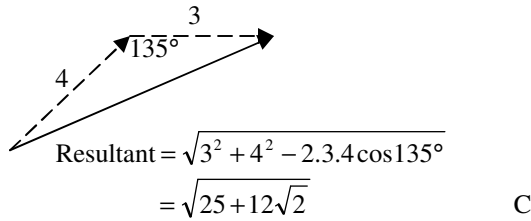
Q13 Let $Q = 100x$ kg be the amount of salt in the tank at time t minutes (Note: t seconds in the question). Rate of salt input = 0, and rate of salt output = $10x$ kg per min.

$$\therefore \frac{dQ}{dt} = -10x, \quad 100 \frac{dx}{dt} = -10x, \quad \therefore 10 \frac{dx}{dt} + x = 0$$

A

Q14 $\vec{u} = \vec{v} = \vec{w}$ when $m = 1$. $\therefore 2\vec{u} - \vec{v} - \vec{w} = \vec{0}$ as an example. Hence they are linearly dependent. B

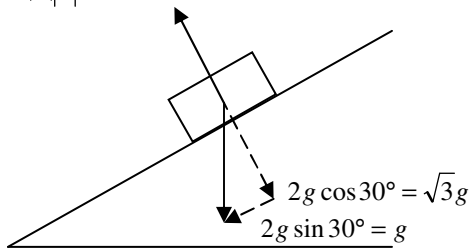
Q15



Q16 $\vec{c} \cdot \vec{a} = 0$, $\vec{c} \cdot \vec{b} = 0$, $\therefore \vec{a}$ and $\vec{b}$ are perpendicular to $\vec{c}$. D

Q17 $\vec{c} + \vec{b} = \vec{a}$, $\therefore \vec{c} = \vec{a} - \vec{b}$
 $\therefore \vec{c} \cdot \vec{c} = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = \vec{a} \cdot \vec{a} - 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}$
 $\therefore |\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos 120^\circ$
 $\therefore |\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{a}||\vec{b}|$

Q18



Force of friction $= \mu N = 0.1 \times \sqrt{3}g = \frac{\sqrt{3}g}{10}$

$F = ma$, $g - \frac{\sqrt{3}g}{10} = 2a$, $g \left(1 - \frac{\sqrt{3}}{10}\right) = 2a$ B

Q19 Vertical: $u = 20 \sin 45^\circ = +10\sqrt{2}$, $v = -10\sqrt{2}$, $a = -g$.

Substitute into $v = u + at$, $-10\sqrt{2} = 10\sqrt{2} - gt$, $t = \frac{20\sqrt{2}}{g}$ A

Q20 $v = x$, $\frac{dx}{dt} = x$, $\frac{dt}{dx} = \frac{1}{x}$, $t = \log_e x + c$.

When $t = 3$, $x = 1$, $\therefore c = 3$ and $t = \log_e x + 3$.

Hence $x = e^{t-3}$. E

Q21 $u = +4$, $a = +2$, $s = +21$, to find v , substitute into $v^2 = u^2 + 2as$. $\therefore v = +10$.

Magnitude of momentum $= 5 \times 10 = 50 \text{ kg ms}^{-1}$ D

Q22 To find the distance from the starting point, firstly find the displacement from the starting point = signed area bounded by the graph and the t -axis.

$s = \frac{1}{2} \times 10 \times 2 - \frac{1}{2} (1 + 4) 5 + \frac{1}{2} \times 10 \times 4 = 17.5$ B

Section 2

Q1a



Q1b Distance $= \int_0^9 t^{\frac{3}{2}} dt = \left[\frac{2t^{\frac{5}{2}}}{5} \right]_0^9 = 97.2 \text{ m}$

Q1c Distance $= \int_{39}^{51} 27 \cos \left(\frac{\pi}{24} (t - 39) \right) dt$
 $= \left[\frac{24 \times 27 \sin \left(\frac{\pi}{24} (t - 39) \right)}{\pi} \right]_{39}^{51} = \frac{648}{\pi} = 206.3 \text{ m}$ D

Q1d Average speed $= \frac{\text{total distance}}{\text{time taken}}$
 $= \frac{97.2 + 27 \times 30 + 206.3}{51} = 21.8 \text{ ms}^{-1}$

Q1e Let $t^{\frac{3}{2}} = \frac{200}{9}$, $t = t_1 = \left(\frac{200}{9} \right)^{\frac{2}{3}} \approx 7.9 \text{ s}$

Let $27 \cos \left(\frac{\pi}{24} (t - 39) \right) = \frac{200}{9}$, use calc. to find $t = t_2 = 43.6 \text{ s}$

Q1f Let T be the time in seconds, where $9 < T < 39$ (refer to the graph).

Distance by motorcycle = distance by car

$20T = 97.2 + 27(T - 9)$,

$T = 20.829 \approx 20.8 \text{ s}$ and distance $= 20T = 20 \times 20.829 \approx 417 \text{ m}$

Q2a Let $z = 0$, $|-1| = 1$, $\left| -\frac{1}{2} - \frac{\sqrt{3}}{2}i \right| = 1$.

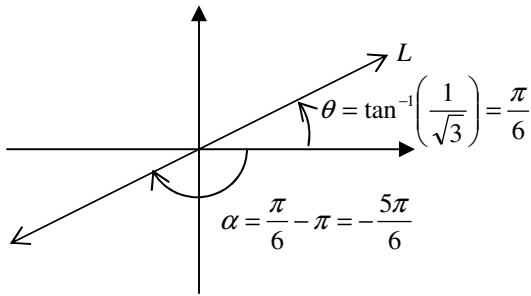
$\therefore (0,0)$ lies on L .

Q2b Let $z = x + yi$. $|(x-1) + yi| = \left| \left(x - \frac{1}{2} \right) + \left(y - \frac{\sqrt{3}}{2} \right) i \right|$

$(x-1)^2 + y^2 = \left(x - \frac{1}{2} \right)^2 + \left(y - \frac{\sqrt{3}}{2} \right)^2$.

Expand and simplify to $y = \frac{1}{\sqrt{3}}x$.

Q2c



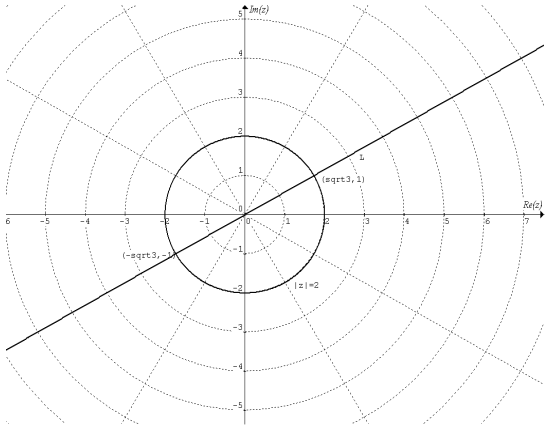
Q2d $|z| = 2$ is $x^2 + y^2 = 4$ (1)

L is $y = \frac{1}{\sqrt{3}}x$ (2)

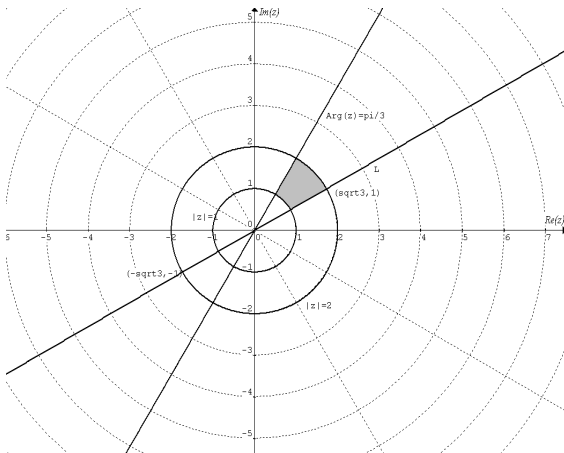
Substitute (2) into (1): $x^2 + \frac{x^2}{3} = 4$, $\therefore x^2 = 3$, $x = \pm\sqrt{3}$ and $y = \pm 1$.

The points of intersection are $(-\sqrt{3}, -1)$ and $(\sqrt{3}, 1)$.

Q2e



Q2f



Shaded area = $\frac{1}{12}(\pi 2^2 - \pi 1^2) = \frac{\pi}{4}$ square units.

Q3a $\tilde{r} = 5 \sin\left(\frac{\pi}{6}t\right)\tilde{i} + 5 \cos\left(\frac{\pi}{6}t\right)\tilde{j} + \left(24.5 - \frac{t^2}{8}\right)\tilde{k}$.

The height above the ground at time t is given by $24.5 - \frac{t^2}{8}$.
At $t = 0$, height = 24.5 metres.

Q3b Let $24.5 - \frac{t^2}{8} = 0$, $t = 14$ s.

Q3c Period of one loop = $\frac{2\pi}{\frac{\pi}{6}} = 12$ s, time taken = 12 s.

Q3d $\tilde{r} = \frac{5\pi}{6} \cos\left(\frac{\pi}{6}t\right)\tilde{i} - \frac{5\pi}{6} \sin\left(\frac{\pi}{6}t\right)\tilde{j} - \frac{t}{4}\tilde{k}$

Q3e At $t = 14$,

$\tilde{r} = \frac{5\pi}{6} \cos\left(\frac{7\pi}{3}\right)\tilde{i} - \frac{5\pi}{6} \sin\left(\frac{7\pi}{3}\right)\tilde{j} - \frac{7}{2}\tilde{k}$
 $= 1.309\tilde{i} - 2.267\tilde{j} - 3.5\tilde{k}$.

Speed = $|\tilde{r}| = \sqrt{1.309^2 + 2.267^2 + 3.5^2} \approx 4.4 \text{ ms}^{-1}$.

Q3f $\tilde{a} = \tilde{r} = -\frac{5\pi^2}{36} \sin\left(\frac{\pi}{6}t\right)\tilde{i} - \frac{5\pi^2}{36} \cos\left(\frac{\pi}{6}t\right)\tilde{j} - \frac{1}{4}\tilde{k}$

$|\tilde{a}| = \sqrt{\left(\frac{5\pi^2}{36} \sin\left(\frac{\pi}{6}t\right)\right)^2 + \left(\frac{5\pi^2}{36} \cos\left(\frac{\pi}{6}t\right)\right)^2 + \frac{1}{16}}$
 $= \sqrt{\left(\frac{5\pi^2}{36}\right)^2 \left(\sin^2\left(\frac{\pi}{6}t\right) + \cos^2\left(\frac{\pi}{6}t\right)\right) + \frac{1}{16}}$
 $= \sqrt{\left(\frac{5\pi^2}{36}\right)^2 + \frac{1}{16}}$ is a constant.

Q3gi $\tilde{r} = \frac{5\pi}{6} \cos\left(\frac{\pi}{6}t\right)\tilde{i} - \frac{5\pi}{6} \sin\left(\frac{\pi}{6}t\right)\tilde{j} - \frac{t}{4}\tilde{k}$,

$\therefore |\tilde{r}| = \sqrt{\frac{25\pi^2}{36} + \frac{t^2}{16}}$.

Distance from start to finish = $\int_0^{14} \sqrt{\frac{25\pi^2}{36} + \frac{1}{16}t^2} dt$.

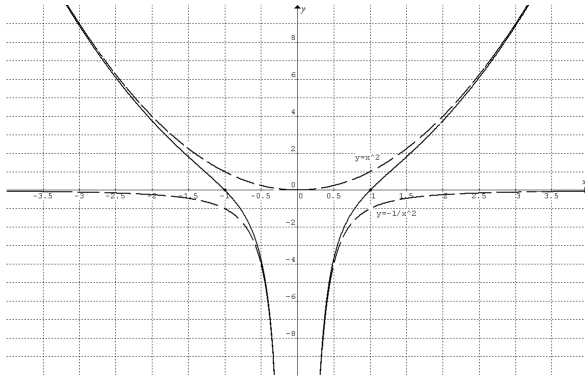
Q3gii Evaluate the definite integral by graphics calc.
Distance ≈ 45.7 metres.

Q4a Let $\frac{x^4 - 1}{x^2} = -10$, use graphics calc. to find
 $x = \pm 0.315 \approx \pm 0.3$.

Let $\frac{x^4 - 1}{x^2} = 10$, use graphics calc. to find $x \approx \pm 3.2$.
 $\therefore [b, a] \approx [0.3, 3.2]$, $\therefore a \approx 3.2$ and $b \approx 0.3$.

Q4b Let $\frac{x^4-1}{x^2} = 0$ to find the x -intercepts. $x = \pm 1$.

$y = \frac{x^4-1}{x^2} = x^2 - \frac{1}{x^2}$. Sketch by addition of ordinates.



Q4c $x^4 - yx^2 - 1 = 0$, $(x^2)^2 - yx^2 - 1 = 0$,

$$\therefore x^2 = \frac{-(-y) \pm \sqrt{(-y)^2 - 4(1)(-1)}}{2(1)} = \frac{y \pm \sqrt{y^2 + 4}}{2}.$$

Since $x^2 > 0$, [note: $x \neq 0$ (refer to $y = \frac{x^4-1}{x^2}$)],

$$x^2 = \frac{y - \sqrt{y^2 + 4}}{2} \text{ is rejected.}$$

$$\therefore x^2 = \frac{y + \sqrt{y^2 + 4}}{2}.$$

$$\text{Q4di } V = \int_{-10}^{10} \pi x^2 dy = \int_{-10}^{10} \frac{\pi}{2} (y + \sqrt{y^2 + 4}) dy$$

Q4dii Use graphics calc. to evaluate the definite integral,
 $V = 174.7 \text{ cm}^3$.

$$\text{Q4e } \frac{dV}{dt} = +1.5 \text{ cm}^3 \text{ s}^{-1}, \frac{dV}{dy} = \pi x^2.$$

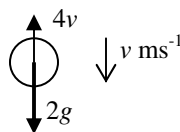
$$\frac{dy}{dt} = \frac{dy}{dV} \times \frac{dV}{dt} = \frac{1}{\pi x^2} \times \frac{dV}{dt}.$$

When the surface is 6 cm from the top, $y = 4$ and

$$x^2 = \frac{4 + \sqrt{16 + 4}}{2} = 4.236.$$

$$\therefore \frac{dy}{dt} = \frac{1}{\pi \times 4.236} \times 1.5 \approx 0.11 \text{ cm per second.}$$

Q5a



$$a = \frac{F}{m} = \frac{2g - 4v}{2} = g - 2v.$$

$$\text{Q5b } a = g - 2v, \frac{dv}{dt} = g - 2v, \frac{dt}{dv} = \frac{1}{g - 2v}, t = \int \frac{1}{g - 2v} dv,$$

$$t = -\frac{\log_e(g - 2v)}{2} + c.$$

$$v = 0 \text{ when } t = 0, \therefore c = \frac{\log_e g}{2} \text{ and } t = 0.5 \log_e \left(\frac{g}{g - 2v} \right).$$

Q5c $a = g - 2v \rightarrow 0$ when $v \rightarrow \frac{g}{2}$, the limiting velocity.

Q5d When $v = \frac{g}{4}$,

$$t = 0.5 \log_e \left(\frac{g}{g - \frac{g}{2}} \right) = 0.5 \log_e 2 = \log_e \sqrt{2} \text{ s after its release.}$$

Q5e $v = \frac{g}{2} (1 - e^{-2t})$, $\frac{dx}{dt} = \frac{g}{2} (1 - e^{-2t})$, where x metres is the displacement from the surface.

$$x = \int_0^{180} \frac{g}{2} (1 - e^{-2t}) dt = 879.6, \text{ evaluated by graphics calc.}$$

The ocean is 880 metres at that location.

$$\text{Q5f When } v = \frac{g}{3}, t = 0.5 \log_e \left(\frac{g}{g - \frac{2g}{3}} \right) = 0.5 \log_e 3 = \log_e \sqrt{3},$$

$$x = \int_0^{\log_e \sqrt{3}} \frac{g}{2} (1 - e^{-2t}) dt \approx 1.1, \text{ evaluated by graphics calc.}$$

The device is 1.1 m below the surface.

$$\text{Q5g } \frac{dx}{dt} = \frac{g}{2} (1 - e^{-2t}),$$

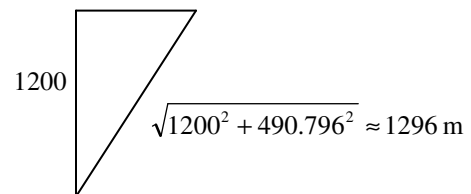
$$x = \frac{g}{2} \int (1 - e^{-2t}) dt = \frac{g}{2} \left(t + \frac{e^{-2t}}{2} \right) + c.$$

$$x = 0 \text{ when } t = 0, \therefore x = \frac{g}{2} \left(t + \frac{e^{-2t}}{2} \right) - \frac{g}{4}.$$

$$x = 1200, 1200 = \frac{g}{2} \left(t + \frac{e^{-2t}}{2} \right) - \frac{g}{4}.$$

By graphics calc. $t = 245.398 \text{ s}$.

$$2t = 490.796$$



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