INSIGHT

Trial Exam Paper

2009

SPECIALIST MATHEMATICS

Written examination 2

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions.
- mark allocations
- tips on how to approach the questions.

SECTION 1

Question 1

The parametric equations $x = 2\sec(t + 4) - 2$ and $y = 3\tan(t + 4) + 1$ define a relation given by

A. $\frac{(x+2)^2}{4} - \frac{(y+1)^2}{9} = 1$

B. $\frac{(x+2)^2}{4} + \frac{(y-1)^2}{9} = 1$

C. $\frac{(x+2)^2}{3} - \frac{(y-1)^2}{2} = 1$

D. $\mathbf{\frac{(x+2)^2}{4} - \frac{(y-1)^2}{9} = 1}$

E. $\frac{(x+2)^2}{2} - \frac{(y-1)^2}{3} = 1$

Answer is D.

Worked solution

$x = 2\sec(t+4) - 2$ and $y = 3\tan(t+4) + 1$

$\sec(t+4) = \frac{x+2}{2}$ $\tan(t+4) = \frac{y-1}{3}$

$\sec^2(t+4) - \tan^2(t+4) = 1$

$\frac{(x+2)^2}{4} - \frac{(y-1)^2}{9} = 1$

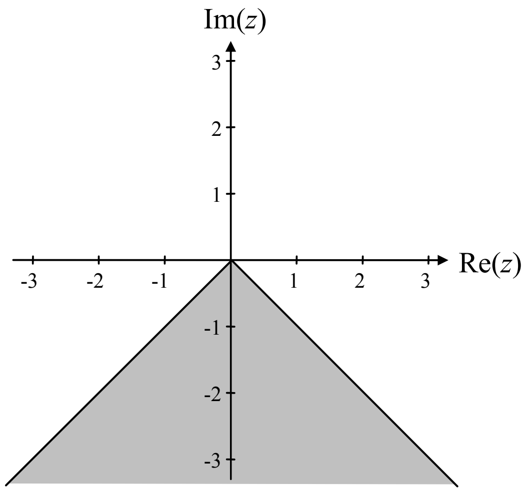
Tip

- *Use the identity* $\sec^2\theta - \tan^2\theta = 1$.

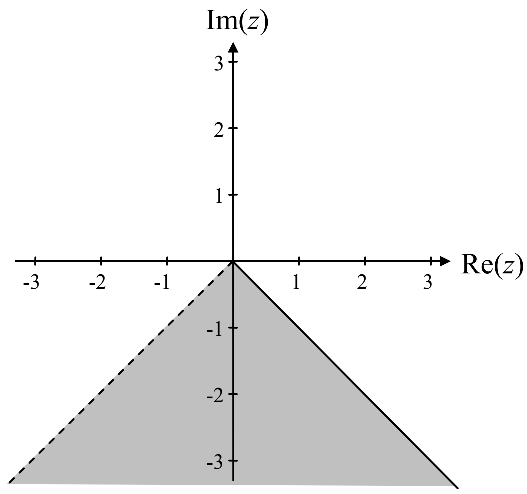
Question 2

The region of the complex plane defined by $\{z\colon -\frac{\pi}{4} \leq \text{Arg}\ i(z-1) < \frac{\pi}{4}\}$ is

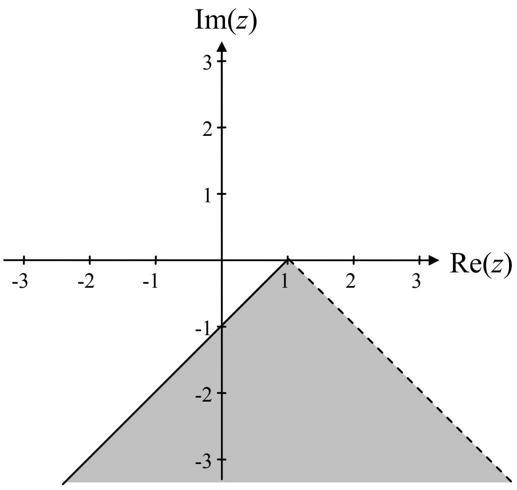
A.



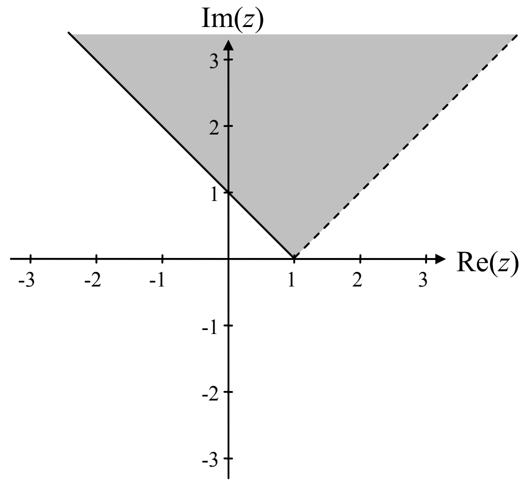
B.



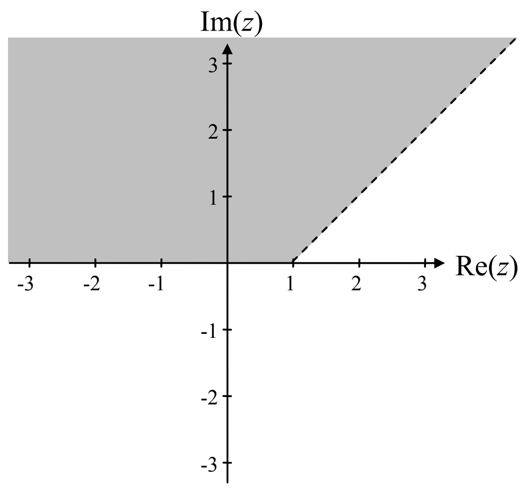
C.



D.



E.



Answer is C.

Worked solution

$$\text{Arg}(i(z-1)) = \text{Arg}(i) + \text{Arg}(z-1)$$

$$= \frac{\pi}{2} + \text{Arg}(z-1)$$

$$-\frac{\pi}{4} \le \frac{\pi}{2} + \text{Arg}(z-1) < \frac{\pi}{4}$$

$$-\frac{3\pi}{4} \le \text{Arg}(z-1) < -\frac{\pi}{4}$$

This describes the region between and below two rays, each starting from the point (1, 0) and making angles of $-\frac{\pi}{4}$ (not included) and $-\frac{3\pi}{4}$ (included) with the positive real axis.

Tip

- $\text{Arg}(ab) = \text{Arg}(a) + \text{Arg}(b)$

Question 3

The maximal domain and range of the function $f(x) = 3\arctan(2x - \pi)$ are given by

A. $d_f = (\pi, 3\pi)$ and $r_f = R$

B. $d_f = R$ and $r_f = [-\frac{\pi}{2}, \frac{\pi}{2}]$

C. $\boldsymbol{d_f = R}$ **and** $\boldsymbol{r_f = (-\frac{3\pi}{2}, \frac{3\pi}{2})}$

D. $d_f = R$ and $r_f = (\frac{\pi}{4}, \frac{3\pi}{4})$

E. $d_f = (-\frac{\pi}{2}, \frac{\pi}{2})$ and $r_f = R$

Answer is C.

Worked solution

$d_f = R$ and $r_f = 3(-\frac{\pi}{2}, \frac{\pi}{2})$

$$= \left(-\frac{3\pi}{2}, \frac{3\pi}{2}\right)$$

Question 4

If $z^2 - z - 2$ is a factor of $P(z) = z^4 - 3z^3 + 10z^2 - 6z - 20,\ \ z \in C$, then all of the factors must be

A. **$z - 2, z + 1, z - 1 + 3i$ and $z - 1 - 3i$**

B. $z - 2, z + 1, z - 1 + 3i$ and $z + 1 - 3i$

C. $z + 2, z - 1, z - 3 + i$ and $z - 3 - i$

D. $z - 2, z + 1, z + 3 + i$ and $z + 3 - i$

E. $z - 2, z + 1, z + 2$ and $z + 5$

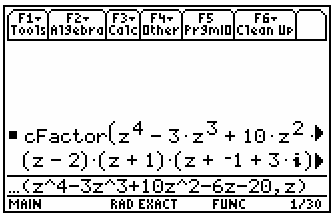
Answer is A.

Worked solution

$$\begin{aligned} P(z) &= z^4 - 3z^3 + 10z^2 - 6z - 20 \\ &= (z^2 - z - 2)(z^2 - 2z + 10) \\ &= (z-2)(z+1)((z-1)^2 + 9) \\ &= (z-2)(z+1)(z-1+3i)(z-1-3i) \end{aligned}$$

The factors are $z - 2$, $z + 1$, $z - 1 + 3i$ and $z - 1 - 3i$.

On calculator:

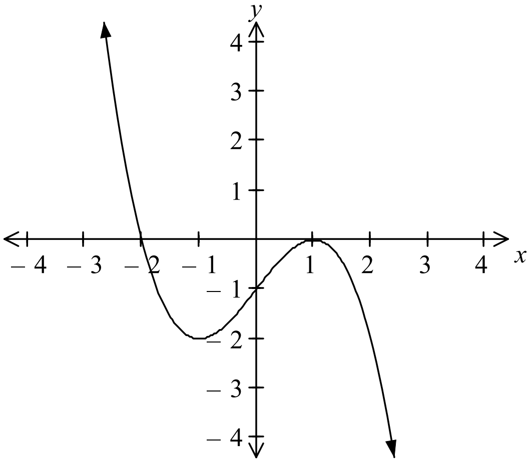


Tip

- *The second quadratic factor can be found by long division of $z^2 - z - 2$ into $P(z) = z^4 - 3z^3 + 10z^2 - 6z - 20$.*

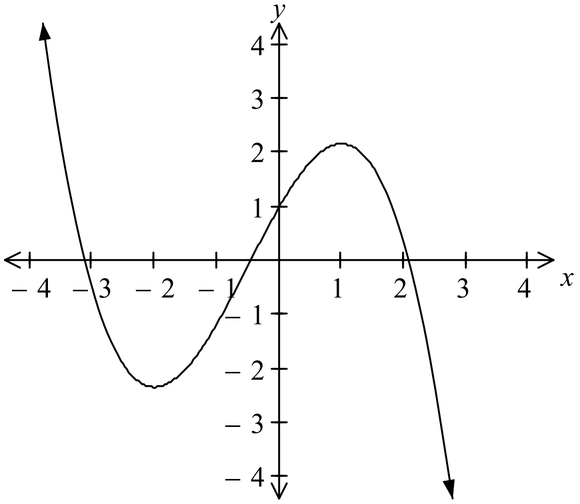
Question 5

The graph of $y = f'(x)$ is shown below.

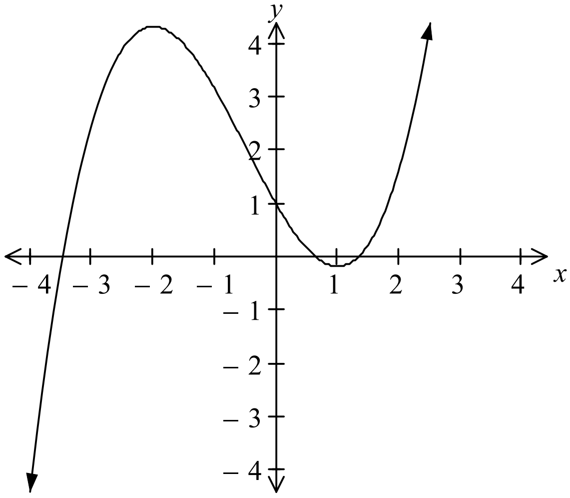


If $f(0) = 1$, then the graph of $y = f(x)$ could be

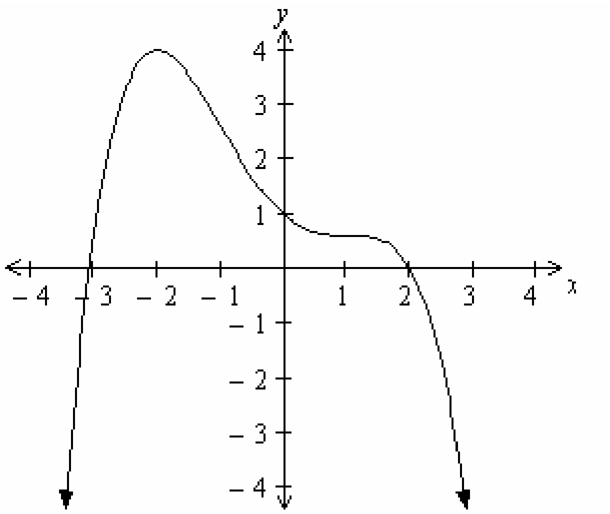
A.



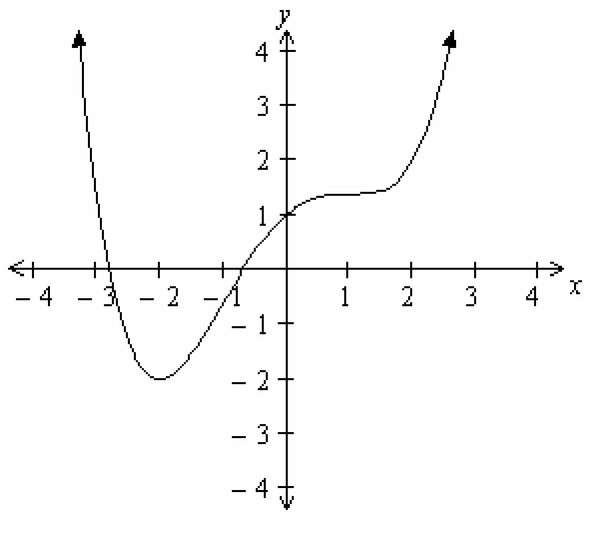
B.



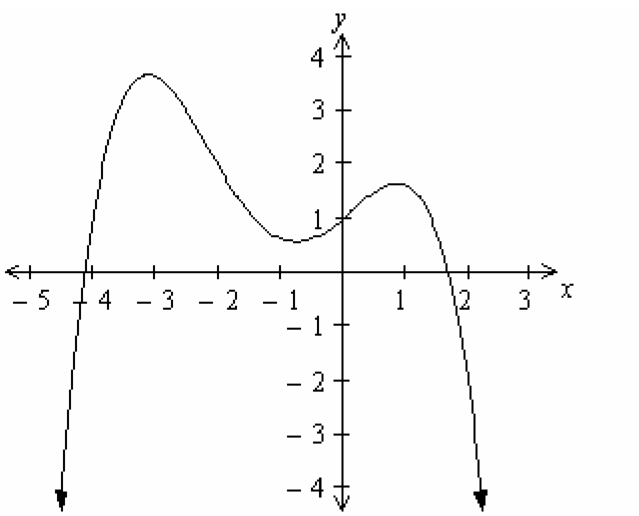
C.



D.



E.



SECTION 1 – Question 5 – continued
TURN OVER

Answer is C.

Worked solution

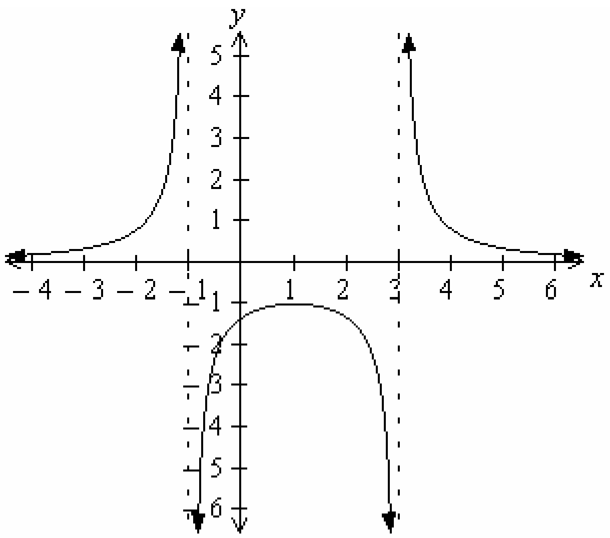
Sign diagram summary of the behaviour of $f'(x)$ and $f(x)$.

x	$x<-2$	$x=-2$	$-2<x<1$	$x=1$	$x>1$
$f'(x)$	>0	$=0$	<0	$=0$	<0
$y=f(x)$	increasing	local maximum	decreasing	stationary point of inflection (negative gradient)	decreasing

The graph of $y=f(x)$ also passes through (0, 1).

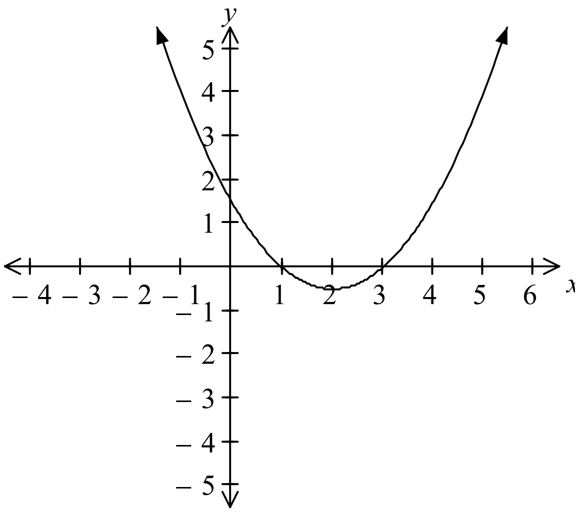
Question 6

The graph of $y=\dfrac{1}{f(x)}$ is shown below.

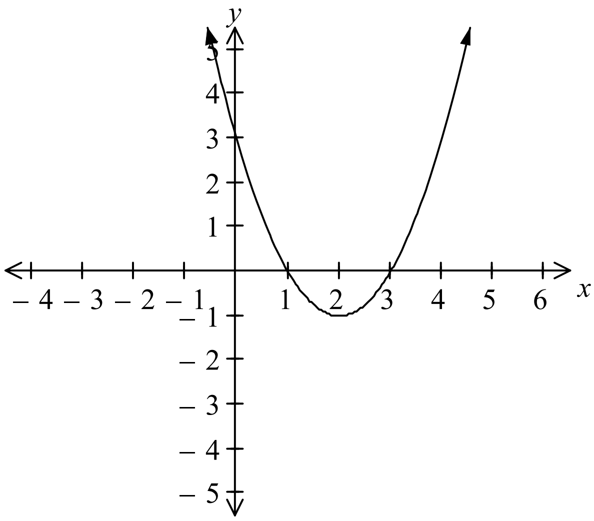


The graph of $y = f(x)$ could be

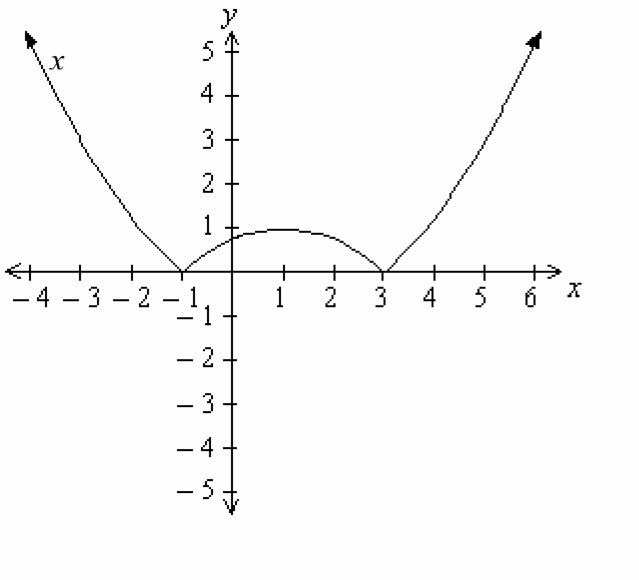
A.



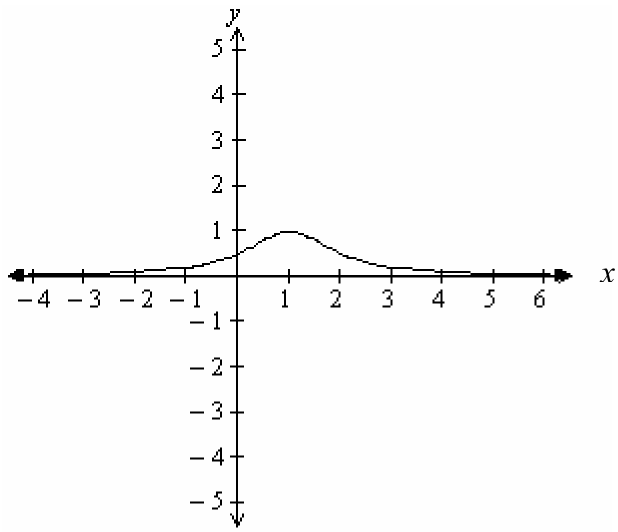
B.



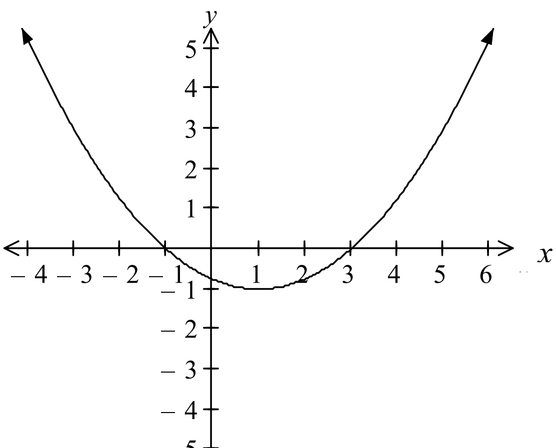
C.



D.



E.



Answer is E.

Worked solution

The graph of $y = \frac{1}{f(x)}$ has vertical asymptotes $x = -1$ and $x = 3$, so the graph of $y = f(x)$ has x-intercepts at $x = -1$ and $x = 3$.

The graph of $y = \frac{1}{f(x)} > 0$ for $x < -1$ and $x > 3$, so the graph of $y = f(x) > 0$ for $x < -1$ and $x > 3$

The graph of $y = \frac{1}{f(x)} < 0$ for $-1 < x < 3$, so the graph of $y = f(x) < 0$ for $-1 < x < 3$.

The graph of $y = \frac{1}{f(x)}$ has a local maximum at (1, –1), so the graph of $y = f(x)$ has a local minimum at (1, –1).

Question 7

The solutions to $z^2 = a + \sqrt{3}ai$, where $z \in C$ and $a \in R^+$, are

A. $\pm\frac{\sqrt{2}}{2}(\sqrt{3}+i)$

B. $\pm\frac{\sqrt{2a}}{2}\left(\sqrt{3}+i\right)$

C. $\pm\frac{\sqrt{2}}{2}(1+\sqrt{3}\,i)$

D. $\pm\frac{\sqrt{2}a}{2}(1+\sqrt{3}\,i)$

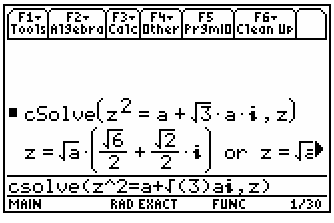
E. $\pm\frac{\sqrt{2}}{2}(1-\sqrt{3}\,i)$

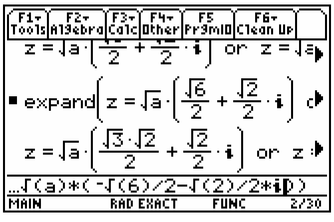
Answer is B.

Worked solution

$$\begin{aligned} z^2 &= a + \sqrt{3}\,ai \\ &= a(1+\sqrt{3}\,i) \\ &= a \times 2\operatorname{cis}\frac{\pi}{3} \\ &= 2a\operatorname{cis}\frac{\pi}{3} \\ z &= \pm\sqrt{2a}\operatorname{cis}\frac{\pi}{6} \\ &= \pm\sqrt{2a}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \\ &= \pm\sqrt{2a}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) \\ &= \pm\frac{\sqrt{2a}}{2}(\sqrt{3}+i) \end{aligned}$$

On calculator:





Question 8

For the vectors $\underset{\sim}{a} = 3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{c} = x\underset{\sim}{i} - 7\underset{\sim}{j} + 10\underset{\sim}{k}$ to be linearly dependent, the value of x must be

A. 4
B. **7**
C. 2
D. -7
E. –2

Answer is B.

Worked solution

Let $\underset{\sim}{c} = n\underset{\sim}{a} + m\underset{\sim}{b}$

$x = 3n + m$ (1) $\underset{\sim}{i}$ components

$-7 = -n + 2m$ (2) $\underset{\sim}{j}$ components

$10 = 2n - 2m$ (3) $\underset{\sim}{k}$ components

$n = 3$ (2) + (3)

$-3 + 2m = -7$ (2)

$2m = -4$

$m = -2$

$x = 3 \times 3 + -2$ (1)

$x = 7$

Question 9

The graph of the relation $\{z: \quad z\bar{z} - 2\,\text{Re}(z) = 8, \quad z \in C\}$ would be

A. a circle with centre (0, 0) and radius $2\sqrt{2}$.
B. a circle with centre (–1, 0) and radius 3.
C. a straight line with gradient 2 and *y*-intercept of 8.
D. a straight line with gradient 1 and *y*-intercept of 8.
E. **a circle with centre (1, 0) and radius 3.**

Answer is E.

Worked solution

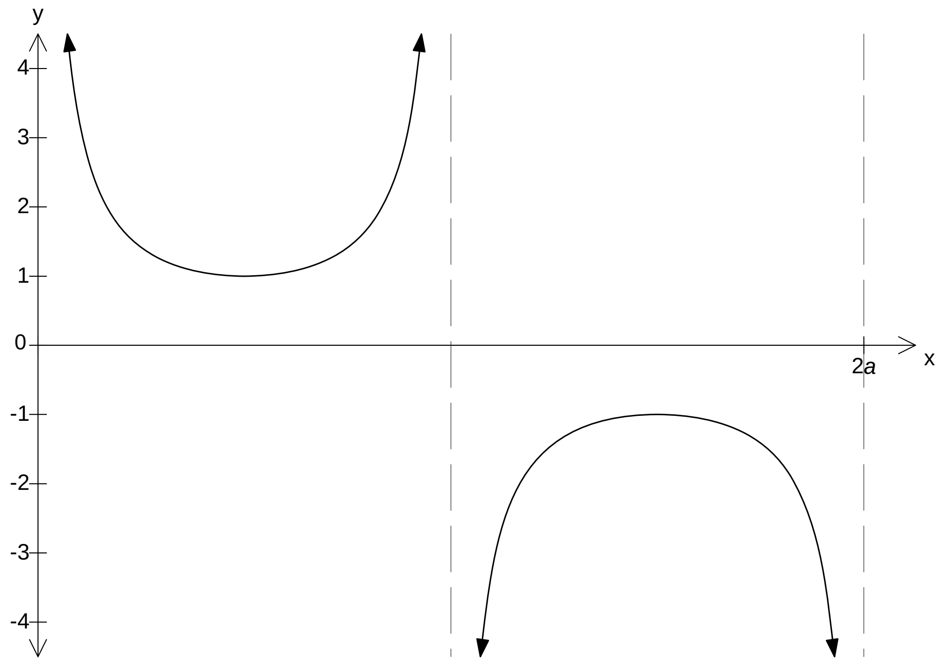
Let $z = x + yi$

$z\bar{z} - 2\,\text{Re}(z) = 8$ gives:

$$(x+yi)(x-yi) - 2x = 8$$
$$x^2 + y^2 - 2x = 8$$
$$x^2 - 2x + y^2 = 8$$
$$x^2 - 2x + 1 + y^2 = 8 + 1$$
$$(x-1)^2 + y^2 = 9$$

This is a circle of centre (1, 0) and radius of 3.

Question 10



The rule for the function graphed above, where $a > 0$, could be

A. $\mathbf{y = cosec\left(\frac{\pi x}{a}\right)}$

B. $y = \sec\left(\frac{\pi x}{a}\right)$

C. $y = \sec\left(\frac{\pi}{a}\left(x - \frac{a}{2}\right)\right)$

D. $y = -\text{cosec}\left(\frac{\pi x}{a}\right)$

E. $y = \text{cosec}\left(\frac{\pi}{a}\left(x + \frac{a}{2}\right)\right)$

Answer is A.

Worked solution

This graph has a period of $2a$ units and turning points at $y = \pm 1$.

It fits the form $y = \text{cosec}\,(n\,x)$.

$$\text{So } y = +\,\text{cosec}\,(n\,x)$$

$$\text{Period} = \frac{2\pi}{n} = 2a$$

$$n = \frac{2\pi}{2a} = \frac{\pi}{a}$$

$$y = \text{cosec}\left(\frac{\pi\,x}{a}\right)$$

Question 11

$\int_0^1 \left(\frac{2-3x}{4-x^2}\right) dx$ is equal to

A. $\log_e\left(\frac{9}{2}\right)$

B. $\log_e 18$

C. 0

D. $\log_e 72$

E. $\mathbf{\log_e\left(\frac{9}{8}\right)}$

Answer is E.

Worked solution

$$\int_0^1 \left(\frac{2-3x}{4-x^2}\right) dx$$

$$\int_0^1 \left(\frac{2}{x+2} - \frac{1}{2-x}\right) dx$$

$$= \left[2\log_e|x+2| + \log_e|2-x|\right]_0^1$$

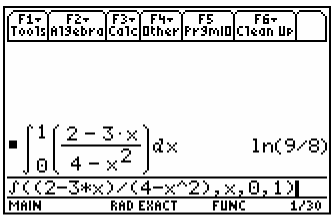
$$= \left[(2\log_e 3 + \log_e 1) - (2\log_e 2 + \log_e 2)\right]$$

$$= 2\log_e 3 - 3\log_e 2$$

$$= \log_e 9 - \log_e 8$$

$$= \log_e\left(\frac{9}{8}\right)$$

On calculator:



SECTION 1 – continued
TURN OVER

Question 12

The gradient of the tangent to the curve $2x\log_e(y) - x = y$ at the point where $y = e$ is

A. 3

B. **–1**

C. –3

D. 1

E. $\frac{1}{2}$

Answer is B.

Worked solution

$$2x\log_e(y) - x = y$$

$$\frac{d}{dx}(2x\log_e(y) - x) = \frac{dy}{dx}$$

$$2\log_e(y) + 2x\frac{d}{dy}(\log_e y)\frac{dy}{dx} - 1 = \frac{dy}{dx}$$

$$2\log_e(y) + \frac{2x}{y}\frac{dy}{dx} - 1 = \frac{dy}{dx}$$

$$\frac{dy}{dx}\left(\frac{2x}{y} - 1\right) = 1 - 2\log_e(y)$$

$$\frac{dy}{dx} = \left(\frac{1 - 2\log_e(y)}{\frac{2x}{y} - 1}\right)$$

At $y = e,\ 2x - x = e$

$$x = e$$

$$\frac{dy}{dx} = \frac{1-2}{2-1} = -1$$

Question 13

Using a suitable substitution, $\int_0^1 \frac{\cos^{-1}\left(\frac{x}{2}\right)}{\sqrt{4-x^2}}\,dx$ is equal to

A. $2\int_2^0 u\,du$

B. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} u\,du$

C. $\frac{1}{2}\int_0^2 u\,du$

D. $2\int_0^{\frac{\pi}{4}} u\,du$

E. $\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{u}\,du$

Answer is B.

Worked solution

$$\int_0^1 \frac{\cos^{-1}\left(\frac{x}{2}\right)}{\sqrt{4-x^2}}\,dx$$

$$= \int_0^1 -\cos^{-1}\left(\frac{x}{2}\right) \times \frac{-1}{\sqrt{4-x^2}}\,dx$$

Let $u = \cos^{-1}\left(\frac{x}{2}\right)$

$$\frac{du}{dx} = \frac{-1}{\sqrt{4-x^2}}$$

terminals: $x = 0,\ u = \cos^{-1}(0) = \frac{\pi}{2}$

$x = 1,\ u = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

$$\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} -u\frac{du}{dx}\,dx$$

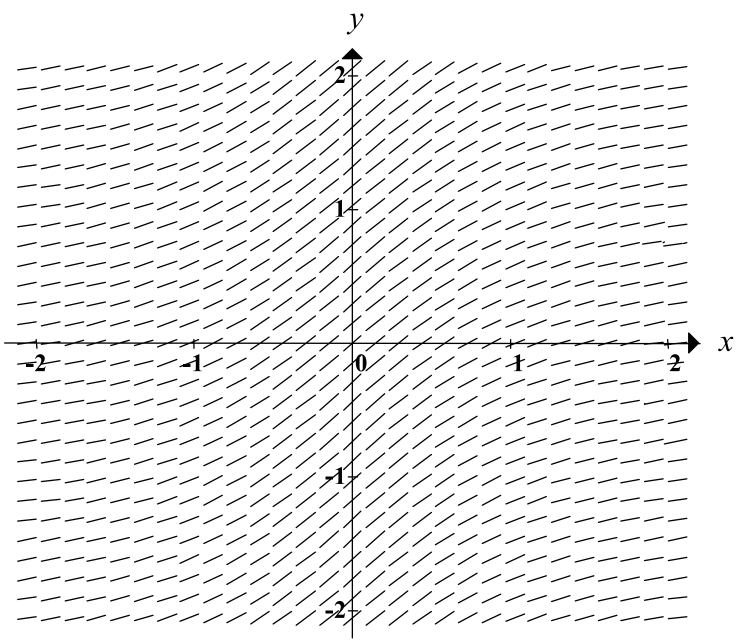
$$= -\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} u\,du$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} u\,du$$

Tip

- *Changing the variable in the integrand requires the terminals to be changed also.*

Question 14



The direction (slope) field for a certain first-order differential equation is shown above.

The differential equation could be

A. $\boldsymbol{\dfrac{dy}{dx} = \dfrac{1}{1+x^2}}$

B. $\dfrac{dy}{dx} = \tan^{-1} x$

C. $\dfrac{dy}{dx} = 1 + x^2 + y^2$

D. $\dfrac{dy}{dx} = |x+1|$

E. $\dfrac{dy}{dx} = \dfrac{1}{|x+y+1|}$

SECTION 1 – Question 14 – continued

Answer is A.

Worked solution

For any value of x the gradient is constant and so $\frac{dy}{dx} = f(x)$.

The shape of the curve is $f(x) = a\tan^{-1}(x) + c$ and so $\frac{dy}{dx} = \frac{a}{a^2 + x^2}$.

If $a = 1$, $\frac{dy}{dx} = \frac{1}{1+x^2}$ is the only suitable option.

Tip

- *If the vertical gradients are equal, or parallel, then the differential equation is a function of x only.*

Question 15

If $\frac{dy}{dx} = \log_e(x)$ and $y(1) = 2$, then the value of y when $x = 3$ can be found by evaluating

A. $1 + \int_2^3 \log_e(t)\, dt$

B. $2 + \int_1^3 \frac{1}{t}\, dt$

C. $\mathbf{2 + \int_1^3 \log_e(t)\, dt}$

D. $1 - \int_2^3 \log_e(t)\, dt$

E. $3 + \int_1^2 \log_e(t)\, dt$

Answer is C.

Worked solution

$\frac{dy}{dx} = \log_e(x)$ and $y(1) = 2$

$$y(3) = y(1) + \int_1^3 \frac{dy}{dt}\, dt$$

$$= 2 + \int_1^3 \frac{dy}{dt}\, dt$$

$$= 2 + \int_1^3 \log_e(t)\, dt$$

Question 16

The position vectors of two moving particles, *R* and *S*, at any time *t* seconds are given by

$\underset{\sim}{r} = at\underset{\sim}{i} - 4\underset{\sim}{j}$ and $\underset{\sim}{s} = t^2\underset{\sim}{i} + 2t\underset{\sim}{j},\ t \geq 0,\ a \in R$, respectively.

The angle between the directions of the two particles at $t = 1$ is

A. 69.3°

B. **45°**

C. 35.3°

D. 19.5°

E. dependent on the value of *a*.

Answer is B.

Worked solution

$$\underset{\sim}{r} = at\underset{\sim}{i} - 4\underset{\sim}{j} \quad \text{and} \quad \underset{\sim}{s} = t^2\underset{\sim}{i} + 2t\underset{\sim}{j}$$

$$\underset{\sim}{r}' = a\underset{\sim}{i} \quad \text{and} \quad \underset{\sim}{s}' = 2t\underset{\sim}{i} + 2t\underset{\sim}{j}$$

$$\underset{\sim}{r}'(1) = a\underset{\sim}{i} \quad \text{and} \quad \underset{\sim}{s}'(1) = 2\underset{\sim}{i} + 2\underset{\sim}{j}$$

$$\underset{\sim}{r}'(1)\underset{\sim}{s}'(1) = a\underset{\sim}{i}(2\underset{\sim}{i} + 2\underset{\sim}{j})$$

$$= 2a$$

$$\left|\underset{\sim}{r}'(1)\right|\left|\underset{\sim}{s}'(1)\right| = a\sqrt{2^2 + 2^2}$$

$$= 2a\sqrt{2}$$

$$\theta = \cos^{-1}\left(\frac{2a}{2a\sqrt{2}}\right)$$

$$= \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= 45°$$

Tip

- *Find the angle between the velocity vectors when t = 1.*

Question 17

The volume of a tank is given by $V = 0.4\pi h^{\frac{5}{2}}$, where *h* cm is the depth of water in the tank at time *t* minutes. Water leaks from the tank at a rate of 16 cm^3/minute. The depth of water in the tank when the height is decreasing at a rate of $\frac{2}{\pi}$ cm/minute is

A. 16 cm
B. 8 cm
C. 4π cm
D. **4 cm**
E. 8π cm

Answer is D.

Worked solution

$$V = 0.4\pi h^{\frac{5}{2}}$$

$$\frac{dV}{dh} = 0.4\pi \times \frac{5}{2} h^{\frac{3}{2}}$$

$$= \pi h^{\frac{3}{2}}$$

$$\frac{dV}{dt} = -16$$

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV} = \frac{dV}{dt} \div \frac{dV}{dh}$$

$$\frac{dh}{dt} = -\frac{16}{\pi h^{\frac{3}{2}}}$$

$$\frac{dh}{dt} = -\frac{16}{\pi h^{\frac{3}{2}}} = -\frac{2}{\pi}$$

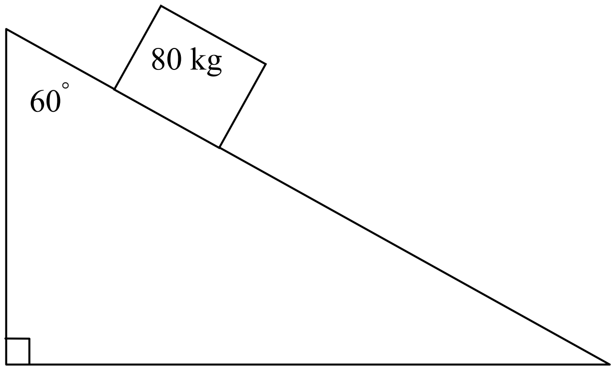
$$2h^{\frac{3}{2}} = 16$$

$$h^{\frac{3}{2}} = 8$$

$$h = 8^{\frac{2}{3}}$$

$$h = 4 \text{ cm}$$

Question 18



A skier of mass 80 kilograms slides from rest down a straight slope inclined at 60° to the vertical. Assuming it is a smooth slope, the speed of the skier after moving 100 metres down the slope is nearest to

A. 41.2 m/s
B. 22.1 m/s
C. **31.3 m/s**
D. 44.3 m/s
E. 10 m/s

Answer is C.

Worked solution

The resultant force on the skier in the direction of motion down the slope is:

$$R = ma$$
$$= 80g\sin 30° = 80a$$
$$40g = 80a$$
$$a = \frac{g}{2}$$
$$= 4.9 \text{ m/s}^2$$

$u = 0$, $a = 4.9$, $s = 100$

$$v^2 = u^2 + 2as$$
$$v = \sqrt{u^2 + 2as}$$
$$= \sqrt{0 + 2 \times 4.9 \times 100}$$
$$= \sqrt{980}$$
$$v \approx 31.3$$

The speed of the skier is approximately 31.3 m/s.

Question 19

A mass of 4 kilograms is at rest when two forces, $\underset{\sim}{F}_1 = (\underset{\sim}{i} - 3\underset{\sim}{j})$ newtons and $\underset{\sim}{F}_2 = (2\underset{\sim}{i} - \underset{\sim}{j})$ newtons, act on it. The time taken for the mass to travel 10 metres is

A. 1 s
B. 2 s
C. **4 s**
D. 5 s
E. 8 s

Answer is C.

Worked solution

$$\underset{\sim}{F}_1 + \underset{\sim}{F}_2 = \underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{i} - \underset{\sim}{j}$$
$$= 3\underset{\sim}{i} - 4\underset{\sim}{j}$$
$$F = \left|\underset{\sim}{F}_1 + \underset{\sim}{F}_2\right| = \sqrt{3^2 + 4^2} = 5$$
$$F = ma$$
$$5 = 4a$$
$$a = 1.25$$
$$u = 0, s = 10$$
$$s = ut + \frac{1}{2}at^2$$
$$10 = 0 + 0.625t^2$$
$$t^2 = 16$$
$$t = 4$$

Question 20

The velocity of a particle moving in a straight line is given by $v(x)=\cos(x^2)$, where x is the displacement from the origin O.

The acceleration of the particle is

A. $a(x)=-2x\sin(x^2)$

B. $a(x)=\cos(2x)$

C. $a(x)=-2x\tan(x^2)$

D. $\boldsymbol{a(x)=-x\sin(2x^2)}$

E. $a(x)=-2x\tan(x^2)\sec(x^2)$

Answer is D.

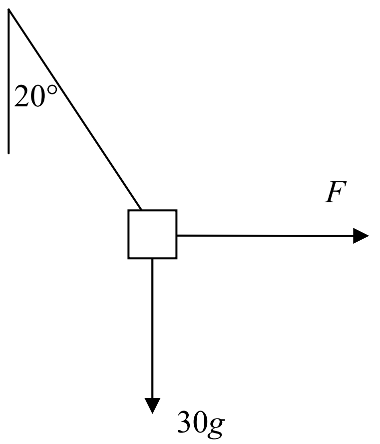
Worked solution

$$\begin{aligned} a &= v\frac{dv}{dx} \\ &= \cos(x^2)(-2x\sin(x^2)) \\ &= -2x\cos(x^2)\sin(x^2) \\ a &= -x\sin(2x^2) \end{aligned}$$

Tip

- *Simplify using the double angle formula* $2\sin\theta\cos\theta=\sin 2\theta$.

Question 21



The magnitude of the horizontal force, F newtons, required to hold a 30 kilogram child in equilibrium on a swinging rope, as shown in the diagram above, is

A. $\dfrac{30g}{\tan 70^\circ}$

B. $\dfrac{30g \sin 70^\circ}{\sin 20^\circ}$

C. $30g \sin 20^\circ$

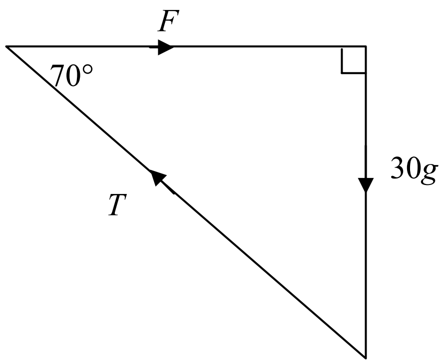
D. $\dfrac{30g}{\sin 70^\circ}$

E. $\mathbf{30g \tan 20^\circ}$

Answer is E.

Worked solution

As there are three forces acting in equilibrium the situation can be represented by a triangle of forces.



$$\tan 70^\circ = \frac{30g}{F}$$

$$F = \frac{30g}{\tan 70^\circ}$$

$$= \frac{30g}{\cot 20^\circ}$$

$$F = 30g \tan 20^\circ$$

Tips

- *Lami's theorem could be used instead of a triangle of forces.*
- $\tan\theta = \cot(90-\theta) = \dfrac{1}{\tan(90-\theta)}$.

Question 22

A lift travelling upwards accelerates at a m/s^2 ($a > 0$) with a person of mass 100 kilograms standing on a set of weight scales in the lift. It then decelerates at twice the magnitude of the acceleration. The magnitude of the change in the reading on the scales will be

A. $100a$ kg

B. $200g$ kg

C. **$300a$ kg**

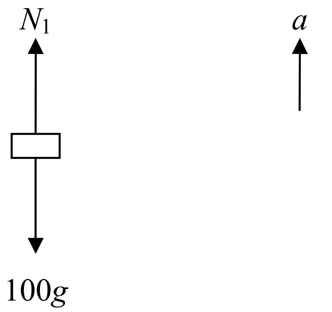
D. $100(g + a)$ kg

E. $-100a$ kg

Answer is C.

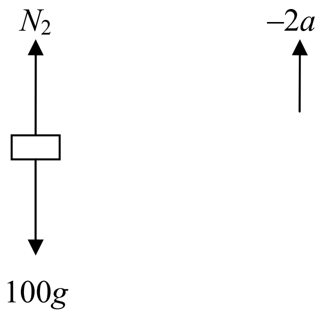
Worked solution

When accelerating:



$$R = N_1 - 100g = 100a$$
$$N_1 = 100g + 100a$$

When decelerating:



$$R = N_2 - 100g = 100 \times -2a$$
$$N_2 - 100g = -200a$$
$$N_2 = 100g - 200a$$

The change in the reading on the scales is $N_1 - N_2$.

$$N_1 - N_2 = 100g + 100a - (100g - 200a)$$
$$= 300a$$

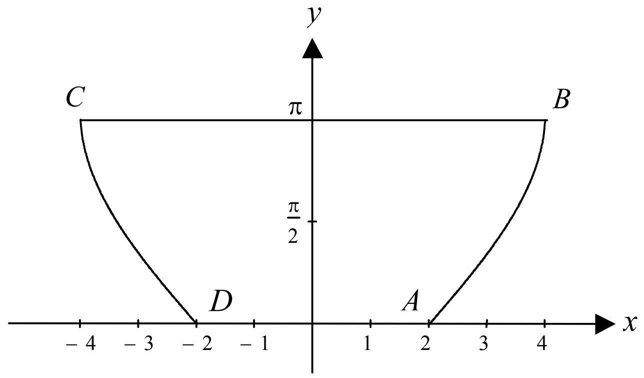
END OF SECTION 1

SECTION 2

Question 1

The diagram below shows the profile of a symmetrical small bowl *ABCD*. The bowl is generated by rotating the area between the curve *AB* and the *y*-axis about the *y*-axis. The top and base of the bowl have radii of 4 cm and 2 cm, respectively, and the height of the bowl is π cm.

The curve *AB* can be modelled by the function $y = a\sin^{-1}(bx - c),\ \ x \in [2, 4]$.



a. Show that $a = 2$, $b = \frac{1}{2}$ and $c = 1$.

Worked solution

$y = a\sin^{-1}(bx - c)$

Vertical dilation is by factor 2. 1M

$a = 2$

Horizontal dilation is by factor 2. 1M

$b = \frac{1}{2}$

$y = 2\sin^{-1}\left(\frac{1}{2}x - c\right)$

$= 2\sin^{-1}\left(\frac{1}{2}(x - 2c)\right)$

Horizontal translation is +2. 1M

$2c = 2$

$c = 1$

3 marks

Mark allocation

- 1 mark for each value.

Tip

- *Can also solve for a, b, c using points* (2, 0) *and* (4, π) *and the fact that* $-1 \le bx - c \le 1$

b. If h cm is the height of water in the bowl at any time, express the volume of water, V cm^3, in terms of h.

Worked solution

$$x = 2 + 2\sin\left(\frac{y}{2}\right)$$

$$x^2 = \left(2 + 2\sin\left(\frac{y}{2}\right)\right)^2$$

$$= 4 + 8\sin\left(\frac{y}{2}\right) + 4\sin^2\left(\frac{y}{2}\right)$$ 1M

$$\text{Volume} = \pi\int_0^h x^2\,dy$$

$$= \pi\int_0^h \left(4 + 8\sin\left(\frac{y}{2}\right) + 4\sin^2\left(\frac{y}{2}\right)\right)dy$$ 1M

$$= \pi\int_0^h \left(4 + 8\sin\left(\frac{y}{2}\right) + \frac{4(1-\cos(y))}{2}\right)dy$$

$$= \pi\int_0^h \left(4 + 8\sin\left(\frac{y}{2}\right) + 2 - 2\cos(y)\right)dy$$

$$= \pi\int_0^h \left(6 + 8\sin\left(\frac{y}{2}\right) - 2\cos(y)\right)dy$$

$$= 2\pi\int_0^h \left(3 + 4\sin\left(\frac{y}{2}\right) - \cos(y)\right)dy$$ 1M

$$= 2\pi\left[3y - 8\cos\left(\frac{y}{2}\right) - \sin(y)\right]_0^h$$

$$= 2\pi\left[(3h - 8\cos\left(\frac{h}{2}\right) - \sin(h)) - (0 - 8 - 0)\right]$$

$$= 2\pi\left[3h - 8\cos\left(\frac{h}{2}\right) - \sin(h) + 8)\right]$$

The volume of the bowl, in cm^3, is $V = 2\pi\left[3h - 8\cos\left(\frac{h}{2}\right) - \sin(h) + 8)\right]$ 1A

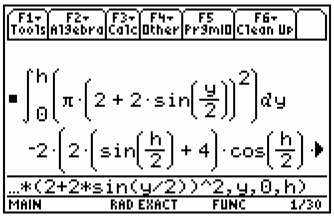
Integrate on CAS to give

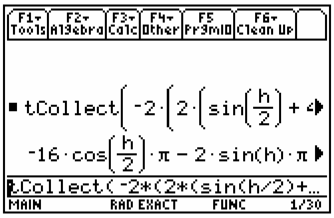
$$V = -2\left(\left(2\sin\left(\frac{h}{2}\right)+4\right)\cos\left(\frac{h}{2}\right)-3h-8\right)\pi$$

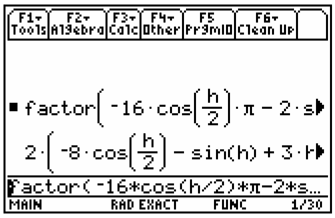
Followed by t collect

$$-16\cos\left(\frac{h}{2}\right)\pi - 2\sin(h)\pi + 6h\pi + 16\pi$$

Then factor on calculator:







4 marks

Mark allocation

- 1 mark for correctly expressing x^2 in terms of y.
- 1 mark for correctly expressing the volume as a definite integral.
- 1 mark for expressing the integrand correctly as a function that can be antidifferentiated by rule.
- 1 mark for the correct answer.

Tip

- *The volume is obtained by rotating the area between* $y = 2\sin^{-1}\left(\frac{1}{2}x - 1\right)$ *and the y-axis and the lines y = 0 and y = h about the y-axis. Hence, x has to be expressed as a function of y.*

c. **Hence**, find the exact volume of water in a full bowl.

Worked solution

$$V = 2\pi\left[3h - 8\cos\left(\frac{h}{2}\right) - \sin(h) + 8)\right]$$

The bowl is full when $h = \pi$.

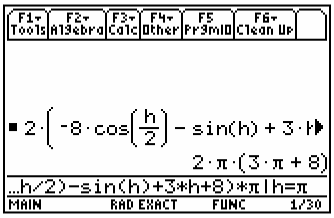
$$V = 2\pi\left[3\pi - 8\cos\left(\frac{\pi}{2}\right) - \sin(\pi) + 8)\right]$$

$$= 2\pi(3\pi - 0 - 0 + 8)$$

$$= 2\pi(3\pi + 8)$$

When the bowl is full it has a volume of $2\pi(3\pi + 8)$ cm^3. 1A

On calculator:



1 mark

Mark allocation

- 1 mark for the correct answer.

d. To the nearest millimetre, what would be the height of the water when the bowl is filled to half its capacity?

Worked solution

$V_{\text{full}} = 2\pi(3\pi + 8)$

$V_{\text{half full}} = \pi(3\pi + 8)$

$$V = 2\pi\left[3h - 8\cos\left(\frac{h}{2}\right) - \sin(h) + 8)\right] = \pi(3\pi + 8)$$

$$2\left[3h - 8\cos\left(\frac{h}{2}\right) - \sin(h) + 8)\right] = 3\pi + 8$$

$$6h - 16\cos\left(\frac{h}{2}\right) - 2\sin(h) + 16 - 3\pi - 8 = 0$$

$$6h - 16\cos\left(\frac{h}{2}\right) - 2\sin(h) + 8 - 3\pi = 0 \qquad \text{1M}$$

Solve this equation using a graphics calculator.

$h = 1.99175$

The bowl is half full when the height is 2.0 cm or 20 mm. 1A

2 marks

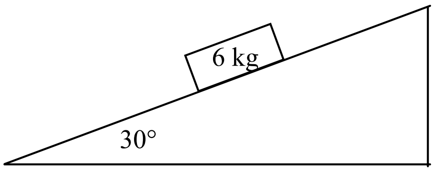
Mark allocation

- 1 mark for setting up the correct equation to solve for h.
- 1 mark for the correct answer.

Total 3 + 4 + 1 + 2 = 10 marks

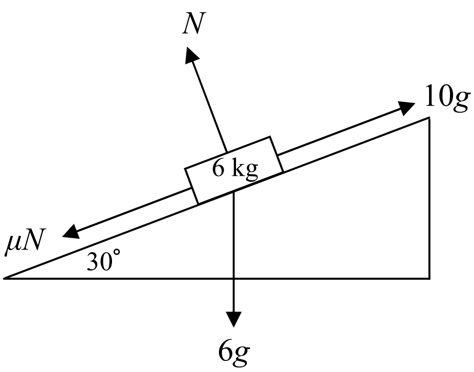
Question 2

A miniature racing car of mass 6 kilograms is propelled from rest up a rough ramp 19.6 metres long and inclined at an angle of 30° to the horizontal. The car is powered up the ramp by a constant force of 10*g* newtons. This causes the car to accelerate at 9.8 m/s^2.



a. Label the forces acting on the car as it moves up the ramp.

Worked solution



1A

1 mark

Mark allocation

- 1 mark for labelling the four forces correctly.

b. Show that at the top of the ramp the car is g metres above the ground and its speed is $2g$ m/s when it leaves the ramp.

Worked solution

$$\begin{aligned}\text{Height} &= 19.6\sin 30° \\ &= 2g \times 0.5 \qquad \text{1M} \\ &= g \text{ m}\end{aligned}$$

$u = 0,\ a = g,\ s = 2g$

$$\begin{aligned}v^2 &= u^2 + 2as \\ &= 0 + 2g \times 2g \\ v &= \sqrt{4g^2} \qquad \text{1M} \\ &= 2g\end{aligned}$$

The height of the car at the top of the ramp is 9.8 metres and its speed is 19.6 m/s.

2 marks

Mark allocation

- 1 mark for correct working to find the height.
- 1 mark for correct working to find the speed.

c. Calculate the exact value of the coefficient of friction.

Worked solution

$$\begin{aligned}N &= 6g\cos 30° \\ &= 3g\sqrt{3}\end{aligned}$$

Resultant force, $R = 10g - \mu N - 6g\sin 30° = 6a$ 1M

$$\begin{aligned}10g - 3g\sqrt{3}\,\mu - 3g &= 6g \\ 7g - 3g\sqrt{3}\,\mu &= 6g \\ 3g\sqrt{3}\,\mu &= g \\ 3\sqrt{3}\,\mu &= 1 \\ \mu &= \frac{1}{3\sqrt{3}} \\ \mu &= \frac{\sqrt{3}}{9} \qquad \text{1A}\end{aligned}$$

2 marks

Mark allocation

- 1 mark for the correct equation of motion.
- 1 mark for the correct answer.

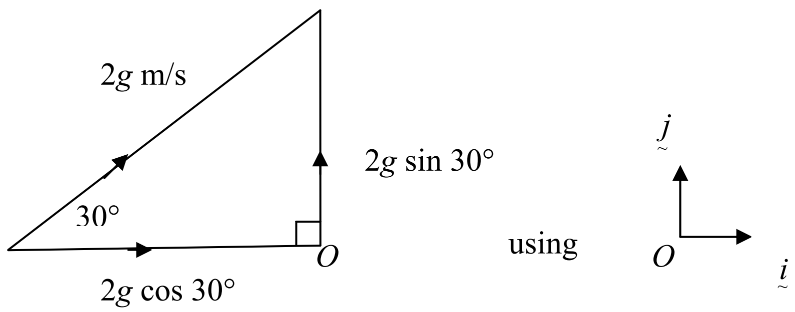
Tip

- *Resolve the forces parallel and perpendicular to the plane.*

When the car leaves the ramp it is only subject to the force of gravity.
Take $\underset{\sim}{i}$ as the unit vector in the horizontal direction and $\underset{\sim}{j}$ as the unit vector in the vertical direction from the point on the ground, directly below the top of the ramp.

d. Determine the velocity vector $\underset{\sim}{v}$ and the position vector $\underset{\sim}{r}$ of the car at any time *t* seconds.

Worked solution



$\underset{\sim}{a} = -g\,\underset{\sim}{j}$

$\underset{\sim}{v} = 2g\cos 30°\,\underset{\sim}{i} + (2g\sin 30° - gt)\,\underset{\sim}{j}$

$= \sqrt{3}\,g\,\underset{\sim}{i} + (g - gt)\,\underset{\sim}{j}$ 1A

$\underset{\sim}{r} = \sqrt{3}\,gt\,\underset{\sim}{i} + (gt - \frac{1}{2}gt^2 + g)\,\underset{\sim}{j}$ 1A

2 marks

Mark allocation

- 1 mark for each correct answer.

e. Find the exact Cartesian equation of the path of the car after it leaves the ramp.

Worked solution

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} = \sqrt{3}\,gt\,\underset{\sim}{i} + (gt - \frac{1}{2}gt^2 + g)\,\underset{\sim}{j}$

$x = \sqrt{3}\,gt$

$t = \dfrac{x}{g\sqrt{3}}$

$y = gt - \dfrac{1}{2}gt^2 + g$

$= g\dfrac{x}{g\sqrt{3}} - \dfrac{1}{2}g\left(\dfrac{x}{g\sqrt{3}}\right)^2 + g$ 1M

$= \dfrac{x}{\sqrt{3}} - \dfrac{1}{2}g\dfrac{x^2}{3g^2} + g$

$y = -\dfrac{x^2}{6g} + \dfrac{x}{\sqrt{3}} + g$ 1A

2 marks

Mark allocation

- 1 mark for correctly substituting the parametric equation of $t(x)$ into expression for $y(t)$.
- 1 mark for the correct answer.

f. Find the exact magnitude of the momentum of the car when it hits the ground.

Worked solution

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} = \sqrt{3}\,gt\,\underset{\sim}{i} + (gt - \frac{1}{2}gt^2 + g)\,\underset{\sim}{j}$

When the car hits the ground:

$$y = gt - \frac{1}{2}gt^2 + g = 0$$

$$t - \frac{1}{2}t^2 + 1 = 0$$

$$t^2 - 2t - 2 = 0$$

$$t = \frac{2 \pm \sqrt{2^2 - 4(1)(-2)}}{2}$$

$$= \frac{2 \pm \sqrt{12}}{2}$$

$$= \frac{2 \pm 2\sqrt{3}}{2}$$

$t = 1 + \sqrt{3}$ only as $t \geq 0$ 1M

$$\underset{\sim}{v}(t) = \sqrt{3}g\,\underset{\sim}{i} + (g - gt)\,\underset{\sim}{j}$$

$$\underset{\sim}{v}(1+\sqrt{3}) = \sqrt{3}g\,\underset{\sim}{i} + (g - g(1+\sqrt{3}))\,\underset{\sim}{j}$$

$$= \sqrt{3}g\,\underset{\sim}{i} - \sqrt{3}\,g\,\underset{\sim}{j}$$

$$v = \sqrt{3g^2 + 3g^2}$$

$= \sqrt{6}g$ 1M

$$p = mv$$

$$= 6\sqrt{6}g$$

The momentum of the car on impact with the ground is $6\sqrt{6}g$ kg m/s. 1A

3 marks

Mark allocation

- 1 mark for finding the correct value of t for the time when the car hits the ground.
- 1 mark for finding the correct speed of the car at impact.
- 1 mark for the correct answer.

Total 1 + 2 + 2 + 2 + 2 + 3 = 12 marks

Question 3

a. Given $w = a + bi$, where $a, b \in R$ and $b > 0$.

If $w + \overline{w} = 2$ and $w\,\overline{w} = 2$, show that $w = 1 + i$.

Worked solution

$$w = a + bi$$

$$w + \overline{w} = 2a = 2$$

$$a = 1 \qquad \text{1M}$$

$$w \times \overline{w} = a^2 + b^2 = 2$$

$$1 + b^2 = 2$$

$$b^2 = 1$$

$$b = 1 \text{ since } b > 0 \qquad \text{1M}$$

$$w = 1 + i$$

2 marks

Mark allocation

- 1 mark for finding the correct value of *a*.
- 1 mark for finding the correct value of *b*.

b. If $v = 1 + \sqrt{3}\,i$,

i. Find $\dfrac{v}{w}$ in simplest exact Cartesian form.

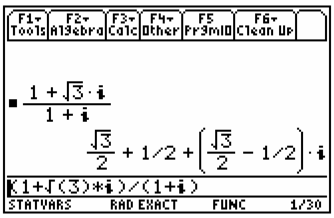
Worked solution

$$\frac{v}{w} = \frac{1 + \sqrt{3}\,i}{1 + i}$$

$$= \frac{1 + \sqrt{3}\,i}{1 + i} \times \frac{1 - i}{1 - i}$$

$$= \frac{1 + \sqrt{3} + (\sqrt{3} - 1)i}{2} \qquad \text{1A}$$

On calculator:



1 mark

Mark allocation

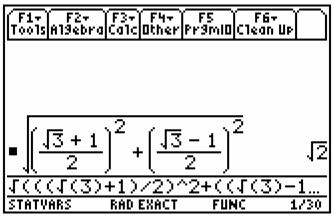
- 1 mark for correct answer.

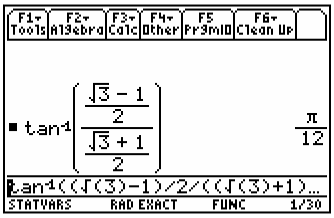
ii. Find $\dfrac{v}{w}$ in polar form.

Worked solution

$$v = 1 + \sqrt{3}\, i$$
$$= 2 \text{ cis } \frac{\pi}{3}$$
$$w = 1 + i$$
$$= \sqrt{2} \text{ cis } \frac{\pi}{4}$$
$$\frac{v}{w} = \frac{2 \text{ cis } \frac{\pi}{3}}{\sqrt{2} \text{ cis } \frac{\pi}{4}} \qquad \text{1M}$$
$$= \frac{2}{\sqrt{2}} \text{ cis}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$
$$= \sqrt{2} \text{ cis}\left(\frac{\pi}{12}\right) \qquad \text{1A}$$

On calculator:





2 marks

Mark allocation

- 1 mark for correctly expressing $\dfrac{v}{w}$ in polar form.
- 1 mark for the correct answer.

c. **Hence**, express $\tan\left(\frac{\pi}{12}\right)$ in the form $a-\sqrt{b}$, where a and b are positive integers.

Worked solution

$$\frac{z}{w}=\sqrt{2}\ \text{cis}\left(\frac{\pi}{12}\right)=\frac{1+\sqrt{3}+(\sqrt{3}-1)i}{2} \qquad \text{1M}$$

$$\tan\left(\frac{\pi}{12}\right)=\frac{\sqrt{3}-1}{\sqrt{3}+1} \qquad \text{1M}$$

$$=\frac{\sqrt{3}-1}{\sqrt{3}+1}\times\frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$=\frac{3-2\sqrt{3}+1}{3-1}$$

$$=\frac{4-2\sqrt{3}}{2}$$

$$=2-\sqrt{3} \qquad \text{1A}$$

3 marks

Mark allocation

- 1 mark for correctly equating the Cartesian form with the polar form.
- 1 mark for correctly writing the exact value for $\tan\left(\frac{\pi}{12}\right)$.
- 1 mark for expressing the answer in the form required.

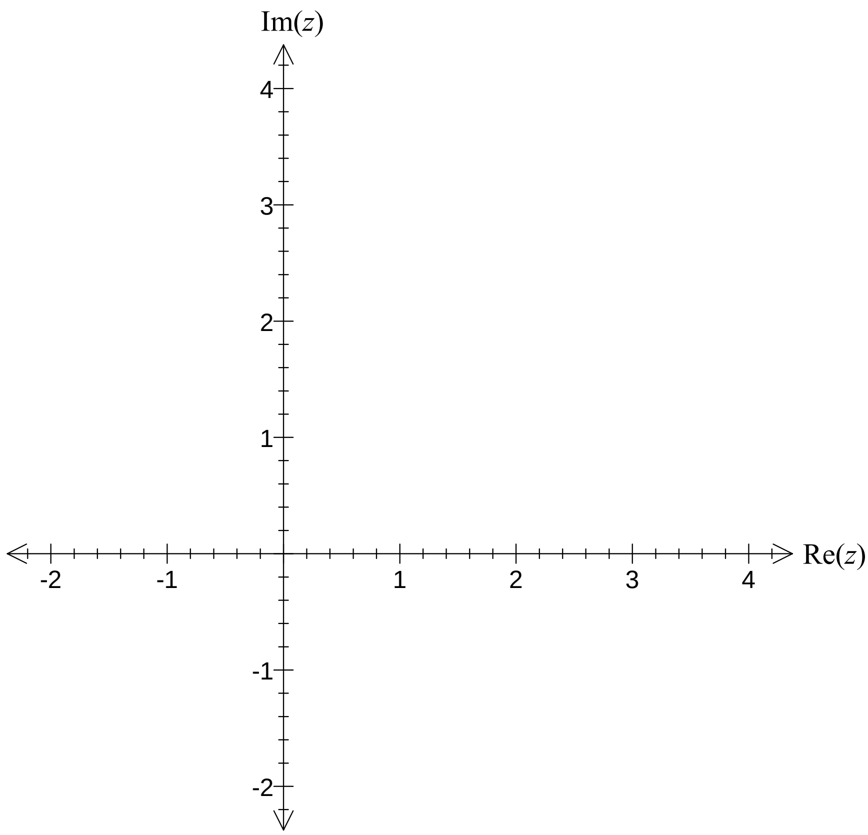
Tip

- *For* $z = x + yi = r\ \text{cis}\ \theta$, *tan* $\theta = \frac{y}{x}$.

d. S is a subset of the complex plane, which is defined as

$$S = \{z : \ |z - w| = 1, \quad z \in C\}$$

Plot the points v and w and sketch the relation defined by S on the Argand diagram below.

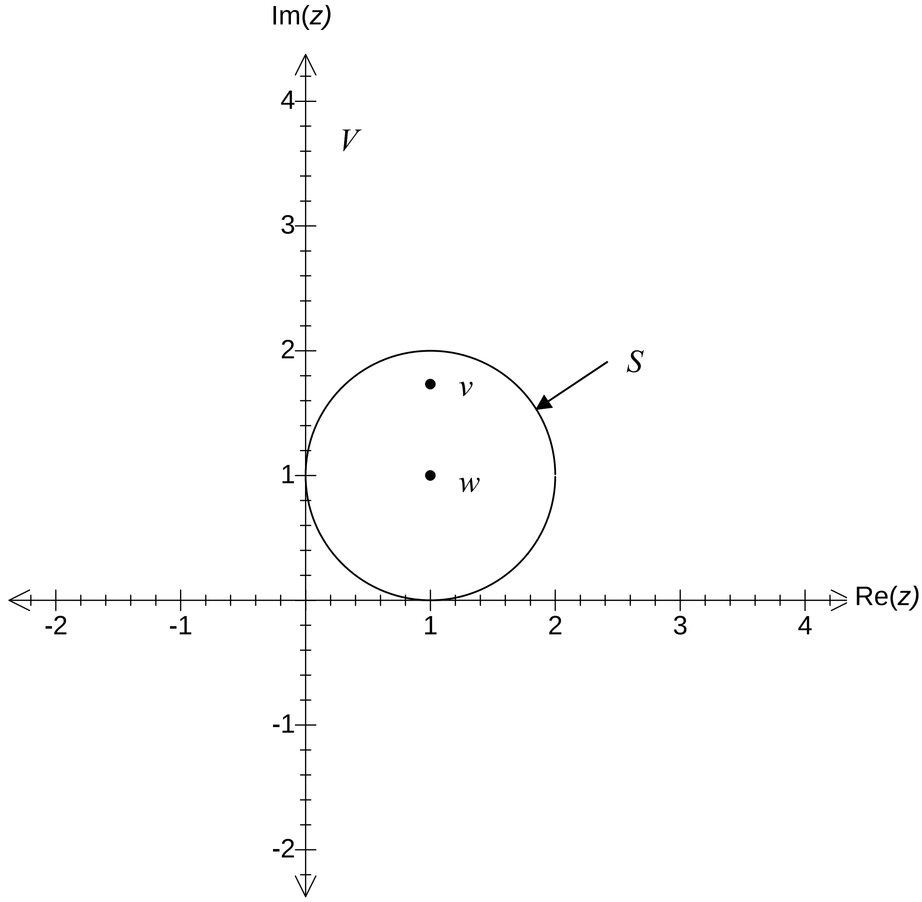


Worked solution

w is the point $1 + i$ and v is the point $1+\sqrt{3}\,i$.

S defines a locus of points z, where the distance from the fixed point $w = 1 + i$ is always 1 unit.

This is a circle with centre (1, 1) and a radius of 1 unit.



1A, 1A

2 marks

Mark allocation

- 1 mark for correctly marking both points v and w.
- 1 mark for representing S as a circle with centre (1, 1) and a radius of 1.

Tip

- *The locus of S could also be derived in Cartesian form.*

e. T is a subset of the complex plane defined by

$$T = \{z : \ |z - v| = |z - w|, \quad z \in C\}$$

i. Express the equation for the relation defined by T in Cartesian form.

Worked solution

T defines a straight line, which is the perpendicular bisector of the line joining the points v and w.

Finding the Cartesian equation of T

$$|z - 1 - \sqrt{3}i| = |z - 1 - i|$$

$$\sqrt{(x-1)^2 + \left(y - \sqrt{3}\right)^2} = \sqrt{(x-1)^2 + (y-1)^2}$$

$$\left(y - \sqrt{3}\right)^2 = (y-1)^2$$

$$y^2 - 2\sqrt{3}y + 3 = y^2 - 2y + 1$$

$$\left(2\sqrt{3} - 2\right)y = 2$$

$$y = \frac{1}{\sqrt{3} - 1}$$

$$y = \frac{\sqrt{3} + 1}{2} \text{ (rationalised)}$$

1 mark

Mark allocation

- 1 mark for the correct Cartesian equation of T.

ii. Part of T is a chord to the relation $S = \{z : \ |z - w| = 1, \quad z \in C\}$

Find the exact length of this chord in the form $a^{\frac{b}{c}}$, where a, b and c are integers.

Worked solution

$$(x-1)^2+(y-1)^2=1 \qquad (S)$$

$$(x-1)^2+\left(\frac{1+\sqrt{3}}{2}-1\right)^2=1 \qquad (S\cap T) \qquad \text{1M}$$

$$(x-1)^2+\left(\frac{\sqrt{3}-1}{2}\right)^2=1$$

$$(x-1)^2+\left(\frac{3-2\sqrt{3}+1}{4}\right)=1$$

$$(x-1)^2+\left(\frac{4-2\sqrt{3}}{4}\right)=1$$

$$(x-1)^2+1-\frac{\sqrt{3}}{2}=1$$

$$(x-1)^2=\frac{\sqrt{3}}{2}$$

$$x-1=\pm\sqrt{\frac{\sqrt{3}}{2}}$$

$$x=1\pm\sqrt{\frac{\sqrt{3}}{2}}$$

$$\text{Chord length } =\left(1+\sqrt{\frac{\sqrt{3}}{2}}\right)-\left(1-\sqrt{\frac{\sqrt{3}}{2}}\right) \qquad \text{1M}$$

$$=2\sqrt{\frac{\sqrt{3}}{2}}$$

$$=\sqrt{\frac{4\sqrt{3}}{2}}$$

$$=\sqrt{2\sqrt{3}}$$

$$=\sqrt{\sqrt{12}}$$

$$=12^{\frac{1}{4}} \qquad \text{1A}$$

The length of the chord is $12^{\frac{1}{4}}$ units.

3 marks

Mark allocation

- 1 mark for the correct equation to find the values of x where $S \cap T$.
- 1 mark for finding the correct values of x where $S \cap T$.
- 1 mark for the correct answer.

Total = 2 + 3 + 3 + 2 + 4 = 14 marks

Question 4

A tank contains 100 litres of sugar solution with a concentration of 0.05 kg/L. A sugar solution of concentration 0.1 kg/L flows into the tank at a rate of 2 L/min. The mixture, which is kept uniform by stirring, flows out of the tank at a rate of 2 L/min. After t minutes the tank contains x kilograms of sugar.

a. Show that the differential equation for x in terms of t is $\frac{dx}{dt} = \frac{10 - x}{50}$ kg/min.

Worked solution

$$\text{Input rate} = 0.1 \times 2 = 0.2 \text{ kg/min}$$

$$\text{Output rate} = \frac{x}{100} \times 2 = \frac{x}{50} \text{ kg/min}$$

$$\frac{dx}{dt} = 0.2 - \frac{x}{50} \qquad \text{1M}$$

$$= \frac{10 - x}{50} \text{ kg/min}$$

1 mark

Mark allocation

- 1 mark for correct setting up of the differential equation.

b. Solve this differential equation to give x as a function of t.

Worked solution

$$\frac{dx}{dt} = \frac{10 - x}{50}$$

$$\frac{dt}{dx} = \frac{50}{10 - x}$$

$$t = \int \frac{50}{10 - x} dx$$

$$t = -50 \log_e k(10 - x), \quad k \in R \qquad \text{1M}$$

When $t = 0$, $x = 100 \times 0.05 = 5$

$$k(10 - 5) = 1$$

$$k = \frac{1}{5} \qquad \text{1M}$$

$$t = -50 \log_e \left(\frac{10 - x}{5} \right)$$

$$\log_e \left(\frac{10 - x}{5} \right) = -0.02t$$

$$\frac{10 - x}{5} = e^{-0.02t}$$

$$10 - x = 5e^{-0.02t}$$

$$x = 10 - 5e^{-0.02t} \qquad \text{1A}$$

3 marks

SECTION 2 – Question 4 – continued

Mark allocation

- 1 mark for solving the antiderivative correctly.
- 1 mark for correctly evaluating the constant of antidifferentiation.
- 1 mark for the correct answer.

c. Calculate the amount of sugar in the tank after 2 minutes. Give your answer correct to three decimal places.

Worked solution

$$x = 10 - 5e^{-0.02\times 2}$$
$$= 10 - e^{-0.04}$$
$$= 5.196$$

There is 5.196 kilograms of sugar in the tank after 2 minutes. 1A

1 mark

Mark allocation

- 1 mark for the correct answer.

d. If this situation continued for a long period of time, how much sugar would be present in the tank?

Worked solution

$$x = 10 - 5e^{-0.02t} = 10 - \frac{5}{e^{0.02t}}$$

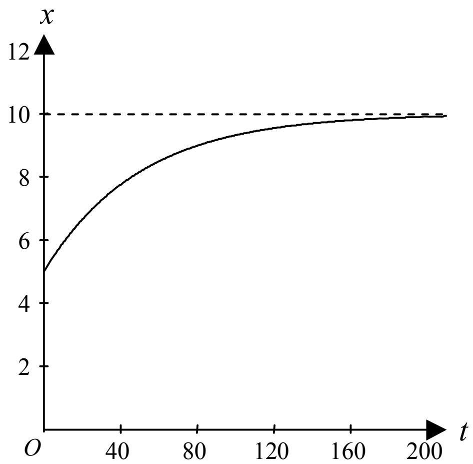
As $t \to \infty,\ x \to 10 - \frac{5}{e^{\infty}} = 10 - 0 = 10$

After a long period of time there will be almost 10 kilograms of sugar in the tank. 1A

1 mark

Tip

- *A graph (below) could assist in answering this question.*



Mark allocation

- 1 mark for the correct answer.

e. If the outflow from the tank was 1 L/min instead of 2 L/min, set up the new differential equation for x in terms of t.

Worked solution

Volume, $V = 100 + 2t - t = 100 + t$

Input rate $= 0.1 \times 2 = 0.2$ kg/min

Output rate $= \dfrac{x}{100+t} \times 1 = \dfrac{x}{100+t}$ kg / min

$$\frac{dx}{dt} = 0.2 - \frac{x}{100+t} \qquad \text{1A}$$

1 mark

Mark allocation

- 1 mark for the correct answer.

f. For the differential equation from part **e.** use Euler's method, with increments of 1 minute, to predict the amount of sugar in the tank after 2 minutes. Give your answer correct to three decimal places.

Worked solution

$$x_{n+1} = x_n + h\frac{dx}{dt}$$

$$h = 1,\ \frac{dx}{dt} = 0.2 - \frac{x}{100+t},\ x_0 = 5,\ t_0 = 0$$

$$x_1 = 5 + 1\left(0.2 - \frac{5}{100}\right) = 5 + 0.15$$

$$= 5.15 \qquad \text{1M}$$

$$x_2 = 5.15 + 1\left(0.2 - \frac{5.15}{101}\right) = 5.15 + 0.2 - 0.05099$$

$$= 5.29901$$

The predicted amount of sugar after 2 minutes is 5.299 kg. 1A

2 marks

Mark allocation

- 1 mark for evaluating x_1 correctly.
- 1 mark for the correct answer.

Total = 1 + 3 + 1 + 1 + 1 + 2 = 9 marks

Question 5

At 10 a.m. an aircraft is flying at an altitude of $(e^2 - e)$ km, 500 km north and 440 km east of a point $T(0, 0, 0)$, which is its touchdown point on a horizontal runway.

The position of the aircraft relative to the point T is given by the vector

$$\underset{\sim}{r}(t) = \left(a + \frac{2420}{t+5}\right)\underset{\sim}{i} + (500 - 24t + 0.28t^2)\underset{\sim}{j} + (e^{c-0.02t} - e)\underset{\sim}{k}, \text{ where } a,\ c \in R.$$

$\underset{\sim}{r}$ is in kilometres and t is the time in minutes after 10 a.m.

$\underset{\sim}{i}$ is the unit vector in an easterly direction, $\underset{\sim}{j}$ is the unit vector in a northerly direction and

$\underset{\sim}{k}$ is the unit vector representing the altitude of the aircraft.

(Treat the aircraft as a point in this problem.)

a. Show that $a = -44$ and $c = 2$.

Worked solution

$$r(t) = \left(a + \frac{2420}{t+5}\right)\underset{\sim}{i} + (500 - 24t + 0.28t^2)\underset{\sim}{j} + (e^{c-0.02t} - e)\underset{\sim}{k}$$

$$\underset{\sim}{r}(0) = (a + 484)\underset{\sim}{i} + 500\underset{\sim}{j} + (e^c - e)\underset{\sim}{k} = 440\underset{\sim}{i} + 500\underset{\sim}{j} + (e^2 - e)\underset{\sim}{k}$$

Equating $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components:

$a + 484 = 440$ and $e^c - e = e^2 - e$ 1M

$a = -44$ $c = 2$

1 mark

Mark allocation

- 1 mark for two correct equations to verify the values of a and c.

b. Show that the aircraft touches down at point T at 10.50 a.m.

Worked solution

$$\underset{\sim}{r}(t) = \left(-44 + \frac{2420}{t+5}\right)\underset{\sim}{i} + (500 - 24t + 0.28t^2)\,\underset{\sim}{j} + (e^{2-0.02t} - e)\,\underset{\sim}{k}$$

$$\underset{\sim}{r}(50) = \left(-44 + \frac{2420}{55}\right)\underset{\sim}{i} + (500 - 24\times 50 + 0.28\times 50^2)\,\underset{\sim}{j} + (e^{2-0.02\times 50} - e)\,\underset{\sim}{k}$$

$$= 0\underset{\sim}{i} + 0\underset{\sim}{j} + 0\underset{\sim}{k} \qquad \text{1M}$$

The aircraft touches down at T at 10.50 a.m.

1 mark

Mark allocation

- 1 mark for showing $r(50) = 0\underset{\sim}{i} + 0\underset{\sim}{j} + 0\underset{\sim}{k}$.

c. Show that the exact velocity of the aircraft at touchdown is $\underset{\sim}{r}' = -0.8\underset{\sim}{i} + 4\underset{\sim}{j} - 0.02e\underset{\sim}{k}$.

Worked solution

$$\underset{\sim}{r}(t) = \left(-44 + \frac{2420}{t+5}\right)\underset{\sim}{i} + (500 - 24t + 0.28t^2)\,\underset{\sim}{j} + (e^{2-0.02t} - e)\,\underset{\sim}{k}$$

$$\underset{\sim}{r}'(t) = \left(\frac{-2420}{(t+5)^2}\right)\underset{\sim}{i} + (-24 + 0.56t)\,\underset{\sim}{j} + (-0.02e^{2-0.02t})\,\underset{\sim}{k} \qquad \text{1M}$$

$$\underset{\sim}{r}'(50) = \left(\frac{-2420}{(50+5)^2}\right)\underset{\sim}{i} + (-24 + 0.56\times 50)\,\underset{\sim}{j} + \left(-0.02e^{2-0.02\times 50}\right)\underset{\sim}{k}$$

$$= -0.8\underset{\sim}{i} + 4\underset{\sim}{j} - 0.02e\underset{\sim}{k} \qquad \text{1A}$$

2 marks

Mark allocation

- 1 mark for correctly differentiating the position vector to find the velocity vector.
- 1 mark for the correct answer.

d. Find the vertical angle to the runway at which the aircraft lands. Give your answer to the nearest hundredth of a degree.

Worked solution

The angle required is the angle between the velocity vector $\underset{\sim}{r}'(50) = -0.8\underset{\sim}{i} + 4\underset{\sim}{j} - 0.02e\underset{\sim}{k}$ and the horizontal components of the velocity vector $-0.8\underset{\sim}{i} + 4\underset{\sim}{j}$.

$$\cos\theta = \frac{\underset{\sim}{r}'(50)(-0.8\underset{\sim}{i} + 4\underset{\sim}{j})}{\left|\underset{\sim}{r}'(50)\right|\left|-0.8\underset{\sim}{i} + 4\underset{\sim}{j}\right|} \qquad \text{1M}$$

$$= \frac{(-0.8\underset{\sim}{i} + 4\underset{\sim}{j} - 0.02e\underset{\sim}{k})(-0.8\underset{\sim}{i} + 4\underset{\sim}{j})}{\left|-0.8\underset{\sim}{i} + 4\underset{\sim}{j} - 0.02e\underset{\sim}{k}\right|\left|-0.8\underset{\sim}{i} + 4\underset{\sim}{j}\right|}$$

$$= \frac{0.64 + 16}{\sqrt{0.8^2 + 4^2 + 0.0004e^2}\sqrt{0.8^2 + 4^2}}$$

$$= \frac{16.64}{\sqrt{16.64 + 0.0004e^2}\sqrt{16.64}}$$

$$\approx 0.99991 \qquad \text{1M}$$

$$\theta \approx \cos^{-1}(0.99991)$$

$$\approx 0.76$$

The aircraft lands at an angle of 0.76° to the runway. 1A

3 marks

Mark allocation

- 1 mark for setting up cos (θ) in terms of the correct vectors.
- 1 mark for evaluating cos (θ) correctly.
- 1 mark for the correct answer.

Tips

- *The direction of motion is determined by the velocity vector.*
- *The angle could also be calculated as*

$$\theta = \tan^{-1}\left(\frac{0.02e}{\sqrt{0.8^2 + 4^2}}\right)$$

- *The velocity of the aircraft immediately after touchdown is* $\underset{\sim}{v}(t) = (-0.8\underset{\sim}{i} + 4\underset{\sim}{j})(1 - t)$ *km/min, where* $t \in [0,1]$ *is the time, in minutes, after touchdown.*

e. Relative to the point *T*, find the position vector $\underset{\sim}{p}(t)$ of the aircraft on the runway when the aircraft stops.

Worked solution

$\underset{\sim}{v}(t) = (-0.8\underset{\sim}{i} + 4\underset{\sim}{j})(1-t)$

$\quad = (-0.8 + 0.8t)\underset{\sim}{i} + (4-4t)\underset{\sim}{j}$

$\underset{\sim}{p}(t) = (-0.8t + 0.4t^2 + c_1) + (4t - 2t^2 + c_2)\underset{\sim}{j}$

$\underset{\sim}{p}(0) = c_1\underset{\sim}{i} + c_2\underset{\sim}{j} = 0\underset{\sim}{i} + 0\underset{\sim}{j}$ since the initial position on the runway is $T(0, 0, 0)$

$c_1 = c_2 = 0$

$\underset{\sim}{p}(t) = (-0.8t + 0.4t^2) + (4t - 2t^2)\underset{\sim}{j}$ 1M

$\underset{\sim}{v}(t) = (-0.8\underset{\sim}{i} + 4\underset{\sim}{j})(1-t) = 0\underset{\sim}{i} + 0\underset{\sim}{j}$

$\Rightarrow t = 1$ 1M

$\underset{\sim}{p}(1) = (-0.8 + 0.4)\underset{\sim}{i} + (4-2)\underset{\sim}{j}$

$\quad = -0.4\underset{\sim}{i} + 2\underset{\sim}{j}$ 1A

3 marks

Mark allocation

- 1 mark for correctly antidifferentiating $\underset{\sim}{v}(t)$ to obtain $\underset{\sim}{p}(t)$.
- 1 mark finding the value of *t* when the aircraft stops.
- 1 mark for the correct answer.

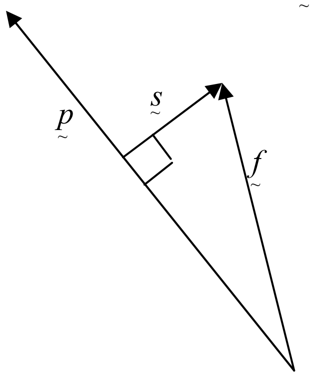
f. A stationary fire engine is positioned 1.5 kilometres north and 200 metres west of *T*. Determine, to the nearest metre, the minimum distance between the fire engine and the aircraft on its path after touchdown.

Worked solution

The position of the fire engine is $\underset{\sim}{f} = -0.2\underset{\sim}{i} + 1.5\underset{\sim}{j}$.

The final position of the aircraft is $\underset{\sim}{p} = -0.4\underset{\sim}{i} + 2\underset{\sim}{j}$.

The vector *s*, representing the shortest distance between *p* and *f*, is the vector resolute of *f* perpendicular to *p*.



$$\underset{\sim}{s} = \underset{\sim}{f} - \left(\frac{\underset{\sim}{f}\cdot\underset{\sim}{p}}{|\underset{\sim}{p}||\underset{\sim}{p}|}\right)\underset{\sim}{p}$$

$$= 0.2\underset{\sim}{i} + 1.5\underset{\sim}{j} - \frac{(-0.2i + 1.5j)(-0.4i + 2j)}{\sqrt{0.4^2 + 2^2}\sqrt{0.4^2 + 2^2}}(-0.4\underset{\sim}{i} + 2\underset{\sim}{j}) \qquad \text{1M}$$

$$= -0.2\underset{\sim}{i} + 1.5\underset{\sim}{j} - \frac{3.08}{4.16}(-0.4\underset{\sim}{i} + 2\underset{\sim}{j})$$

$$= -0.2\underset{\sim}{i} + 1.5\underset{\sim}{j} - \frac{1}{4.16}(-1.232\underset{\sim}{i} - 6.16\underset{\sim}{j})$$

$$= \frac{1}{4.16}(0.4\underset{\sim}{i} + 0.08\underset{\sim}{j}) \qquad \text{1M}$$

$$s = \frac{1}{4.16}\sqrt{0.4^2 + 0.08^2}$$

$$\cong 0.098 \qquad \text{1A}$$

The minimum distance between the aircraft and the fire engine is 98 metres.

3 marks

Mark allocation

- 1 mark for correctly setting up the vector representing the shortest distance.
- 1 mark for simplifying this vector.
- 1 mark for the correct answer.

Tips

- *The aircraft travels in a straight line along the runway after landing because the velocity vector* $\underset{\sim}{v}(t) = (-0.8\underset{\sim}{i} + 4\underset{\sim}{j})(1 - t)$ *is always parallel to* $(-0.8\underset{\sim}{i} + 4\underset{\sim}{j})$.
- *Draw a vector diagram first to help clarify the vector resolute required.*

Total 1 + 1 + 2 + 3 + 3 + 3 = 13 marks

END OF SOLUTIONS