



INSIGHT

Trial Exam Paper

2009

SPECIALIST MATHEMATICS

Written examination 1

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions.
- mark allocations
- tips on how to approach the questions.

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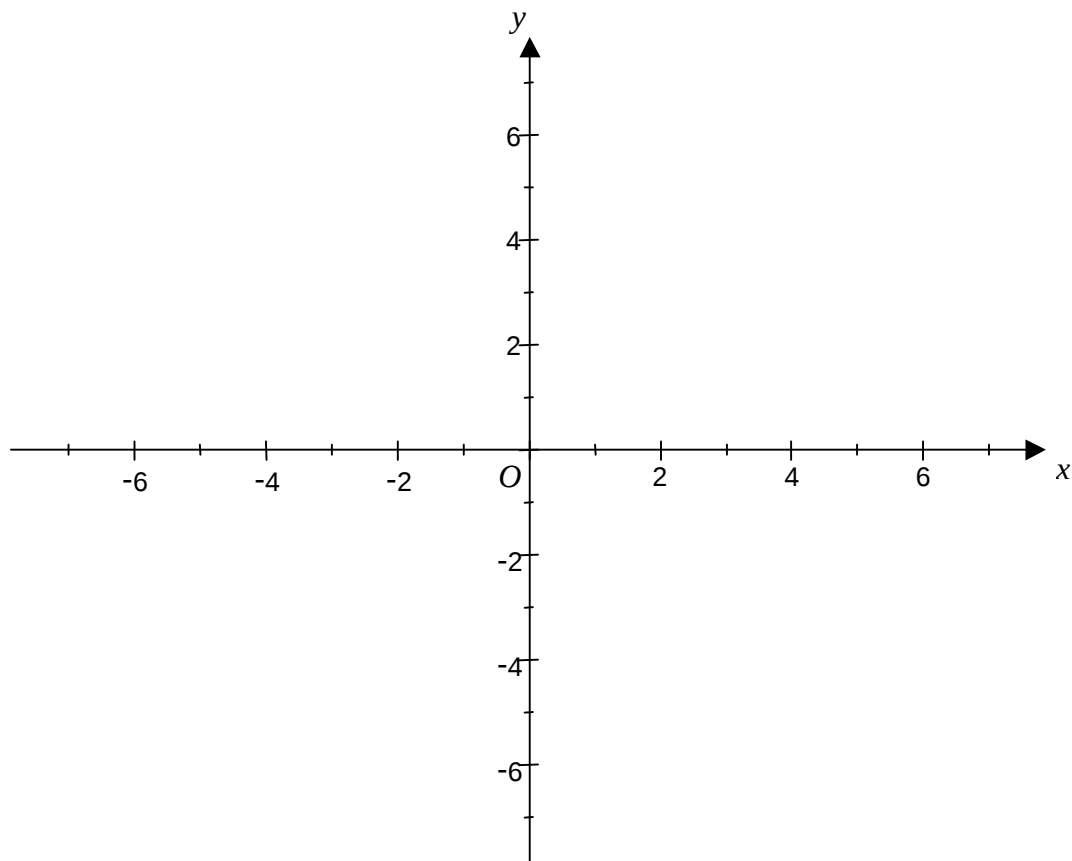
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Question 1

Sketch the graph of $y = \frac{1}{2-x} - x$ on the axes below. Give the exact coordinates of any stationary points and intercepts and the equations of any straight line asymptotes.

**Worked solution**

$y = \frac{1}{2-x} - x$ has a vertical asymptote at $x = 2$ and an oblique asymptote at $y = -x$.

Stationary points occur where $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{1}{(2-x)^2} - 1$$

$$0 = \frac{1}{(2-x)^2} - 1$$

$$(2-x)^2 = 1$$

$$2-x = \pm 1$$

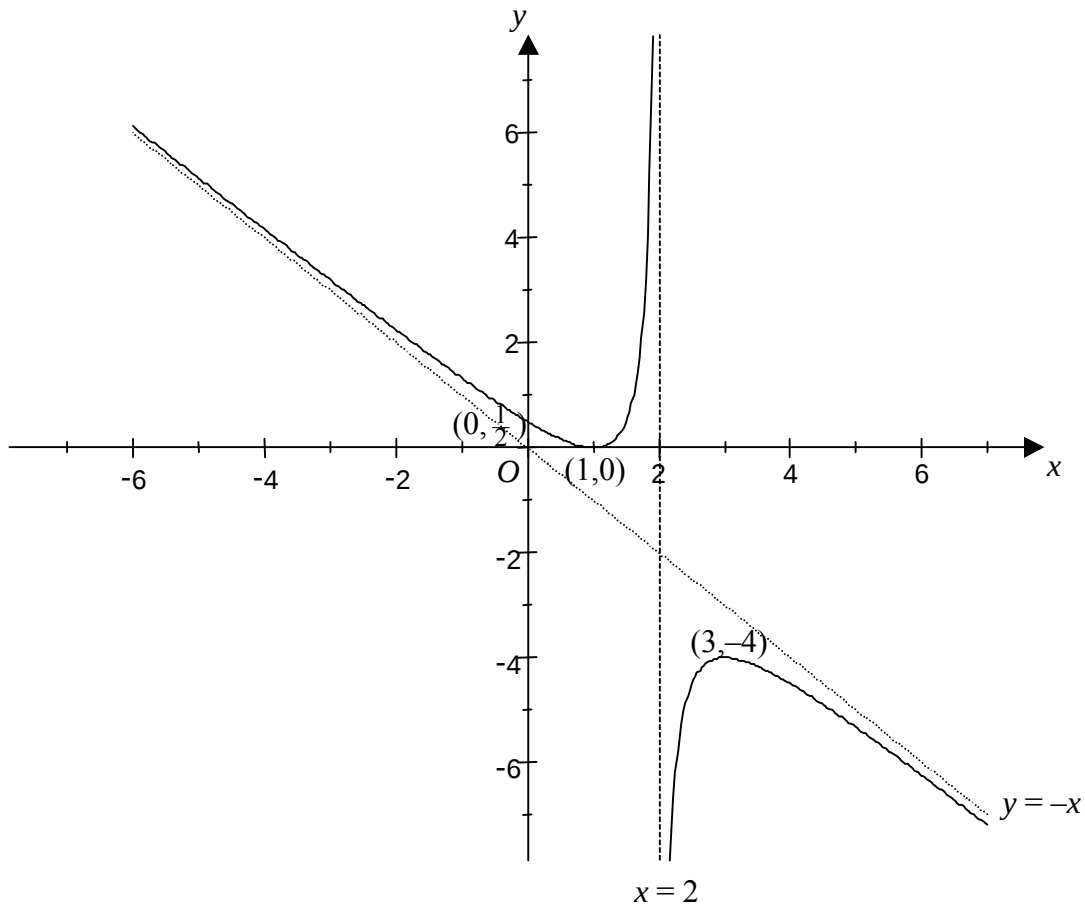
$$x = 1, 3$$

Stationary points are at $(1, 0)$ and $(3, -4)$.

x -intercept, $x = 1, y = 0$

y -intercept, $x = 0, y = \frac{1}{2}$

Question 1 – continued
TURN OVER



4 marks

Mark allocation

- 1 mark for the equations of both straight line asymptotes.
- 1 mark for the coordinates of both stationary points.
- 1 mark for correct intercepts.
- 1 mark for correct shape.

Question 2

Find all solutions of $z^3 + z + 10 = 0$, $z \in \mathbb{C}$.

Worked solution

Let $P(z) = z^3 + z + 10$

Find a factor of $P(z)$

$$P(1) \neq 0, P(-1) \neq 0, P(2) \neq 0$$

$$P(-2) = (-2)^3 + (-2) + 10 = 0$$

$\therefore z + 2$ is a factor

Find the quadratic factor

$$\begin{array}{r} z^2 - 2z + 5 \\ z + 2 \overline{) z^3 + 0z^2 + z + 10} \\ \underline{z^3 + 2z^2} \\ -2z^2 + z + 10 \\ \underline{-2z^2 - 4z} \\ 5z + 10 \\ \underline{5z + 10} \\ 0 \end{array}$$

$$\begin{aligned} P(z) &= (z + 2)(z^2 - 2z + 5) \\ &= (z + 2)((z^2 - 2z + 1) + 4) \\ &= (z + 2)((z - 1)^2 + 4) \\ &= (z + 2)((z - 1)^2 - (2i)^2) \\ &= (z + 2)(z - 1 + 2i)(z - 1 - 2i) \end{aligned}$$

$$\text{Solving } (z + 2)(z - 1 + 2i)(z - 1 - 2i) = 0$$

$$z = -2, \quad z = 1 - 2i, \quad z = 1 + 2i$$

3 marks

Mark allocation

- 1 mark for $z = -2$.
- 1 mark for correct method.
- 1 mark for three correct solutions.

TURN OVER

Question 3

Find the cube roots of $4i - 4\sqrt{3}$ in polar form.

Worked solution

Let $z^3 = -4\sqrt{3} + 4i = r \operatorname{cis}(\theta)$

Where $r = \sqrt{(-4\sqrt{3})^2 + 4^2}$ and $\tan(\theta) = \frac{4}{-4\sqrt{3}} = -\frac{1}{\sqrt{3}}$

$$r = \sqrt{16 \times 3 + 16}$$

$$\theta = \pi - \frac{\pi}{6}$$

$$r = 8$$

$$\theta = \frac{5\pi}{6}$$

$$\therefore z^3 = 8 \operatorname{cis}\left(\frac{5\pi}{6}\right)$$

$$z = \left(8 \operatorname{cis}\left(\frac{5\pi}{6}\right)\right)^{\frac{1}{3}}$$

$$z = 8^{\frac{1}{3}} \operatorname{cis}\left(\frac{1}{3}\left(\frac{5\pi}{6} + 2k\pi\right)\right) \quad \text{by De Moivre's Theorem}$$

$$k = 0 \quad z = 2 \operatorname{cis}\left(\frac{5\pi}{18}\right)$$

$$k = 1 \quad z = 2 \operatorname{cis}\left(\frac{5\pi}{18} + \frac{2\pi}{3}\right) = 2 \operatorname{cis}\left(\frac{17\pi}{18}\right)$$

$$k = -1 \quad z = 2 \operatorname{cis}\left(\frac{5\pi}{18} - \frac{2\pi}{3}\right) = 2 \operatorname{cis}\left(-\frac{7\pi}{18}\right)$$

In polar form the cube roots of $4i - 4\sqrt{3}$ are: $2 \operatorname{cis}\left(-\frac{7\pi}{18}\right)$, $2 \operatorname{cis}\left(\frac{5\pi}{18}\right)$, $2 \operatorname{cis}\left(\frac{17\pi}{18}\right)$

3 marks

Mark allocation

- 1 mark for finding $8 \operatorname{cis}\left(\frac{5\pi}{6}\right)$.
- 1 mark for applying De Moivre's Theorem.
- 1 mark for three correct solutions.

Question 4

Find the point of intersection of the normals to the curve $x^2y + y^2 = 5$ at $y = 1$.

Worked solution

When $y = 1$, $x^2 \times 1 + 1^2 = 5$

$$x^2 = 4$$

$$x = \pm 2$$

Need to find the gradient of the normals at the points (2, 1) and (-2, 1)

$$x^2y + y^2 = 5$$

Using implicit differentiation to find the gradient of the tangent at these points

$$2xy + x^2 \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x^2 + 2y) = -2xy$$

$$\frac{dy}{dx} = \frac{-2xy}{x^2 + 2y}$$

$$\text{At (2, 1)} \quad \frac{dy}{dx} = \frac{-2 \times 2 \times 1}{2^2 + 2 \times 1} = -\frac{2}{3} \quad \Rightarrow \quad \text{gradient of normal is } \frac{3}{2}$$

$$\text{At (-2, 1)} \quad \frac{dy}{dx} = \frac{-2 \times -2 \times 1}{(-2)^2 + 2 \times 1} = \frac{2}{3} \quad \Rightarrow \quad \text{gradient of normal is } -\frac{3}{2}$$

$$\text{Equation of the normal at (2, 1):} \quad y = \frac{3}{2}x + c \quad \Rightarrow \quad 1 = \frac{3}{2} \times 2 + c, \quad c = -2$$

$$y = \frac{3}{2}x - 2$$

$$\text{Equation of the normal at (-2, 1):} \quad y = -\frac{3}{2}x + c \quad \Rightarrow \quad 1 = -\frac{3}{2} \times (-2) + c, \quad c = -2$$

$$y = -\frac{3}{2}x - 2$$

The normal equations $y = \frac{3}{2}x - 2$ and $y = -\frac{3}{2}x - 2$ intersect on the y-axis at (0, -2)

5 marks

Mark allocation

- 1 mark for finding the points (2, 1) and (-2, 1).
- 1 mark for a correct method used to find derivative.
- 1 mark for finding the correct derivative.
- 1 mark for correct normal equations.
- 1 mark for correct answer (0, -2).

TURN OVER

Question 5

Given $\frac{d}{dx} \left(\arctan \left(\frac{2x}{x^2 - 1} \right) \right) = \frac{a}{x^2 + b}, |x| \neq 1$

Find the real numbers a and b .

Worked solution

$$\text{Let } y = \arctan(u) \quad \text{where } u = \frac{2x}{x^2 - 1} \quad \Rightarrow \quad \frac{du}{dx} = \frac{2(x^2 - 1) - 2x(2x)}{(x^2 - 1)^2}$$

$$\frac{du}{dx} = \frac{-2x^2 - 2}{(x^2 - 1)^2}$$

$$\frac{du}{dx} = \frac{-2(x^2 + 1)}{(x^2 - 1)^2}$$

Applying the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{1 + u^2} \times \frac{-2(x^2 + 1)}{(x^2 - 1)^2}$$

$$\frac{dy}{dx} = \frac{1}{1 + \left(\frac{2x}{x^2 - 1} \right)^2} \times \frac{-2(x^2 + 1)}{(x^2 - 1)^2}$$

$$\frac{dy}{dx} = \frac{-2(x^2 + 1)}{(x^2 - 1)^2 + (2x)^2}$$

$$\frac{dy}{dx} = \frac{-2(x^2 + 1)}{(x^4 - 2x^2 + 1) + 4x^2}$$

$$\frac{dy}{dx} = \frac{-2(x^2 + 1)}{x^4 + 2x^2 + 1}$$

$$\frac{dy}{dx} = \frac{-2(x^2 + 1)}{(x^2 + 1)^2}$$

$$\frac{dy}{dx} = \frac{-2}{x^2 + 1}$$

Hence $a = -2$ and $b = 1$

3 marks

Mark allocation

- 1 mark for applying the chain rule correctly.
- 1 mark for simplifying algebra.
- 1 mark for two correct answers.

Question 6

Find $\int e^{6x} \sqrt{e^{2x} + 1} dx$.

Worked solution

Let $u = e^{2x} + 1$

$$\frac{du}{dx} = 2e^{2x}$$

$$du = 2e^{2x} dx$$

$$\int e^{6x} \sqrt{e^{2x} + 1} dx$$

$$= \frac{1}{2} \int e^{4x} \sqrt{e^{2x} + 1} (2e^{2x} dx)$$

$$= \frac{1}{2} \int (e^{2x})^2 \sqrt{e^{2x} + 1} (2e^{2x} dx)$$

$$\text{If } u = e^{2x} + 1, \Rightarrow e^{2x} = u - 1$$

$$= \frac{1}{2} \int (u - 1)^2 \sqrt{u} du$$

$$= \frac{1}{2} \int (u^2 - 2u + 1) u^{\frac{1}{2}} du$$

$$= \frac{1}{2} \int \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}} \right) du$$

$$= \frac{1}{2} \left(\frac{2}{7} u^{\frac{7}{2}} - 2 \times \frac{2}{5} u^{\frac{5}{2}} + \frac{2}{3} u^{\frac{3}{2}} \right) + c$$

$$= \frac{1}{7} u^{\frac{7}{2}} - \frac{2}{5} u^{\frac{5}{2}} + \frac{1}{3} u^{\frac{3}{2}} + c$$

$$= \frac{1}{7} (e^{2x} + 1)^{\frac{7}{2}} - \frac{2}{5} (e^{2x} + 1)^{\frac{5}{2}} + \frac{1}{3} (e^{2x} + 1)^{\frac{3}{2}} + c \quad \text{where } c \text{ is constant}$$

4 marks

Mark allocation

- 1 mark for selecting correct substitution.
- 1 mark for simplifying integral in terms of u .
- 1 mark for integrating correctly.
- 1 mark for correct answer.

Tip

- Use substitution.

TURN OVER

Question 7

An object in a refrigerator cools according to differential equation $\frac{dT}{dt} = -k(T - 3)$, $k \in R$,

where T °C is the temperature of the object t hours after it being placed in the refrigerator.

A drink with an initial temperature of 18°C is placed in the refrigerator for 1 hour, and it cools to 8°C in that time.

a. Show that $T = 15e^{-kt} + 3$ is a solution to this differential equation.

Worked solution

Given $T = 15e^{-kt} + 3$ (1)

$$\Rightarrow \frac{dT}{dt} = -15k e^{-kt}$$

$$\frac{dT}{dt} = -k(15e^{-kt}) \text{ (2)}$$

From (1) $15e^{-kt} = T - 3$

Substituting into (2)

$$\Rightarrow \frac{dT}{dt} = -k(T - 3)$$

Therefore $T = 15e^{-kt} + 3$ is a solution of $\frac{dT}{dt} = -k(T - 3)$.

Alternative method:

$$\frac{dT}{dt} = -k(T - 3)$$

$$\frac{dT}{dT} = \frac{1}{-k(T - 3)}$$

$$t = -\frac{1}{k} \int \frac{1}{T - 3} dt$$

$$t = -\frac{1}{k} \log_e (T - 3) + c, \quad T > 3$$

$$t = 0, T = 18, \quad 0 = -\frac{1}{k} \log_e (18 - 3) + c$$

$$c = \frac{1}{k} \log_e (15)$$

$$t = -\frac{1}{k} \log_e (T - 3) + \frac{1}{k} \log_e (15)$$

$$t = -\frac{1}{k} \log_e \left(\frac{T - 3}{15} \right)$$

$$e^{-kt} = \frac{T - 3}{15}$$

$$T = 15e^{-kt} + 3$$

2 marks

Mark allocation

- 1 mark for method used.
- 1 mark for showing correct answer.

Question 7 – continued

b. Find the exact value of k .

Worked solution

When $t = 1$, $T = 8$

$$8 = 15e^{-k \times 1} + 3$$

$$e^{-k} = \frac{5}{15}$$

$$k = -\log_e\left(\frac{1}{3}\right)$$

$$k = \log_e(3)$$

1 mark

Mark allocation

- 1 mark for correct answer.

c. Find the exact temperature of the drink after 2 hours.

Worked solution

$$T = 15e^{-kt} + 3$$

$$T = 15e^{-\log_e(3)t} + 3$$

$$T = 15e^{-t \log_e(3)} + 3$$

$$T = 15e^{\log_e(3)^{-t}} + 3$$

$$T = 15(3)^{-t} + 3$$

$$\text{When } t = 2 \quad T = 15(3)^{-2} + 3$$

$$T = \frac{15}{9} + 3$$

$$T = 4\frac{2}{3} \text{ } ^\circ\text{C}$$

1 mark

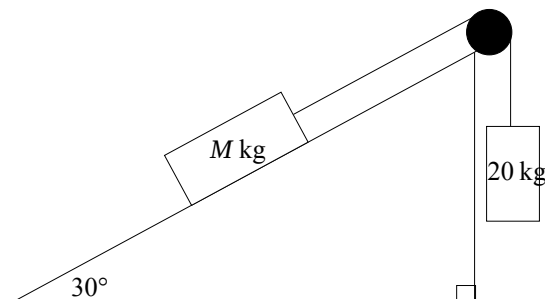
Total $2 + 1 + 1 = 4$ marks

Mark allocation

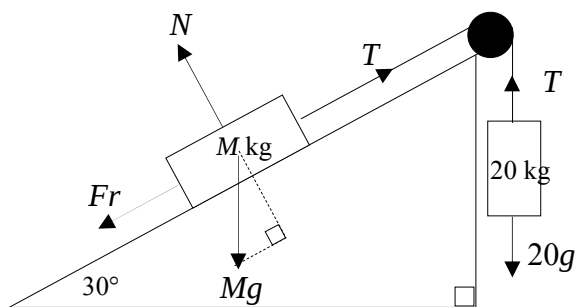
- 1 mark for correct answer.

Question 8

A mass of M kg is connected to a 20 kg mass by a light string passing over a smooth pulley. The rough plane is inclined at 30° to the horizontal level and has coefficient of friction $\mu = \frac{1}{5}$. The tension in the string connecting the two masses is 150 newtons.



- a. Show that the M kg mass is accelerating up the inclined plane at 2.3 m/s^2 .

Worked solution

Resolving forces around the 20 kg mass. Assume this mass is moving downwards

$$20g - T = 20a$$

$$T = 150 \text{ newtons}$$

$$20 \times 9.8 - 150 = 20a$$

$$20a = 46$$

$$a = 2.3 \text{ m/s}^2$$

Since the acceleration is positive, the 20 kg mass is moving downwards. The masses are connected so the M kg mass on the inclined plane must be moving upwards with the same acceleration.

2 marks

Mark allocation

- 1 mark for equation of motion for vertical mass.
- 1 mark for correct answer.

Tip

- Show all forces acting on the diagram.

b. Determine the exact value of M .

Worked solution

Resolving forces around the M kg mass on the inclined plane.

This is moving upwards with an acceleration of 2.3 m/s^2 .

$$T - Mg \sin(30^\circ) - Fr = Ma$$

$$N = Mg \cos(30^\circ) = \frac{\sqrt{3}Mg}{2} \text{ newtons}$$

$$150 - \frac{Mg}{2} - \frac{\sqrt{3}Mg}{10} = 2.3M$$

$$Fr = \mu N = \frac{1}{5} \times \frac{\sqrt{3}Mg}{2} = \frac{\sqrt{3}Mg}{10} \text{ newtons}$$

$$\frac{23M}{10} + \frac{Mg}{2} + \frac{\sqrt{3}Mg}{10} = 150$$

$$\frac{M}{10}(23 + 5g + \sqrt{3}g) = 150$$

$$M = \frac{1500}{23 + (5 + \sqrt{3})g} \text{ kg}$$

3 marks

Total 2 + 3 = 5 marks

Mark allocation

- 1 mark for resolving forces correctly.
- 1 mark for correct substitution into the equation of motion.
- 1 mark for correct answer.

Question 9

The position of a particle at time t is given by $\underline{r} = (1 - 2 \sin(\pi t))\underline{i} + (\cos(2\pi t) + 2)\underline{j}$

a. Find the Cartesian equation of the path of the particle.

Worked solution

Find y in terms of x :

$$x = 1 - 2 \sin(\pi t)$$

$$y = \cos(2\pi t) + 2$$

$$\sin(\pi t) = \frac{1-x}{2} \dots\dots(1)$$

$$y = 1 - 2 \sin^2(\pi t) + 2$$

$$y = 3 - 2 \sin^2(\pi t) \dots\dots(2)$$

Substitute (1) into (2)

$$y = 3 - 2 \left(\frac{1-x}{2} \right)^2$$

$$y = -\frac{1}{2}(x-1)^2 + 3$$

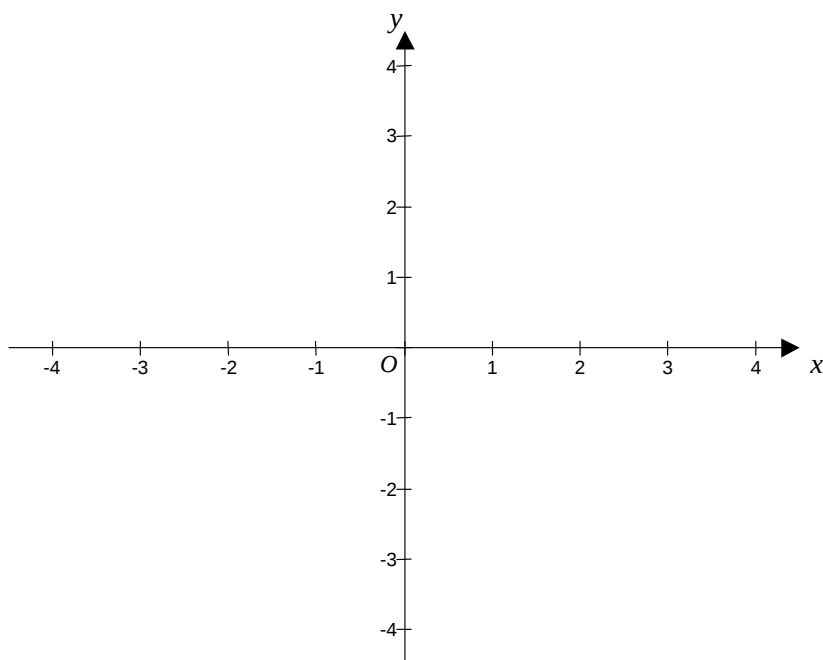
2 marks

Mark allocation

- 1 mark for selecting correct substitution.
- 1 mark for simplifying integral in terms of u .

Question 9 – continued
TURN OVER

- b. Sketch a graph of the path of the particle for $0 \leq t \leq \frac{1}{2}$ indicating its direction of motion.



Worked solution

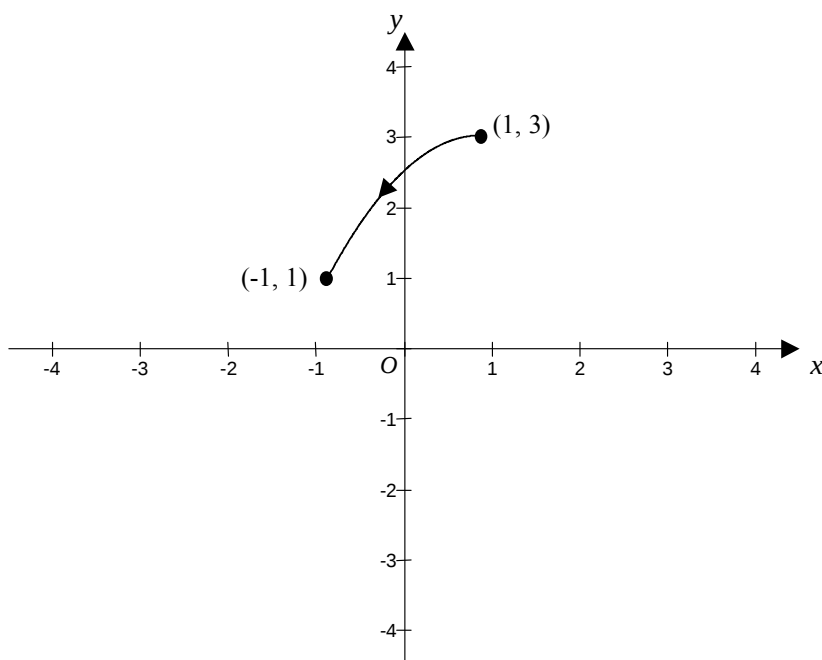
When $t = 0$ $x = 1 - 2 \sin(0)$ $y = \cos(0) + 2$

$x = 1$ $y = 3$

When $t = \frac{1}{2}$ $x = 1 - 2 \sin\left(\frac{\pi}{2}\right)$ $y = \cos(\pi) + 2$

$x = -1$ $y = 1$

At $t = 0$ the particle starts at the point $(1, 3)$. It moves anticlockwise along the parabola $y = -\frac{1}{2}(x-1)^2 + 3$ to reach the point $(-1, 1)$ at $t = \frac{1}{2}$.



2 marks

Mark allocation

- 1 mark for drawing their curve.
- 1 mark for correct answer with direction of motion shown.

c. Determine the speed at which the particle is travelling when $t = \frac{1}{4}$.

Worked solution

Speed = $|\dot{\underline{r}}|$

$$\underline{r} = (1 - 2\sin(\pi t))\underline{i} + (\cos(2\pi t) + 2)\underline{j}$$

$$\dot{\underline{r}} = -2\pi\cos(\pi t)\underline{i} - 2\pi\sin(2\pi t)\underline{j}$$

$$|\dot{\underline{r}}| = \sqrt{(-2\pi\cos(\pi t))^2 + (-2\pi\sin(2\pi t))^2}$$

$$|\dot{\underline{r}}| = 2\pi\sqrt{\cos^2(\pi t) + \sin^2(2\pi t)}$$

$$\text{When } t = \frac{1}{4}, \quad |\dot{\underline{r}}| = 2\pi\sqrt{\cos^2\left(\frac{\pi}{4}\right) + \sin^2\left(\frac{\pi}{2}\right)}$$

$$|\dot{\underline{r}}| = 2\pi\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + 1^2}$$

$$|\dot{\underline{r}}| = \sqrt{6}\pi$$

1 mark

Total $2 + 2 + 1 = 5$ marks

Mark allocation

- 1 mark for correct answer.

Tip

- *Differentiate to find velocity vector.*

TURN OVER

Question 10

A cyclist of mass 72 kg is travelling on a straight track with a velocity of 1 m/s when he passes O . At that instant he applies a variable force of $v^3 + 3v$ newtons, where v m/s is his velocity t seconds after passing O . Calculate the exact distance of the cyclist from O when his velocity reaches $\sqrt{3}$ m/s. Assume air resistance is negligible.

4 marks

Worked solution

$$F = ma$$

$$v^3 + 3v = 72a$$

$$a = \frac{v^3 + 3v}{72}$$

$$v \frac{dv}{dx} = \frac{v^3 + 3v}{72}$$

$$\frac{dv}{dx} = \frac{v^3 + 3v}{72v}$$

$$\frac{dv}{dx} = \frac{v^2 + 3}{72}$$

$$x = \int \frac{72}{v^2 + 3} dv$$

$$x = \frac{72}{\sqrt{3}} \int \frac{\sqrt{3}}{v^2 + 3} dv$$

$$x = \frac{72\sqrt{3}}{3} \tan^{-1}\left(\frac{v}{\sqrt{3}}\right) + c$$

$$x = 24\sqrt{3} \tan^{-1}\left(\frac{v}{\sqrt{3}}\right) + c$$

When $x = 0$, $v = 1$ m/s

$$0 = 24\sqrt{3} \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + c$$

$$c = -24\sqrt{3} \times \frac{\pi}{6}$$

$$c = -4\sqrt{3}\pi$$

$$x = 24\sqrt{3} \tan^{-1}\left(\frac{v}{\sqrt{3}}\right) - 4\sqrt{3}\pi$$

Find x when $v = \sqrt{3}$ m/s

$$x = 24\sqrt{3} \tan^{-1}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) - 4\sqrt{3}\pi$$

$$x = 24\sqrt{3} \tan^{-1}(1) - 4\sqrt{3}\pi$$

$$x = 24\sqrt{3} \times \frac{\pi}{4} - 4\sqrt{3}\pi$$

$$x = 6\sqrt{3}\pi - 4\sqrt{3}\pi$$

$$x = 2\sqrt{3}\pi$$

The cyclist is $2\sqrt{3}\pi$ m from O when his velocity is $\sqrt{3}$ m/s.

Mark allocation

1 mark for finding the acceleration in terms of v .

1 mark for establishing correct integral.

1 mark for finding correct expression for x in terms of v .

1 mark for answer.

Tip

- *Write down the equation of motion and find the acceleration in terms of v .*