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**SPECIALIST MATHS
TRIAL EXAMINATION 2
SOLUTIONS
2009**

Section 1 – Multiple-choice answers

1. B	7. D	13. B	19. A
2. C	8. E	14. C	20. E
3. E	9. C	15. D	21. D
4. E	10. C	16. C	22. B
5. C	11. E	17. A	
6. B	12. D	18. D	

Section 1- Multiple-choice solutions

Question 1

The asymptotes are $y = ax$ and $x = 0$. Only option B offers these asymptotes since

$$y = \frac{ax^3 + 1}{x^2}$$

$$y = ax + \frac{1}{x^2}, a > 0$$

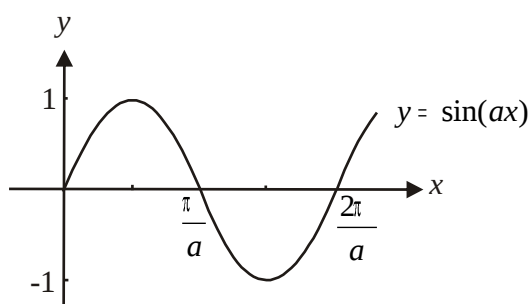
The answer is B.

Question 2

$$y = \operatorname{cosec}(ax)$$

$$= \frac{1}{\sin(ax)}$$

Now $\sin(ax) = 0$ for $x = 0, \frac{\pi}{a}, \frac{2\pi}{a}, \frac{3\pi}{a}, \dots$



So, $y = \frac{1}{\sin(ax)}$ will have asymptotes at $x = 0, \frac{\pi}{a}, \frac{2\pi}{a}, \frac{3\pi}{a}, \dots$

The answer is C.

Question 3

$$y = 2\arcsin(2x + 1) - \pi$$

For the domain,

$$-1 \leq 2x + 1 \leq 1$$

$$-2 \leq 2x \leq 0$$

$$-1 \leq x \leq 0$$

For the range,

$$-\frac{\pi}{2} \leq \arcsin(2x + 1) \leq \frac{\pi}{2}$$

$$-\pi \leq 2\arcsin(2x + 1) \leq \pi$$

$$-2\pi \leq 2\arcsin(2x + 1) - \pi \leq 0$$

The answer is E.

Question 4

$$z = \sqrt{3}\operatorname{cis}\left(-\frac{\pi}{2}\right)$$

$$z^6 = \left(\sqrt{3}\right)^6 \operatorname{cis}\left(-\frac{6\pi}{2}\right) \quad (\text{De Moivre})$$

$$= 27\operatorname{cis}(-3\pi)$$

$$= 27\operatorname{cis}(\pi)$$

$$\operatorname{Arg}(z^6) = \pi$$

Note that $-\pi < \operatorname{Arg}(z^6) \leq \pi$.

The answer is E.

Question 5

The complex number $\bar{z}$, the conjugate of z can be represented by the point T .

The complex number $\bar{z}i$ is obtained by rotating point T by $\frac{\pi}{2}$ radians in an anticlockwise direction.

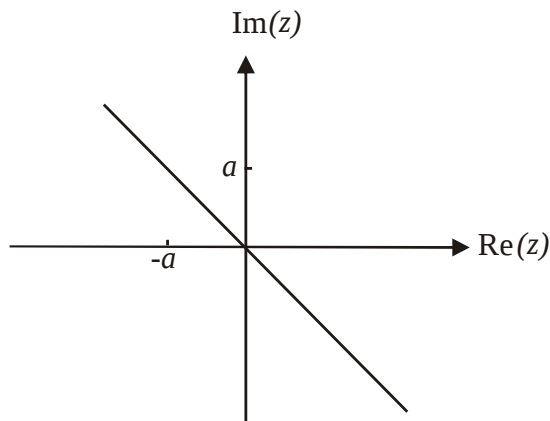
So $\bar{z}i$ is represented by the point R .

The answer is C.

Question 6

Method 1 – graphical approach.

The graph is made up of the set of complex numbers which are the same distance from the complex number ai and $-a$.



These lie on the line that passes through the point representing the complex number $-a + ai$ and has a gradient of -1 .

The answer is B.

Method 2 – algebraic approach

$$|z - ai| = |z + a|, \quad a > 0$$

$$|x + yi - ai| = |x + yi + a|$$

$$\sqrt{x^2 + (y - a)^2} = \sqrt{(x + a)^2 + y^2}$$

$$x^2 + y^2 - 2ay + a^2 = x^2 + 2ax + a^2 + y^2$$

$$-2ay = 2ax$$

$$y = -x$$

The line passes through the complex number $-a + ai$ and has a gradient of -1 .

The answer is B.

Question 7

Since the coefficients of the cubic polynomial are real, the conjugate root theorem applies and so $z = ai$ is another root as is $z = b$ where b is any real number.

$$p(z) = (z - ai)(z + ai)(z - b)$$

$$= (z^2 + a^2)(z - b)$$

$$= z^3 - bz^2 + a^2z - a^2b$$

If $b = 0$,

$$p(z) = z^3 + a^2z$$

If $b = 1$,

$$p(z) = z^3 - z^2 + a^2z - a^2$$

The answer is D.

Question 8

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \sin(x) \cos^3(x) dx \\ &= \int_1^0 -\frac{du}{dx} u^3 dx \\ &= \int_0^1 u^3 du \end{aligned}$$

The answer is E.

$$\text{let } u = \cos(x)$$

$$\frac{du}{dx} = -\sin(x)$$

$$x = \frac{\pi}{2}, u = 0$$

$$x = 0, u = 1$$

Question 9

$$f'(t) = \ln(t+1) \quad t > 0$$

$$\int_0^1 f'(t) dt = f(1) - f(0)$$

$$\text{Since } f(0) = 2,$$

$$\int_0^1 f'(t) dt = f(1) - 2$$

$$\text{so } f(1) = \int_0^1 f'(t) dt + 2$$

$$= \int_0^1 \ln(t+1) dt + 2$$

The answer is C.

Question 10

Let V = volume of water in trough.

$$V = h^2 \times 150 (\text{cm}^3)$$

$$\frac{dV}{dh} = 300h$$

$$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$$

$$= \frac{1}{300h} \times 300$$

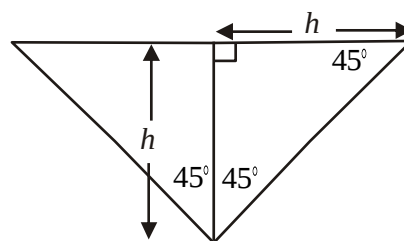
$$= \frac{1}{h}$$

When $h = 20$

$$\frac{dh}{dt} = \frac{1}{20} \text{ cm/min}$$

$$= 0.05 \text{ cm/min}$$

The answer is C.



Question 11

$$\begin{aligned}
 & \int_0^2 \frac{2x-1}{\sqrt{3-x}} dx \\
 &= - \int_3^1 (5-2u) \frac{1}{\sqrt{u}} \times \frac{du}{dx} dx \\
 &= - \int_3^1 (5u^{-\frac{1}{2}} - 2u^{\frac{1}{2}}) du \\
 &= \int_1^3 (5u^{-\frac{1}{2}} - 2u^{\frac{1}{2}}) du
 \end{aligned}$$

The answer is E.

$$\text{let } u = 3 - x$$

$$\frac{du}{dx} = -1$$

$$x = 3 - u$$

$$2x - 1 = 2(3 - u) - 1$$

$$= 6 - 2u - 1$$

$$= 5 - 2u$$

$$\text{Also } x = 2, u = 1$$

$$x = 0, u = 3$$

Question 12

$$\begin{aligned}
 \frac{dH}{dt} &= \text{rate of inflow of } H \text{ (litres / minute)} - \text{rate of outflow of } H \text{ (litres / minute)} \\
 &= 5 - \frac{H}{200} \times 5 \\
 &= 5 - \frac{H}{40} \\
 &= \frac{200 - H}{40}
 \end{aligned}$$

Initially there is 40% of 200 = 80 litres of hydrogen in the cylinder.

$$\text{So } \frac{dH}{dt} = \frac{200 - H}{40}, t = 0, H = 80.$$

The answer is D.

Question 13

The particles are in the same position when

$$t^2 = 7t - 10 \quad \text{AND} \quad 3t - 4 = 11$$

$$t^2 - 7t + 10 = 0$$

$$3t = 15$$

$$(t - 5)(t - 2) = 0$$

$$t = 5$$

$$t = 5, 2$$

At $t = 5$ the particles are in the same position.

The answer is B.

Question 14

$$\vec{a} \cdot \vec{b} = (\vec{i} + 2\vec{j} + 2\vec{k})(2\vec{i} - \vec{j} + 2\vec{k})$$

$$= 2 - 2 + 4 = 4$$

$$\text{Also, } \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\text{So, } 4 = \sqrt{1+4+4} \cdot \sqrt{4+1+4} \cdot \cos\theta$$

$$\text{So, } \cos\theta = \frac{4}{9}, \text{ and therefore } \theta = 63^\circ 37'$$

The answer is C.

Question 15

Since PQRS is a rhombus the diagonals cross at right angles so

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$$

$$\vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{b} = 0$$

$$\vec{a} \cdot \vec{a} = \vec{b} \cdot \vec{b}$$

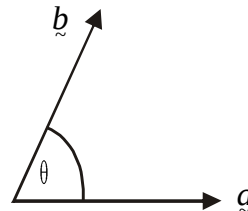
So A is true.

Since PQRS is a rhombus and $|\vec{a}| = 1$ then $|\vec{b}| = 1$ so $|\vec{a}| = |\vec{b}|$ so B is true.

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$= \cos\theta$$

$$\text{since } |\vec{a}| = |\vec{b}| = 1$$



So option C is true but option D isn't because

$\theta \neq 180 - \theta$ since $\theta \neq 90^\circ$. So option D is false.

For option E, since $\theta \neq 90^\circ$, $\vec{a}$ and $\vec{b}$ are never at right angles so $\vec{a} \cdot \vec{b} \neq 0$

Option E is true.

The answer is D.

Question 16

$$\vec{r} = 2\sqrt{t}\vec{i} + (5-t)\vec{j}$$

$$\begin{aligned} \text{distance from origin} &= \sqrt{(2\sqrt{t})^2 + (5-t)^2} \\ &= \sqrt{4t + 25 - 10t + t^2} \\ &= \sqrt{t^2 - 6t + 25} \end{aligned}$$

Method 1

$$\begin{aligned} \text{This is a minimum when } 2t - 6 &= 0 \\ t &= 3 \end{aligned}$$

Method 2

$$(t^2 - 6t + 9) - 9 + 25 = (t - 3)^2 + 16$$

Minimum occurs when $t = 3$.

The answer is C.

Question 17

The total force $\vec{F}$ acting on the body is

$$\begin{aligned}\vec{F} &= \vec{P} + \vec{Q} + \vec{R} \\ &= 4\vec{i} - \vec{j} - 3\vec{i} + 2\vec{j} + 2\vec{i} + 3\vec{j} \\ &= 3\vec{i} + 4\vec{j}\end{aligned}$$

Now $\vec{F} = m\vec{a}$

$$3\vec{i} + 4\vec{j} = 8\vec{a}$$

$$\begin{aligned}\vec{a} &= \frac{3}{8}\vec{i} + \frac{1}{2}\vec{j} \\ |\vec{a}| &= \sqrt{\frac{9}{64} + \frac{1}{4}} \\ &= \sqrt{\frac{25}{64}} \\ &= \frac{5}{8} \text{ m/s}^2.\end{aligned}$$

The answer is A.

Question 18

$$\vec{p} = m\vec{v}$$

$$20 = 4m$$

$$m = 5 \text{ kg}$$

Later when $p = 45$

$$45 = 5 \times v$$

$$v = 9 \text{ m/s}$$

Since acceleration is constant and $u = 4$, $a = 0.5$ and $v = 9$,

$$v^2 = u^2 + 2as$$

$$\text{so, } 81 = 16 + 2 \times \frac{1}{2} \times s$$

$$s = 81 - 16$$

$$= 65$$

The mass covers 65m.

The answer is D.

Question 19

$\int_0^{20} f(t) dt$ gives the displacement of particle B from the start.

$\int_0^{12} f(t) dt - \int_{12}^{20} f(t) dt$ gives the distance travelled.

Therefore the distance between the two particles is given by $\left| \int_0^{20} f(t) dt - 20g(20) \right|$.

The answer is A.

Question 20

$$a = \frac{1}{\sqrt{4-x^2}}$$

$$\frac{d}{dx} \left(\frac{1}{2} v^2 \right) = \frac{1}{\sqrt{4-x^2}}$$

$$\frac{1}{2} v^2 = \int \frac{1}{\sqrt{4-x^2}} dx$$

$$\frac{1}{2} v^2 = \sin^{-1} \left(\frac{x}{2} \right) + c$$

when $x = 0$, $v = 4$

$$c = 8$$

$$\frac{1}{2} v^2 = \sin^{-1} \left(\frac{x}{2} \right) + 8$$

$$v^2 = 2 \sin^{-1} \left(\frac{x}{2} \right) + 16$$

$$\frac{v^2 - 16}{2} = \sin^{-1} \left(\frac{x}{2} \right)$$

$$\frac{x}{2} = \sin \left(\frac{v^2 - 16}{2} \right)$$

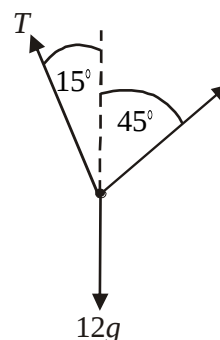
$$x = 2 \sin \left(\frac{v^2 - 16}{2} \right)$$

The answer is E.

Question 21Method 1 – using Lami's TheoremLet T be the tension in the shorter string.

$$\begin{aligned}
 \frac{T}{\sin(80^\circ - 45^\circ)} &= \frac{12g}{\sin(60^\circ)} \quad (\text{Lami's Theorem}) \\
 T &= 12g \div \frac{\sqrt{3}}{2} \times \sin(135^\circ) \\
 &= 12g \times \frac{2}{\sqrt{3}} \times \frac{1}{\sqrt{2}} \\
 &= \frac{24g}{\sqrt{6}} \\
 &= 4\sqrt{6}g \text{ newtons}
 \end{aligned}$$

The answer is D.

Method 2 – resolving horizontally and verticallyLet T_1 be the tension in the shorter string andlet T_2 be the tension in the longer string. T_1 is the tension required.

Resolving horizontally:

$$T_1 \cos(75^\circ) = T_2 \cos(45^\circ) \quad (1)$$

Resolving vertically:

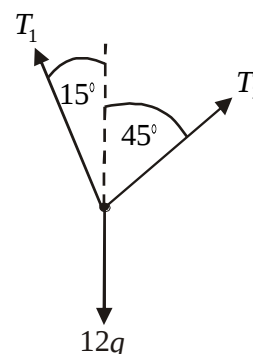
$$T_1 \cos(15^\circ) + T_2 \cos(45^\circ) = 12g \quad (2)$$

$$(1) \text{ in } (2) \text{ gives } T_1 \cos(15^\circ) + T_1 \cos(75^\circ) = 12g$$

$$T_1 (\cos(15^\circ) + \cos(75^\circ)) = 12g$$

$$T_1 = \frac{12g}{\cos(15^\circ) + \cos(75^\circ)}$$

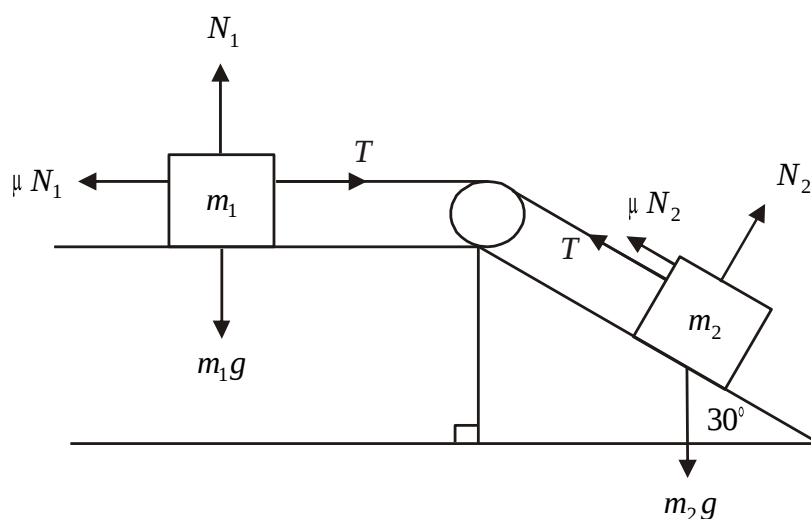
$$T_1 = 96.02 \text{ newtons}$$



The answer is D.

Question 22

Mark in the forces operating.



Around the m_1 kg mass

$$T = \mu N_1 \text{ and } N_1 = m_1 g$$

$$\text{so } T = \mu m_1 g$$

Around the m_2 kg mass

$$T + \mu N_2 = m_2 g \sin(30^\circ) \text{ and } N_2 = m_2 g \cos(30^\circ)$$

$$T + \frac{\mu\sqrt{3}m_2g}{2} = \frac{m_2g}{2} \qquad \qquad = \frac{\sqrt{3}}{2} m_2g$$

$$T = \frac{m_2g - \mu\sqrt{3}m_2g}{2}$$

$$\text{So } \mu m_1 g = \frac{m_2g (-\sqrt{3}\mu)}{2}$$

$$\frac{m_1}{m_2} = \frac{1 - \sqrt{3}\mu}{2\mu}$$

The answer is B.

SECTION 2

Question 1

a. $\underline{r}(t) = 2\cos(t)\underline{i} + 3\sin(t)\underline{j}$

$$x = 2\cos(t) \quad y = 3\sin(t)$$

$$x^2 = 4\cos^2(t) \quad y^2 = 9\sin^2(t)$$

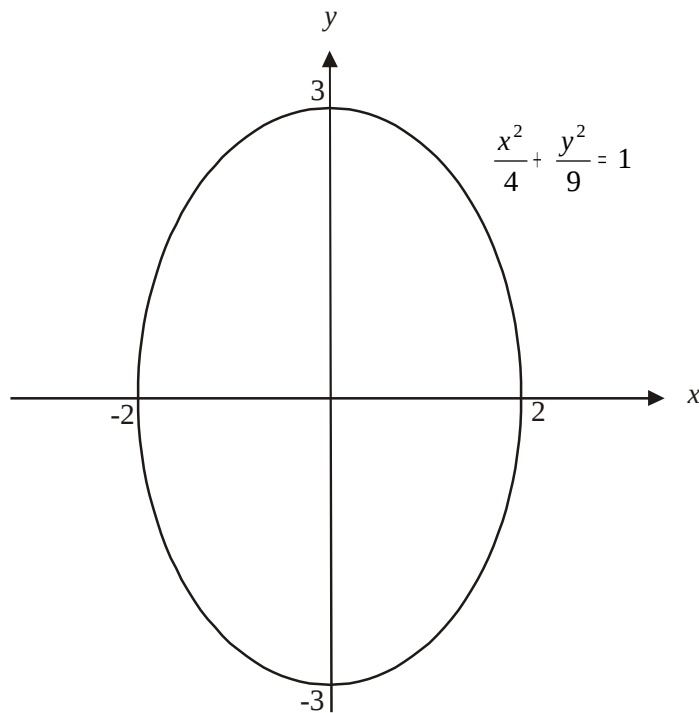
$$\frac{x^2}{4} = \cos^2(t) \quad \frac{y^2}{9} = \sin^2(t)$$

$$\frac{x^2}{4} + \frac{y^2}{9} = \cos^2(t) + \sin^2(t)$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

(1 mark)

b.



(1 mark) for correct shaped graph and intercepts

c. $\underline{r}(0) = 2\cos(0)\underline{i} + 3\sin(0)\underline{j}$

$$= 2\underline{i} + 0\underline{j}$$

The red light starts at the point (2,0) and moves in an anticlockwise direction around the ellipse.

(1 mark) correct starting point

(1 mark) correct direction

d. The period of the x-coordinate of motion is 2π and the period of the y-coordinate of motion is also 2π . So it takes 2π seconds to complete one circuit.

(1 mark)

e. Given $\underline{r}(t) = 2\cos(t)\underline{i} + 3\sin(t)\underline{j}$
 $\underline{v}(t) = -2\sin(t)\underline{i} + 3\cos(t)\underline{j}$ (1 mark)
 $|\underline{v}(t)| = \sqrt{(-2\sin(t))^2 + (3\cos(t))^2}$
 $|\underline{v}(t)| = \sqrt{4\sin^2(t) + 9\cos^2(t)}$

(1 mark)

f. i. Method 1 – “Hence”
 speed is a min/max when
 $-5\sin(t)\cos(t) = 0$ (Note $\sqrt{5\cos^2(t) + 4} \neq 0$)
 $\frac{-5}{2} \times 2\sin(t)\cos(t) = 0$
 $\frac{-5}{2} \times \sin(2t) = 0$
 $\sin(2t) = 0$ $0 < t \leq 2\pi$
 $2t = \pi, 2\pi, 3\pi, 4\pi$ $0 < 2t \leq 4\pi$
 $t = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

(1 mark)

speed = $\sqrt{4\sin^2(t) + 9\cos^2(t)}$
 when $t = \frac{\pi}{2}$, speed = $\sqrt{4 + 0} = 2$
 when $t = \pi$, speed = $\sqrt{0 + 9(-1)^2} = 3$
 when $t = \frac{3\pi}{2}$, speed = $\sqrt{4 \times (-1)^2 + 0} = 2$
 when $t = 2\pi$, speed = $\sqrt{0 + 9} = 3$
 Speed is a maximum when $t = \pi$ sec and $t = 2\pi$ sec.

(1 mark)

Method 2 – “otherwise”

$|\underline{v}(t)| = \sqrt{4\sin^2(t) + 9\cos^2(t)}$ (from part e.)
 $= \sqrt{4\sin^2(t) + 4\cos^2(t) + 5\cos^2(t)}$
 $= \sqrt{4 + 5\cos^2(t)}$ (1 mark)

So speed is a max. when $\cos(t) = \pm 1$.

$$t = \pi, 2\pi \text{ since } 0 < t \leq 2\pi$$

Speed is a maximum when $t = \pi$ sec and $t = 2\pi$ sec. (1 mark)

ii. From part i., maximum speed is $|\underline{v}(\pi)| = \sqrt{4\sin^2(\pi) + 9\cos^2(\pi)} = 3$ m/s.

(or $|\underline{v}(2\pi)| = \sqrt{4\sin^2(2\pi) + 9\cos^2(2\pi)} = 3$ m/s)

(1 mark)

Total 10 marks

Question 2

a.

$$z_1 = \sqrt{3} + i$$

$$r = \sqrt{3+1}$$

$$= 2$$

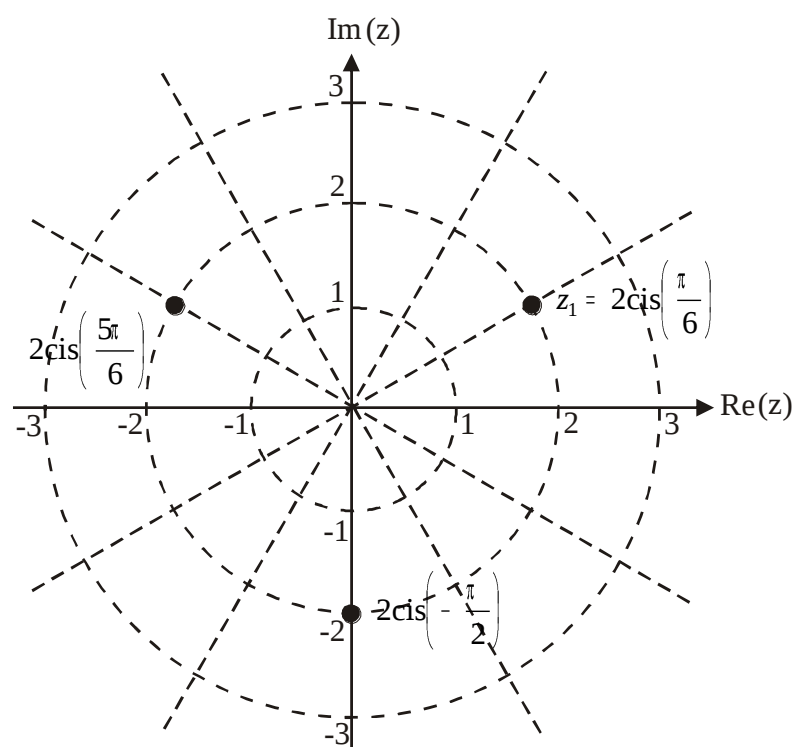
$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{6}$$

$$z_1 = 2\text{cis}\left(\frac{\pi}{6}\right)$$

(1 mark)

b.

(1 mark) - z_1 (1 mark) - $2\text{cis}\left(\frac{5\pi}{6}\right)$ (1 mark) - $2\text{cis}\left(\frac{-\pi}{2}\right)$

c. $z \bar{z} + |z_1| \times \operatorname{Re}(i^2 z) - 2\operatorname{Im}(z) = -1$

Now $i^2 z = -1(x + iy)$

$= -x - iy$

so $\operatorname{Re}(i^2 z) = -x$

Also $2\operatorname{Im}(z) = 2y$

So we have

$(x + iy)(x - iy) + 2 \times -x - 2y = -1$

(1 mark)

$x^2 + y^2 - 2x - 2y = -1$

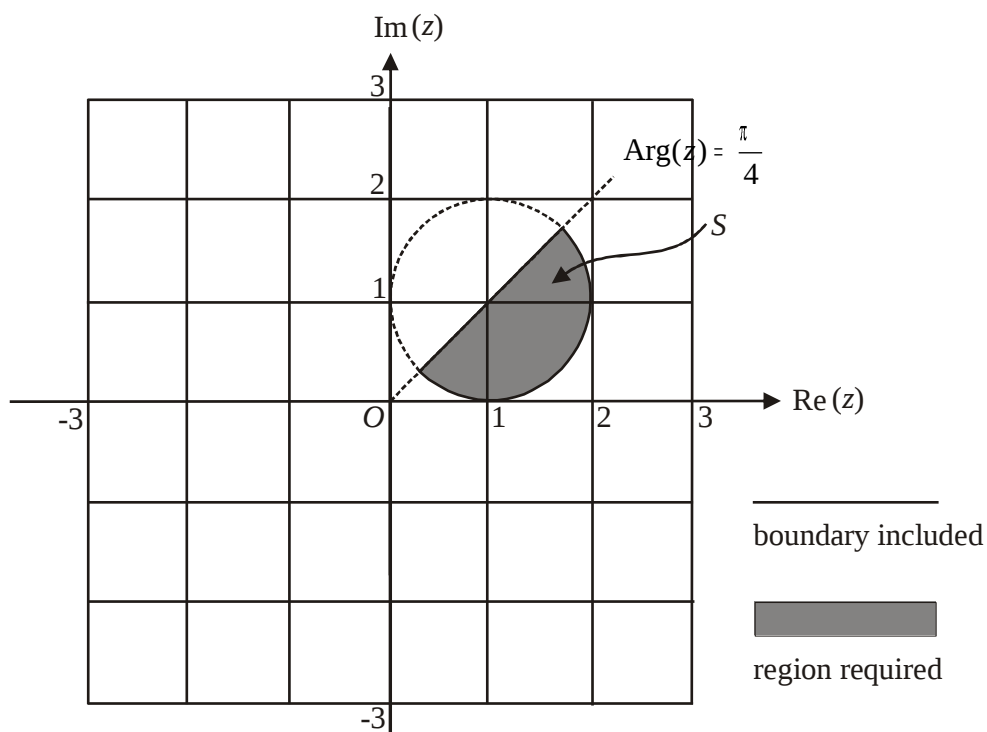
$x^2 - 2x + 1 + y^2 - 2y + 1 = -1 + 2$

$(x-1)^2 + (y-1)^2 = 1$

as required

(1 mark) – completing the squares

d.

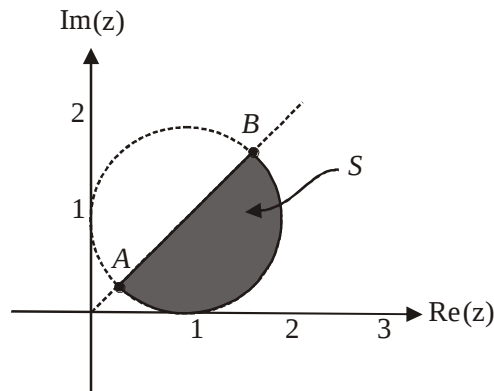


(1 mark) – correct circular boundary

(1 mark) – correct linear boundary

(1 mark) - correct boundary marking and shading

e. Method 1



At point A on the diagram above $|z_2|$ is a minimum and at point B, $|z_2|$ is a maximum.
Find the point of intersection of the Cartesian equations

$$(x-1)^2 + (y-1)^2 = 1 \text{ and } y = x$$

$$(x-1)^2 + (x-1)^2 = 1$$

$$2(x-1)^2 = 1$$

$$(x-1)^2 = \frac{1}{2}$$

$$x-1 = \pm \frac{1}{\sqrt{2}}$$

$$x = 1 \pm \frac{1}{\sqrt{2}}$$

(1 mark)

So A is point $\left(1 - \frac{1}{\sqrt{2}}, 1 - \frac{1}{\sqrt{2}}\right)$ since $y = x$

So minimum value of $|z_2|$ is

$$\sqrt{\left(1 - \frac{1}{\sqrt{2}}\right)^2 + \left(1 - \frac{1}{\sqrt{2}}\right)^2}$$

$$= \sqrt{2\left(1 - \frac{2}{\sqrt{2}} + \frac{1}{2}\right)}$$

$$= \sqrt{2\left(\frac{3}{2} - \sqrt{2}\right)}$$

$$= \sqrt{3 - 2\sqrt{2}}$$

(1 mark)

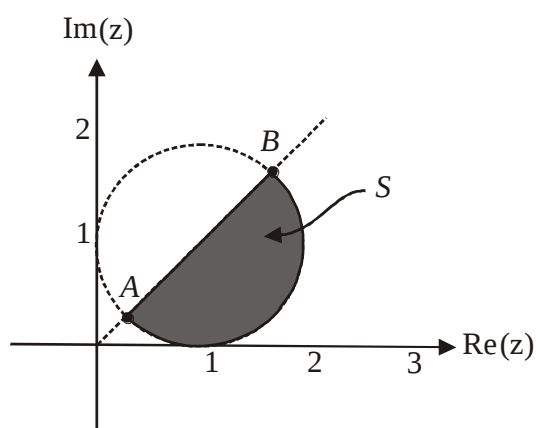
B is the point $\left(1 + \frac{1}{\sqrt{2}}, 1 + \frac{1}{\sqrt{2}}\right)$

So, similarly, maximum value of $|z_2|$ is

$$\sqrt{\left(1 + \frac{1}{\sqrt{2}}\right)^2 + \left(1 + \frac{1}{\sqrt{2}}\right)^2}$$

$$= \sqrt{3 + 2\sqrt{2}}$$

(1 mark)

Method 2

At point A on the diagram above $|z_2|$ is a minimum and at point B , $|z_2|$ is a maximum.
 The centre of the circle is a distance of $\sqrt{2}$ units from the origin so **(1 mark)**

A is $\sqrt{2} - 1$ units from $(0, 0)$ and so the minimum value of $|z_2|$ is $\sqrt{2} - 1$.
(1 mark)

Similarly, B is $\sqrt{2} + 1$ units from $(0, 0)$ and so the maximum value of $|z_2|$ is $\sqrt{2} + 1$.
(1 mark)

(Note that the answers obtained using Method 1 and Method 2 are equivalent since

$$(\sqrt{2} - 1)^2 = 2 - 2\sqrt{2} + 1 = 3 - 2\sqrt{2} \quad \text{and} \quad (\sqrt{2} + 1)^2 = 2 + 2\sqrt{2} + 1 = 3 + 2\sqrt{2}$$

So $\sqrt{2} - 1 = \sqrt{3 - 2\sqrt{2}} \quad \text{and} \quad \sqrt{2} + 1 = \sqrt{3 + 2\sqrt{2}} \quad)$

Total 12 marks

Question 3

a. $L - 700g = 700a$

Since $a = 0.5$,

$$L = 700 \times 0.5 + 700g$$

$$= 7210 \text{ newtons}$$

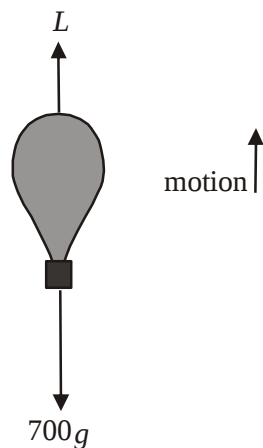
b. Since acceleration is constant, use

$$s = ut + \frac{1}{2}at^2$$

$$50 = 0 + \frac{1}{2} \times 0.5t^2$$

$$t^2 = 200$$

$$t = 10\sqrt{2} \text{ secs since } t \geq 0$$

**(1 mark)**

c. Method 1

$$v = u + at$$

$$= 0 + 0.5 \times 10\sqrt{2}$$

$$= 5\sqrt{2} \text{ m/s}$$

(1 mark)

Method 2

$$v^2 = u^2 + 2as$$

$$v^2 = 0 + 2 \times 0.5 \times 50$$

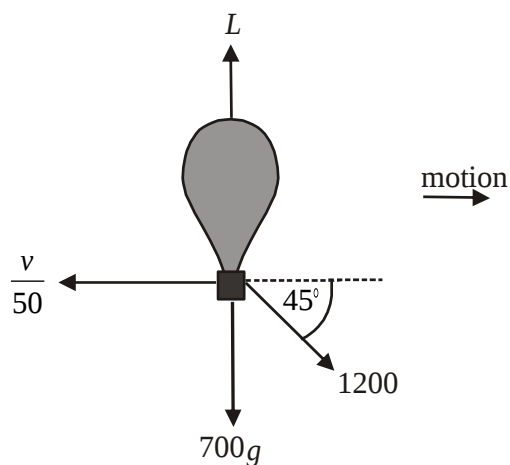
$$= 50$$

$$v = \sqrt{50}$$

$$= 5\sqrt{2} \text{ m/s}$$

(1 mark)

d.

**(1 mark)**

- e. The equation of motion is given by $\vec{R} = m \vec{a}$.

$$(1200 \cos(45^\circ) - \frac{v}{50}) \vec{i} + (L - 700g - 1200 \sin(45^\circ)) \vec{j} = 700a \vec{i}$$

(1 mark)

Resolving horizontally,

$$1200 \cos 45^\circ - \frac{v}{50} = 700a$$

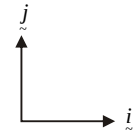
$$700a = \frac{1200}{\sqrt{2}} - \frac{v}{50}$$

$$700a = \frac{60000 - \sqrt{2}v}{50\sqrt{2}}$$

$$a = \frac{60000 - \sqrt{2}v}{35000\sqrt{2}}$$

$$= \frac{60000\sqrt{2} - 2v}{70000}$$

$$a = \frac{30000\sqrt{2} - v}{35000}$$



(1 mark)

- f. $a = \frac{30000\sqrt{2} - v}{35000}$ from part e.

$$v \frac{dv}{dx} = \frac{30000\sqrt{2} - v}{35000}$$

$$\frac{dv}{dx} = \frac{30000\sqrt{2} - v}{35000v}$$

$$\frac{dx}{dv} = \frac{35000v}{30000\sqrt{2} - v}$$

$$x = \int_0^5 \frac{35000v}{30000\sqrt{2} - v} dv$$

$$= 10.3128$$

(1 mark)

Distance covered by balloon is 10.31m (correct to 2 decimal places).

(1 mark)

- g. The distance travelled by the balloon is given by

$$x = \int_0^{15.5685} \frac{35000v}{30000\sqrt{2} - v} dv$$

= 100 (to the nearest whole number) as required

(1 mark)

h.From part **e.**,

$$a = \frac{30\,000\sqrt{2} - v}{35\,000}$$

$$\frac{dv}{dt} = \frac{30\,000\sqrt{2} - v}{35\,000}$$

$$\frac{dt}{dv} = \frac{35\,000}{30\,000\sqrt{2} - v}$$

$$t = 35\,000 \int_0^{15.5685} \frac{1}{30\,000\sqrt{2} - v} dv$$

$$= 12.85 \text{ secs (correct to 2 decimal places)}$$

(1 mark) – correct integrand**(1 mark)** – correct terminals**(1 mark)** – correct answer**Total 12 marks**

Question 4

- a. The left hand branch can be drawn using the direction field and passing through the point $(-1, -2)$.

$$\text{Now, } \frac{dy}{dx} = 2x - \frac{1}{x^2}$$

$$y = \int \left(2x - \frac{1}{x^2} \right) dx$$

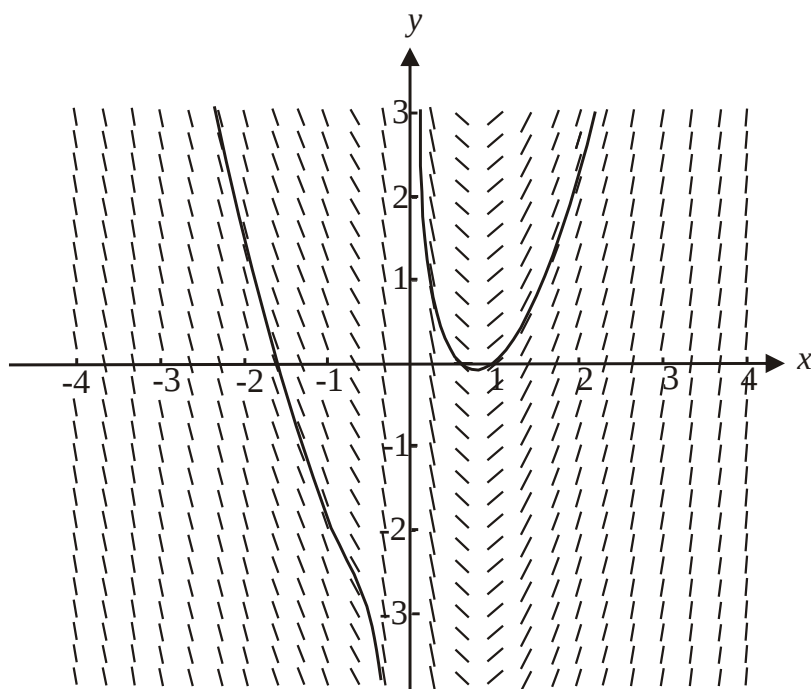
$$= x^2 + \frac{1}{x} + c$$

For this particular solution, $x = -1$ and $y = -2$

$$-2 = 1 - 1 + c \text{ so } c = -2$$

So this solution is $y = x^2 + \frac{1}{x} - 2$ which passes through the point $(1, 0)$ for example.

Use this point and the direction field to sketch the second branch.



(1 mark) – correct left branch

(1 mark) – correct value of c (correct right branch)

(1 mark) – correct right branch

b. From the formulae sheet,

If $\frac{dy}{dx} = f(x)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n)$

So $x_0 = -1$ and $y_0 = -2$

$$\begin{aligned} \text{and } x_1 &= -1 + 0.25 \text{ and } y_1 = -2 + 0.25 \times \left(2 \times -1 - \frac{1}{(-1)^2} \right) \\ &= -0.75 \qquad \qquad = -2 - 0.75 \\ & \qquad \qquad \qquad = -2.75 \end{aligned}$$

(1 mark)

$$\begin{aligned} \text{So } x_2 &= -0.75 + 0.25 \qquad y_2 = -2.75 + 0.25 \times \left(2 \times -0.75 - \frac{1}{(-0.75)^2} \right) \\ &= -0.5 \qquad \qquad = -3.57 \text{ (to 2 dec. places)} \end{aligned}$$

(1 mark)

c.

$$\begin{aligned} \frac{dy}{dx} &= 2x - \frac{1}{x^2} \\ \frac{d^2y}{dx^2} &= 2 + \frac{2}{x^3} \\ &= \frac{2x^3 + 2}{x^3} \end{aligned}$$

$$\frac{d^2y}{dx^2} = 0 \text{ when } 2x^3 + 2 = 0$$

$$x^3 = -1$$

$$x = -1$$

A point of inflection occurs when $\frac{d^2y}{dx^2} = 0$ AND $\frac{d^2y}{dx^2}$ changes sign on either side of $x = -1$.

$$\text{When } x = -2, \frac{d^2y}{dx^2} = \frac{-16 + 2}{-8} = \frac{7}{4} > 0$$

$$\text{When } x = -\frac{1}{2}, \frac{d^2y}{dx^2} = \frac{2 \times \frac{-1}{8} + 2}{-\frac{1}{8}} = -14 < 0$$

So a point of inflection occurs at $x = -1$.

(1 mark) – correctly giving $\frac{d^2y}{dx^2} = 0$

(1 mark) –for $x = -1$

(1 mark) showing the change of sign

- d. From the slope field for $x > 0$, we see a family of curves with a minimum between $x = 0$ and $x = 1$.

That minimum occurs when

$$\frac{dy}{dx} = 2x - \frac{1}{x^2} = 0$$

$$2x = \frac{1}{x^2}$$

$$x^3 = \frac{1}{2}$$

$$x = 2^{-\frac{1}{3}}$$

(1 mark)

Now, $y = x^2 + \frac{1}{x} + c$

When $x = 2^{-\frac{1}{3}}$,

$$y = 2^{-\frac{2}{3}} + 2^{\frac{1}{3}} + c$$

$$y = 1.88988... + c$$

The graph touches the x -axis when $y = 0$ so we require $c = -1.8899$ (correct to 4 decimal places).

(1 mark)

Total 10 marks

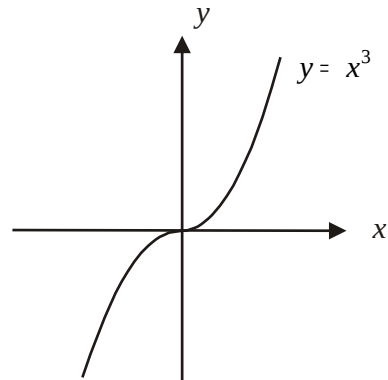
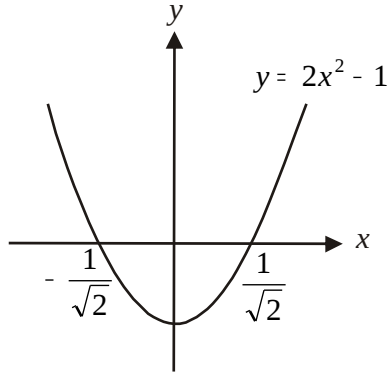
Question 5

a. $f(x) = \sqrt{\frac{x^3}{2x^2 - 1}}$

For a maximal domain, $\frac{x^3}{2x^2 - 1} > 0$.

$f(x)$ is defined when numerator and denominator are both positive or both negative.

Exclude where $2x^2 - 1 = 0$.



From the graphs, this occurs when $x \in \left(-\frac{1}{\sqrt{2}}, 0\right] \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$. (1 mark)

b. i. $V = \pi \int_1^{200} y^2 dx$

$$V = \pi \int_1^{200} \frac{x^3}{2x^2 - 1} dx \quad (1 \text{ mark})$$

ii. Let $u = 2x^2 - 1$ and so $x^2 = \frac{u+1}{2}$ $x=1$ so $u=1$ and $x=200$ so $u=79\,999$

$$\frac{du}{dx} = 4x$$

$$\begin{aligned} \text{so } V &= \pi \int_1^{79\,999} \frac{1}{u} \times \frac{1}{4} \frac{du}{dx} \times \frac{u+1}{2} dx \\ &= \frac{\pi}{8} \int_1^{79\,999} \left(1 + \frac{1}{u}\right) du \end{aligned}$$

(1 mark) – correct integrand

(1 mark) – correct terminals

iii. $V = \frac{\pi}{8} \int_1^{79\,999} \left(1 + \frac{1}{u}\right) du$

$$= \frac{\pi}{8} \left[u + \log_e |u| \right]_1^{79\,999} \quad (1 \text{ mark})$$

$$= \frac{\pi}{8} \{79\,999 + \log_e(79\,999) - (1 + \log_e(1))\}$$

$$= \frac{\pi}{8} (79\,998 + \log_e(79\,999)) \text{ m}^3 \quad (1 \text{ mark})$$

c. height = $200 - 1 = 199 \text{ m}$ (1 mark)

- d. At the base, $f(200) = 10.0000625$, so the diameter is 20m (to the nearest whole metre)

(1 mark)

e.
$$f'(x) = \frac{x^2(2x^2 - 3)}{2(2x^2 - 1)^2 \sqrt{\frac{x^3}{2x^2 - 1}}}$$

For min/max $f'(x) = 0$

$$x^2(2x^2 - 3) = 0$$

$$x = 0, x = \pm\sqrt{\frac{3}{2}}$$

The domain of f that describes a pylon is $x \in [1, 200]$ so $x = \sqrt{\frac{3}{2}}$

(1 mark)

From the graph given, this is consistent with a minimum point.

Now $f\left(\sqrt{\frac{3}{2}}\right) = 0.958415 \dots$

The minimum radius is therefore 0.96m (to 2 decimal places).

(1 mark)

- f. For a point of inflection $f''(x) = 0$

$$-x(4x^4 - 20x^2 - 3) = 0$$

$$x = 0 \text{ or } x = 2.26842 \dots$$

(1 mark)

Now $x = 0$ is outside the required domain of $x \in [1, 200]$.

The top of the pylon coincides with $x = 1$ so the lights are 1.27m (to 2 decimal places) below the top of the pylons.

(1 mark)

(Note that from the earlier given sketch it is evident that a point of inflection occurs around the point where $x = 2.27$. If asked to verify that a point is a point of inflection it needs to be shown that $\frac{d^2y}{dx^2} = 0$ AND that the sign of $\frac{d^2y}{dx^2}$ is different on either side of the point.)

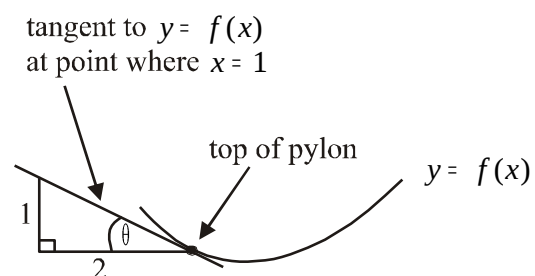
- g. At the top of the pylon $x = 1$.

Now $f'(1) = -\frac{1}{2}$ (1 mark)

Angle required = θ

$$\theta = \tan^{-1}\left(\frac{1}{2}\right)$$

$$\theta = 26.6^\circ \text{ (correct to 1 decimal place)}$$



(1 mark)

Total 14 marks