

Student Name: _____

SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 2



2008 Trial Examination

Reading Time: 15 minutes

Writing Time: 2 hours

QUESTION AND ANSWER BOOK

Structure of book

<i>Section</i>	<i>Number of questions</i>	<i>Number of questions to be answered</i>	<i>Number of marks</i>
1	22	22	22
2	5	5	58
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, an approved graphics calculator or a scientific calculator.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied

- Question book of 22 pages.

Instructions

- Print your name in the space provided on the top of this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other electronic devices into the examination room.

SECTION 1

Instructions for Section 1

Answer **all** questions.

Choose the response that is **correct** for the question.

A correct answer scores 1, an incorrect answer scores 0.

Marks are **not** deducted for incorrect answers.

If more than 1 answer is completed for any question, no mark will be given.

Take the **acceleration due to gravity**, to have magnitude $g \text{ m/s}^2$, where $g = 9.8$

Question 1

The asymptotes of the function $y = \frac{2x^3 - 7x^2 + 5x}{x^2 - 3x}$ are

- A. $x = 3$, $x = 0$ and $y = 2x$
- B. $x = 3$, $x = 0$ and $y = 2x - 1$
- C. $x = 3$, $x = 0$ and $y = 2$
- D. $x = 3$ and $y = 2x - 1$
- E. $x = 0$ and $y = 2x - 1$

Question 2

The equation of a hyperbola with its centre at $(-1, 2)$ and an asymptote $5x - 3y + 11 = 0$ is

- A. $\frac{(x-1)^2}{36} - \frac{(y+2)^2}{100} = 1$
- B. $\frac{(x-1)^2}{9} - \frac{(y+2)^2}{25} = 1$
- C. $\frac{(x+1)^2}{5} - \frac{(y-2)^2}{3} = 1$
- D. $\frac{(x+1)^2}{36} - \frac{(y-2)^2}{100} = 1$
- E. $\frac{(x+1)^2}{25} - \frac{(y-2)^2}{9} = 1$

Question 3

The graph of $y = \frac{-2}{x^2 - (m-1)x - m}$, $m > -1$, is positive when

- A. $x \in (m, \infty)$
- B. $x \in (-1, m)$
- C. $x \in (-\infty, m)$
- D. $x \in (-\infty, -1) \cup (m, \infty)$
- E. $x \in (m, 1)$

SECTION 1- continued

Question 4

If z and w are complex numbers such that $|w| = |3z|$ and $\text{Arg}(w) = \text{Arg}(z) + \frac{\pi}{2}$ where

$\text{Arg}(w) > 0, \text{Arg}(z) < \pi$, which one of the following must be true

- A. $w = 3zi$
- B. $w = 3\bar{z}$
- C. $\bar{w} = 3z$
- D. $\text{Arg}(w) + \text{Arg}(z) = \pi$
- E. $wi = 3z$

Question 5

If $z = 2\text{cis}\left(-\frac{2\pi}{3}\right)$ then z^{-2} is

- A. $-4\text{cis}\left(-\frac{2\pi}{3}\right)$
- B. $-4\text{cis}\left(\frac{4\pi}{3}\right)$
- C. $\frac{1}{4}\text{cis}\left(\frac{2\pi}{3}\right)$
- D. $\frac{1}{4}\text{cis}\left(-\frac{2\pi}{3}\right)$
- E. $\frac{1}{4}\text{cis}\left(\frac{4\pi^2}{9}\right)$

Question 6

For any $n \geq 3$, the complex roots of the equation $z^n = 1$ are vertices of a polygon with perimeter

- A. $2n$
- B. $n \sin \frac{2\pi}{n}$
- C. $2n \sin \frac{2\pi}{n}$
- D. $n \sin \frac{\pi}{n}$
- E. $2n \sin \frac{\pi}{n}$

SECTION 1- continued
TURN OVER

Question 7

For the curve with equation $y = x^2 e^{-x}$, the values of x for which $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} < 0$ are

- A. $(0, 2 - \sqrt{2})$
- B. $(2 - \sqrt{2}, 2 + \sqrt{2})$
- C. $(2 - \sqrt{2}, 2)$
- D. $(2, 2 + \sqrt{2})$
- E. $(2 + \sqrt{2}, 2)$

Question 8

The graph of $f(x) = a \sin^{-1}\left(x - \frac{1}{2}\right) + b$ passes through the points $\left(0, -\frac{\pi}{12}\right)$ and $\left(1, \frac{7\pi}{12}\right)$. The values of a and b are

- A. $a = 2, b = \frac{\pi}{4}$
- B. $a = \frac{-2}{3}, b = \frac{25\pi}{436}$
- C. $a = -2, b = \frac{\pi}{4}$
- D. $a = 2, b = -\frac{\pi}{4}$
- E. $a = \frac{-2}{3}, b = \frac{\pi}{4}$

Question 9

If the scalar resolute of $\mathbf{a} = 2\mathbf{i} + x\mathbf{j} - \mathbf{k}$ in the direction of $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ is 2, then x is equal to

- A. 0
- B. $\sqrt{3}$
- C. 3
- D. $3\sqrt{3}$
- E. 9

SECTION 1- continued

Question 10

If $\mathbf{m} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{n} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and θ is the angle between the direction of $\mathbf{m}$ and $\mathbf{n}$, the exact value of $\cos 2\theta$ is

- A. $-\frac{1}{6}$
 B. $\frac{1}{6}$
 C. $\frac{17}{18}$
 D. $-\frac{17}{18}$
 E. 3

Question 11

A solution of $\int x(2x-3)^3 dx$ is

- A. $\frac{u^5}{5} + \frac{u^4}{4}$
 B. $\frac{(2x-3)^5}{20} + \frac{3(2x-3)^4}{16}$
 C. $\frac{u^5}{20} + \frac{u^4}{16}$
 D. $\frac{(2x-3)^5}{10} + \frac{(2x-3)^4}{8}$
 E. none of these

Question 12

A tank initially contains 100 litres of solution in which 4 kg of salt is dissolved. A solution containing 5 kg of salt per litre is added at the rate of 8 litres per minute. The mixture is drained simultaneously at a rate of 10 litres per minute. There are x kg of salt in the tank after t minutes. This can be described by a differential equation

- A. $\frac{dx}{dt} = 40 - \frac{10x}{100+2t}$
 B. $\frac{dx}{dt} = 100 - \frac{5x}{50-t}$
 C. $\frac{dx}{dt} = 40 - \frac{5x}{50-t}$
 D. $\frac{dx}{dt} = 20 - \frac{5x}{50-t}$
 E. $\frac{dx}{dt} = 40 + \frac{5x}{50+t}$

SECTION 1- continued
TURN OVER

Question 13

A spherical balloon is inflated at a rate $\frac{\pi}{3} \text{ cm}^3$ per minute. The rate at which its surface area is increasing when the radius is 5 cm is

- A. $\frac{\pi^2}{9}$
- B. $\frac{2\pi}{15}$
- C. $\frac{15}{2\pi}$
- D. $\frac{3}{\pi}$
- E. $\frac{9}{\pi^2}$

Question 14

An approximation to the solution of the differential equation $\frac{dy}{dx} = y - x^2$ with $y(4) = 1$ is found using Euler's method with $h = 0.04$. When $x = 4.12$, the value for y is closest to

- A. -0.9122
- B. -0.2386
- C. -0.8435
- D. -1.6277
- E. 0.9984

Question 15

A solution of $\int \sin 2x \sec^2 2x \, dx$ can be written as $a \sec 2x$. The value of a is

- A. $\frac{1}{2}$
- B. 2
- C. $-\frac{1}{2}$
- D. -2
- E. none of these

SECTION 1- continued

Question 16

$$\int_0^2 \frac{x^2}{4+x^2} dx =$$

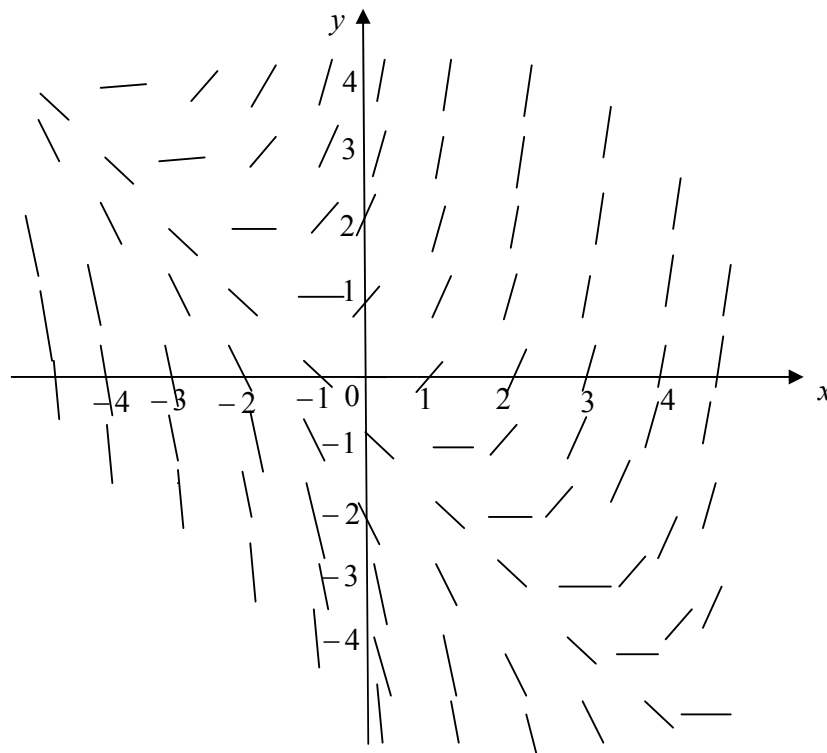
- A. $2 - \ln 2$
- B. $\ln 2 - \frac{1}{2}$
- C. $2 \tan^2 2 + 4 \ln(\cos 2)$
- D. $2 \ln(\sec 2) - \sin 2$
- E. $2 - \frac{\pi}{2}$

Question 17

The velocity, $v \text{ ms}^{-1}$, travelling in a straight road is given by $v = 4 - x^2$ where x is the position of the body at time t seconds. The acceleration of the body is equal to

- A. $-2x$
- B. $(4 - x^2)t$
- C. $-2xt$
- D. $-2x^3$
- E. $-8x + 2x^3$

SECTION 1- continued
TURN OVER

Question 18

The family of solutions of a first order differential equation is shown above. The differential equation could be

- A. $\frac{dy}{dx} = y - x$
- B. $\frac{dy}{dx} = -x^2 + y$
- C. $\frac{dy}{dx} = -\frac{x}{y}$
- D. $\frac{dy}{dx} = x + y$
- E. $\frac{dy}{dx} = -x$

Question 19

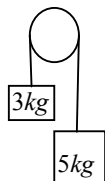
A car of mass 1 tonne, travelling at $\frac{50}{3} \text{ m/s}$ on a level road, has its speed reduced to $\frac{20}{3} \text{ m/s}$ in 5 seconds when the brakes are applied. The total retarding force (assumed constant) is

- A. 7200N
- B. 200N
- C. 2000N
- D. 720N
- E. 1000N

SECTION 1- continued

Question 20

Masses of 3kg and 5kg are hanging at the ends of a light string that passes over a smooth fixed peg as shown in the diagram.



The tension in the string is

- A. $2g$
- B. $\frac{15g}{4}$
- C. $8g$
- D. $\frac{g}{4}$
- E. g

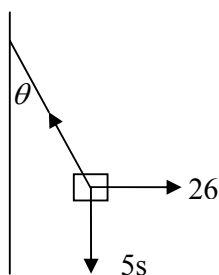
Question 21

A ball is thrown vertically upwards with an initial speed of 29.4 ms^{-1} from the top of a building 20 m high. Neglecting air resistance, the ball is above the top of the building for

- A. 6 seconds
- B. 5 seconds
- C. 4 seconds
- D. 3 seconds
- E. 2 seconds

Question 22

A particle of mass 5kg hangs at the end of a string attached to a fixed point. The particle is held at rest by a horizontal force of magnitude 26N so that the string makes an angle θ° with the vertical as shown in the diagram.



The angle θ is closest to

- A. 89°
- B. 62°
- C. 28°
- D. 11°
- E. 45°

END OF SECTION 1
TURN OVER

SECTION 2

Instructions for Section 2

Answer **all** questions.

A decimal approximation will not be accepted if the question specifically asks for an **exact** answer is required.

For questions worth more than one mark, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams are **not** drawn to scale.

Take the **acceleration due to gravity**, to have magnitude $g \text{ m/s}^2$, where $g = 9.8$

Question 1

Let $p(z) = z^4 - 9z^3 + az^2 - 16z - 48$, where a is a real number. The complex number w is a solution of the equation $p(z) = 0$ such that $w + \bar{w} = 4$ and $w\bar{w} = 8$.

- a. Using this information, write down a quadratic factor of $p(z)$.

1 mark

- b. The other quadratic factor of $p(z)$ is $z^2 - 5z - 6$. Find the value of a .

2 marks

SECTION 2- Question 1- continued

c. Show that $w = 2 + 2i$.

2 marks

d. Find, in polar form

i. w .

ii. $\overline{w}^5$.

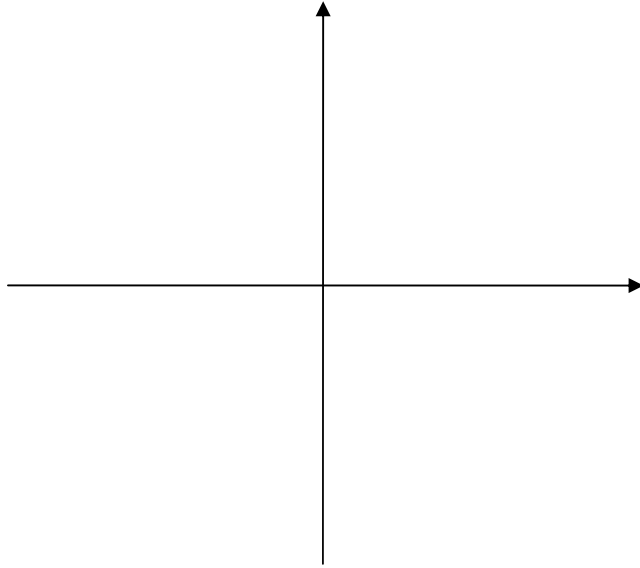
iii. $\sqrt[3]{w}$.

1+1+2 = 4 marks

SECTION 2- Question 1-continued

TURN OVER

- e. Let S and T be subsets of the complex plane where $S = \{z : |z - w| \leq |w|\}$ and $T = \{z : 0 < \text{Arg} z \leq \text{Arg} w\}$. Sketch S and T and shade $S \cap T$.



3 marks
Total 12 marks

SECTION 2- continued

Question 2

- a.** Use integration by substitution to find the indefinite integral $\int x^{n-1} e^{-x^n} dx$, where n is a positive integer.

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3 marks

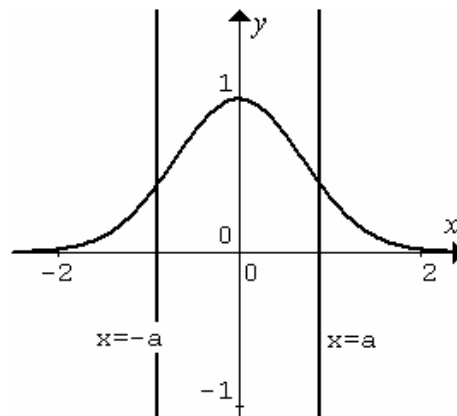
- b. Evaluate the definite integral $\int_0^2 x e^{-x^2} dx$ giving your answer in exact form.

2 marks

SECTION 2- Question 2- continued
TURN OVER

c. The graph of $y = e^{-x^2}$ is shown below.

- i. Shade the region bounded by the curve $y = e^{-x^2}$ and the lines $x = a$, $x = -a$ and $y = 0$.



- ii. Write down an expression which gives the volume of the solid obtained by revolving the shaded region about the y -axis. Do **not** evaluate the volume at this stage.

1 + 2 = 3 marks

SECTION 2- Question 2-continued

- d. Given that $\int \log_e y dy = y \log_e y - y + c$, find the volume of this solid of revolution.

3 marks

- e. What is the limiting value of the volume when $a \rightarrow \infty$?

1 mark

Total 12 marks

SECTION 2- continued
TURN OVER

Question 3

The nautilus is a marine creature that lives around coral reefs (*Figure 1*).

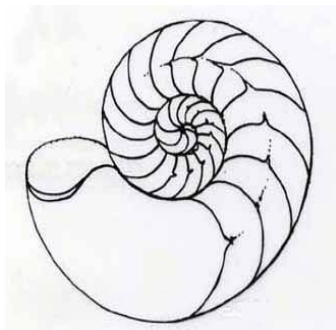


Figure 1

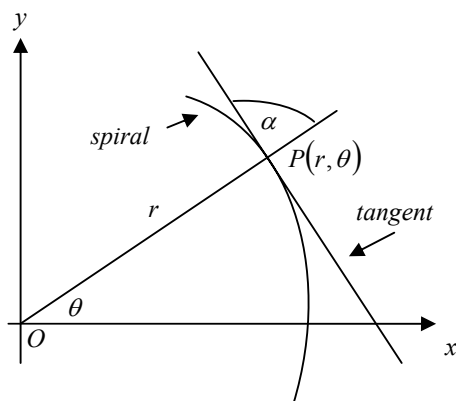


Figure 2

The mathematical model of a nautilus shell is an equiangular spiral (*Figure 2*). The equations of equiangular spirals are of the form $r = Ae^{k\theta}$, where k is a constant. At every point P , the tangent to the spiral makes the same angle, α , with the line OP . The size of the angle α depends on the constant k .

The line OP makes an angle θ with the positive part of the x -axis.

- a. Show that the Cartesian coordinates of the point $P(x, y)$ are

$$x = Ae^{k\theta} \cos \theta, \quad y = Ae^{k\theta} \sin \theta.$$

2 marks

SECTION 2- Question 3- continued

- b. Show that $\frac{dy}{dx} = \frac{k \tan \theta + 1}{k - \tan \theta}$.

4 marks

- c. Explain why the gradient of the tangent to the spiral can be written as $\tan(\alpha + \theta)$.

1 mark

SECTION 2- Question 3- continued
TURN OVER

- d. Hence, or otherwise, find the value of α in terms of k .

3 marks
Total 10 marks

SECTION 2- continued

Question 4

A ship has the position vector $\mathbf{r}_s = 0\mathbf{i} + 0\mathbf{j}$. A missile is spotted at the position $\mathbf{r}_m = 1000\mathbf{i} + 500\mathbf{j}$ moving towards a ship with velocity $\mathbf{v}_m = -30\mathbf{i} + 3\mathbf{j}$. Assume that gravity is the only force acting on the missile.

- a. Show that the position vector of the missile at time t is given by

$$\mathbf{r}_m(t) = (-30t + 1000)\mathbf{i} + (-4.9t^2 + 3t + 500)\mathbf{j}$$

3 marks

An anti-missile can be fired from the ship with a velocity of 100ms^{-1} at an angle θ° to the horizontal. Assume that gravity is the only force acting on the anti-missile

- b. Show that the position vector of the anti-missile at time t is given by

$$\mathbf{r}_a(t) = 100t \cos \theta \mathbf{i} + (-4.9t^2 + 100t \sin \theta)\mathbf{j}$$

3 marks

SECTION 2- Question 4 –continued
TURN OVER

- c. Show that the anti-missile must be fired at an angle of 35.8° in order to intercept the missile.

[illegible]

5 marks

- d.** How many seconds after firing will the missile be intercepted?

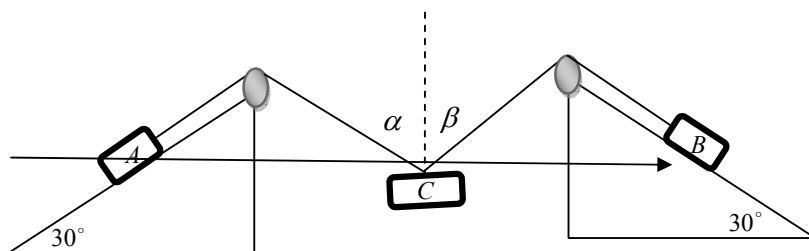
1 mark

Total 12 marks

SECTION 2- continued

Question 5

The diagram shows particles A and B each lying on smooth planes of inclination 30° to the horizontal. A and B are attached to inextensible strings passing over smooth pulleys and are connected to a third particle C hanging freely. The strings make angles of α and β with the vertical as shown. The particles A, B and C have masses respectively $2M$, $1.5M$ and $3M$. The system rests in equilibrium.



- a. On the diagram above, show all forces acting on these three particles.

2 marks

- b. Express the tensions in the strings in terms of M .

2 marks

SECTION 2- Question 5 –continued
TURN OVER

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d. Hence, find the values of α and β accurate to the nearest degree.

Total 12 marks

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