

SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 2



2008 Trial Examination

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: D

Explanation:

The asymptotes of the function $y = \frac{2x^3 - 7x^2 + 5x}{x^2 - 3x}$ can be found by simplifying first, then dividing the polynomials.

$$y = \frac{x(2x^2 - 7x + 5)}{x(x-3)} = \frac{2x^2 - 7x + 5}{x-3} = 2x - 1 + \frac{2}{x-3}$$

The function has a hole at $x = 0$

Question 2

Answer: D

Explanation:

The equation of a hyperbola with its centre at $(-1, 2)$ is $\frac{(x+1)^2}{a^2} - \frac{(y-2)^2}{b^2} = 1$. The equation of its asymptote is $5x - 3y + 11 = 0$, or $y = \frac{5}{3}x + \frac{11}{3}$. It follows that $\frac{b}{a} = \frac{5}{3}$, therefore $a = \frac{3b}{5}$.

From the given alternatives, it has to be $a = 6, b = 10$

Question 3*Answer:* B*Explanation:*

For the graph of $y = \frac{-2}{x^2 - (m-1)x - m}$, $m > -1$ to be positive, it is necessary that

$x^2 - (m-1)x - m$ is negative.

$$x^2 - (m-1)x - m < 0$$

$$x^2 - mx + x - m < 0$$

$$x(x-m) + (x-m) < 0$$

$$(x-m)(x+1) < 0$$

$$x \in (-1, m)$$

Question 4*Answer:* A*Explanation:*

Rotating a complex number anticlockwise for an angle of $\frac{\pi}{2}$ gives the same result as

multiplying the number by i . Because $|w| = |3z|$, it must be $w = 3zi$.

Proof:

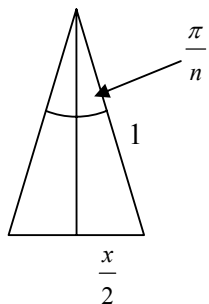
$$\begin{aligned} \text{Let } z &= |z| \operatorname{cis} \varphi. \text{ Then } w = 3|z| \operatorname{cis} \left(\varphi + \frac{\pi}{2} \right) \\ &= 3|z| \left(\cos \left(\varphi + \frac{\pi}{2} \right) + i \sin \left(\varphi + \frac{\pi}{2} \right) \right) = 3|z| (-\sin \varphi + i \cos \varphi) \\ &= 3|z| (i^2 \sin \varphi + i \cos \varphi) = 3|z| (\cos \varphi + i \sin \varphi) i = 3zi \end{aligned}$$

Question 5*Answer:* D*Explanation:*

$$\text{If } z = 2 \operatorname{cis} \left(-\frac{2\pi}{3} \right) \text{ then } z^{-2} = 2^{-2} \operatorname{cis} \left(\frac{4\pi}{3} \right) = \frac{1}{4} \operatorname{cis} \left(-\frac{2\pi}{3} \right)$$

Question 6*Answer:* E*Explanation:*

If $z^n = 1$, then the complex roots $z = \text{cis} \frac{2k\pi}{n}$ are symmetrically placed around the circle of radius 1. The regular polygon obtained from these vertices can be divided into n congruent isosceles triangles with side x and the top angle of $\frac{2\pi}{n}$.



From the diagram, $\sin \frac{\pi}{n} = \frac{\frac{x}{2}}{1}$, $x = 2 \sin \frac{\pi}{n}$. The perimeter is

$$nx = 2n \sin \frac{\pi}{n}$$

Question 7*Answer:* C*Explanation:*

For $y = x^2 e^{-x}$, $\frac{dy}{dx} = 2xe^{-x} - x^2 e^{-x} = e^{-x}(2x - x^2)$ and $\frac{d^2y}{dx^2} = -e^{-x}(2x - x^2) + e^{-x}(2 - 2x) = e^{-x}(x^2 - 4x + 2)$

$\frac{dy}{dx} > 0$ for $2x - x^2 > 0$ which is $x \in (0, 2)$

$\frac{d^2y}{dx^2} < 0$ for $x^2 - 4x + 2 < 0$ which is $x \in (2 - \sqrt{2}, 2 + \sqrt{2})$. The intersection is $x \in (2 - \sqrt{2}, 2)$

Question 8*Answer:* A*Explanation:*

By substituting points:

$$a \sin^{-1}\left(-\frac{1}{2}\right) + b = -\frac{\pi}{12} \quad \text{and} \quad a \sin^{-1}\left(\frac{1}{2}\right) + b = \frac{7\pi}{12}$$

$$-\frac{a\pi}{6} + b = -\frac{\pi}{12} \quad (1)$$

$$\frac{a\pi}{6} + b = \frac{7\pi}{12} \quad (2)$$

Adding equations (1) and (2) gives $2b = \frac{6\pi}{12}$, $b = \frac{\pi}{4}$.Substitute back into (2) $\frac{a\pi}{6} + \frac{\pi}{4} = \frac{7\pi}{12}$, $a = 2$ **Question 9***Answer:* C*Explanation:*The scalar resolute of $\mathbf{a} = 2\mathbf{i} + x\mathbf{j} - \mathbf{k}$ in the direction of $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ is

$$\begin{aligned} \mathbf{a} \cdot \hat{\mathbf{b}} &= \frac{1}{3}(2\mathbf{i} + x\mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) \\ &= \frac{1}{3}(2 + 2x - 2) \end{aligned}$$

From $\frac{1}{3}(2 + 2x - 2) = 2$ the value of x is 3.**Question 10***Answer:* D*Explanation:*

$$\cos \theta = \frac{\mathbf{m} \cdot \mathbf{n}}{|\mathbf{m}||\mathbf{n}|} = \frac{2 - 2 - 1}{\sqrt{6}\sqrt{6}} = -\frac{1}{6}$$

$$\cos 2\theta = 2\cos^2 \theta - 1 = \frac{2}{36} - 1 = -\frac{17}{18}$$

Question 11*Answer:* B*Explanation:*

Let $u = 2x - 3$. Then $\frac{du}{dx} = 2$ and $x = \frac{u+3}{2}$

$$\begin{aligned}\int x(2x-3)^3 dx &= \frac{1}{2} \int \frac{u+3}{2} u^3 du = \frac{1}{4} \int (u^4 + 3u^3) du \\ &= \frac{(2x-3)^5}{20} + \frac{3(2x-3)^4}{16}\end{aligned}$$

Question 12*Answer:* C*Explanation:*

$$\begin{aligned}\frac{dx}{dt} &= 5 \times 8 - \frac{10x}{100-2t} \\ &= 40 - \frac{5x}{50-t}\end{aligned}$$

Question 13*Answer:* B*Explanation:*

We are given $\frac{dV}{dt} = \frac{\pi}{3}$ and need to find $\frac{dS}{dt}$ when $r = 5$, where S is the surface area of the balloon.

Using the chain rule $\frac{dS}{dt} = \frac{dS}{dr} \frac{dr}{dV} \frac{dV}{dt}$ and $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$, we have

$$\begin{aligned}\frac{dV}{dr} &= 4\pi r^2 \text{ and } \frac{dS}{dr} = 8\pi r \\ \frac{dS}{dt} &= 8\pi r \times \frac{1}{4\pi r^2} \times \frac{\pi}{3} = \frac{2\pi}{3r}. \text{ For } r = 5, \frac{dS}{dt} = \frac{2\pi}{15}\end{aligned}$$

Question 14*Answer:* A*Explanation:*

$$x_{n+1} = x_n + h, \quad y_{n+1} = y_n + hf(x_n, y_n), \quad x_0 = 4, y_0 = 1, h = 0.04, \quad f(x, y) = y - x^2$$

$$x_1 = 4.04, y_1 = 1 + 0.04(1 - 4^2) = 0.4$$

$$x_2 = 4.08, y_2 = 0.4 + 0.04(0.4 - 4.04^2) = -0.23864$$

$$x_3 = 4.12, y_3 = -0.23864 + 0.04(-0.23864 - 4.08^2) = -0.9122$$

Question 15*Answer:* A*Explanation:*

$$\int \sin 2x \sec^2 2x \, dx = \int \frac{\sin 2x}{\cos^2 2x} \, dx$$

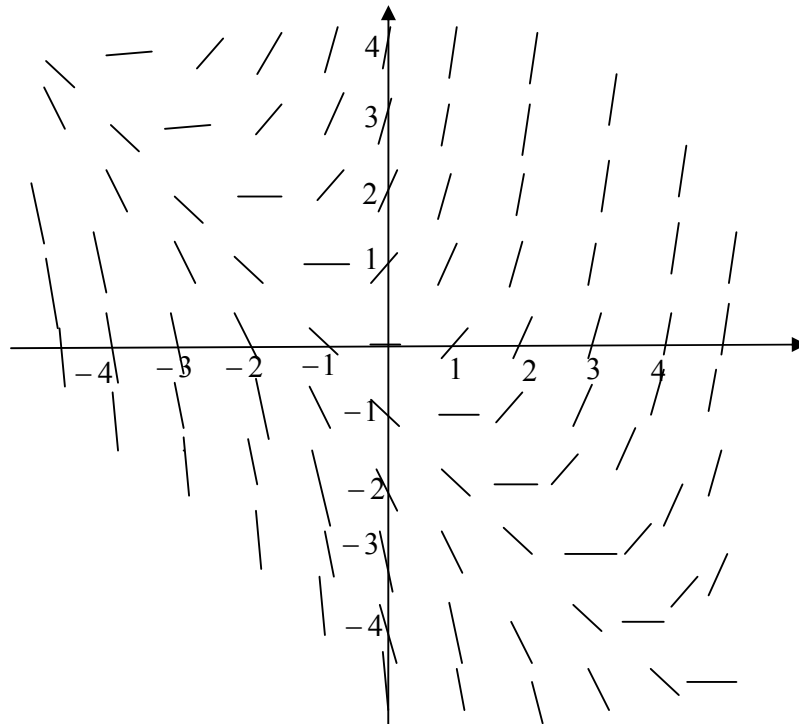
$$\begin{aligned} \text{Using substitution } u = \cos 2x, \quad \int \frac{\sin 2x}{\cos^2 2x} \, dx &= -\frac{1}{2} \int \frac{1}{u^2} \, du = -\frac{1}{2} \times -\frac{1}{u} \\ &= \frac{1}{2 \cos 2x} = \frac{1}{2} \sec 2x \end{aligned}$$

Question 16*Answer:* E*Explanation:*

$$\int_0^2 \frac{x^2}{4+x^2} \, dx = \int_0^2 \left(1 - \frac{4}{4+x^2} \right) \, dx = \left[x - 2 \tan^{-1} \frac{x}{2} \right]_0^2 = 2 - \frac{\pi}{2}$$

Question 17*Answer:* E*Explanation:*

$$\begin{aligned} a &= v \frac{dv}{dx} = (4 - x^2)(-2x) \\ &= -8x + 2x^3 \end{aligned}$$

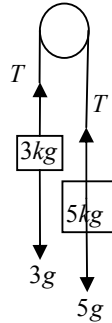
Question 18*Answer:* D*Explanation:*

From the direction field, $\frac{dy}{dx} = 0$ at $(1, -1), (2, -2), (3, -3), \dots$ which is correct only for the differential equation $\frac{dy}{dx} = x + y$.

Question 19*Answer:* C*Explanation:*

$$a = \frac{v_2 - v_1}{t} = \frac{-10}{5} = -2 \text{ ms}^{-2}$$

$$m = 1000 \text{ kg}, F = 1000 \times 2 = 2000 \text{ N}$$

Question 20*Answer:* B*Explanation:*

$$T - 3g = 3a$$

$$-T + 5g = 5a$$

By adding the equations $2g = 8a$, $a = \frac{g}{4}$. Substituting back into $T - 3g = 3a$ gives

$$T = 3g + 3a = \frac{15g}{4}$$

Question 21*Answer:* A*Explanation:*

$$a = -g$$

$$v = -gt + 29.4$$

$$s = -4.9t^2 + 29.4t + 20$$

$$s = 20 \text{ when } -4.9t^2 + 29.4t = 0$$

$$t = 0, t = 6$$

The ball is above the top of the building for 6 seconds.

OR

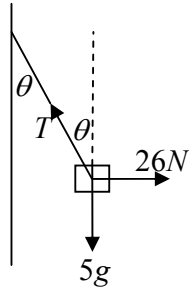
As the ball passes the top of the building on its way down its velocity will be equal and opposite to its initial velocity i.e. -29.4 ms^{-1}

$$a = -g, u = 29.4, v = -29.4$$

$$v = u + at$$

$$t = \frac{v - u}{a} = \frac{-29.4 - 29.4}{-9.8} = 6$$

The ball is above the top of the building for 6 seconds

Question 22*Answer: C**Explanation:*

$$T \cos \theta - 5g = 0 \Rightarrow T \cos \theta = 5g \quad (1)$$

$$T \sin \theta - 26 = 0 \Rightarrow T \sin \theta = 26 \quad (2)$$

Dividing equation (2) by equation (1)

$$\frac{\sin \theta}{\cos \theta} = \frac{26}{5g}, \theta = \tan^{-1} \left(\frac{26}{5g} \right) = 27.95^\circ.$$

SECTION 2

Question 1

- a. If w is a solution of $p(z)$ then $\bar{w}$ is also a solution. Therefore $z - w$ and $z - \bar{w}$ are factors of $p(z)$. Thus $(z - w)(z - \bar{w})$ is a quadratic factor of $p(z)$.

$$(z - w)(z - \bar{w}) = z^2 - (w + \bar{w})z + w\bar{w} = z^2 - 4z + 8$$

Therefore $z^2 - 4z + 8$ is a quadratic factor of $p(z)$.

This could also be done by first solving $w + \bar{w} = 4$ and $w\bar{w} = 8$ simultaneously.

A1

b. $(z^2 - 5z - 6)(z^2 - 4z + 8) = z^4 - 9z^3 + az^2 - 16z - 48 \therefore$

M1

By equating quadratic terms, $8 - 6 + 20 = a$, $a = 22$

A1

- c. Solving $z^2 - 4z + 8 = 0$ gives $z = \frac{4 \pm \sqrt{-16}}{2} = 2 \pm 2i$. Therefore, $w = 2 + 2i$.

This could also be done by solving $w + \bar{w} = 4$ and $w\bar{w} = 8$ simultaneously.

M1 + A1

- d. Find, in polar form

i. $|w| = \sqrt{4 + 4} = 2\sqrt{2}$, $\text{Arg} w = \frac{\pi}{4}$ $w = 2\sqrt{2} \text{cis} \frac{\pi}{4}$

A1

ii. $\bar{w}^5 = \left(2\sqrt{2} \text{cis} \left(-\frac{\pi}{4} \right) \right)^5 = 128\sqrt{2} \text{cis} \frac{3\pi}{4}$

A1

iii. $\sqrt[3]{w} = \left(2\sqrt{2} \right)^{\frac{1}{3}} \text{cis} \frac{\frac{\pi}{4} + 2k\pi}{3}, k = -1, 0, 1 = \sqrt{2} \text{cis} \left(-\frac{7\pi}{12} \right), \sqrt{2} \text{cis} \frac{\pi}{12}, \sqrt{2} \text{cis} \frac{3\pi}{4}$

M1 + A1

e. $S = \{z : |z - (2 + 2i)| \leq 2\sqrt{2}\}$

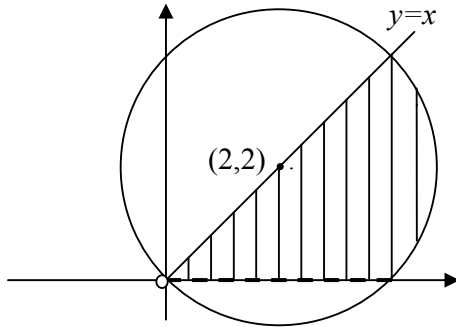
Correct circle drawn

A1

$$T = \left\{ z : 0 < \text{Arg} z \leq \frac{\pi}{4} \right\}$$

Correct lines shown with a dotted line along the x -axis

A1



Correct region shaded with the origin excluded

A1

Total 12 marks

Question 2

- a. Let $u = x^n$. Then $\frac{du}{dx} = nx^{n-1}$

M1

$$\int x^{n-1} e^{-x^n} dx = \frac{1}{n} \int e^{-u} du = -\frac{1}{n} e^{-u} + c = -\frac{1}{n} e^{-x^n} + c$$

M1 + A1

- b. Using the result from a. or by substitution $\int_0^2 x e^{-x^2} dx = \left[-\frac{1}{2} e^{-x^2} \right]_0^2$

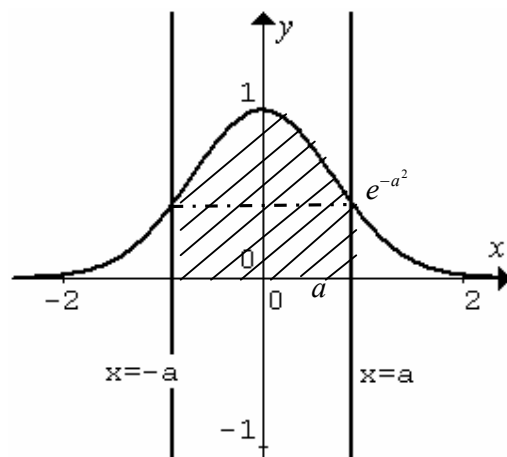
A1

$$= -\frac{1}{2} e^{-4} + \frac{1}{2} = \frac{1 - e^{-4}}{2}$$

A1

c.

i.



Correct shading

A1

- ii. The volume of this solid of revolution consists of a cylinder of radius a and height e^{-a^2} and a solid obtained by revolving part of the curve from e^{-a^2} to 1 about the y -axis.

$$V = a^2 \pi e^{-a^2} + \pi \int_{e^{-a^2}}^1 x^2 dy.$$

M1

From $y = e^{-x^2}$, $x^2 = -\log_e y$. Therefore $V = a^2 \pi e^{-a^2} - \pi \int_{e^{-a^2}}^1 \log_e y dy$

A1

d. $V = a^2 \pi e^{-a^2} - \pi [y \log_e y - y]_{e^{-a^2}}^1$

M1

$$= a^2 \pi e^{-a^2} - \pi [-1 - e^{-a^2} \log_e e^{-a^2} + e^{-a^2}]$$

M1

$$\begin{aligned} &= a^2 \pi e^{-a^2} + \pi - a^2 \pi e^{-a^2} - \pi e^{-a^2} \\ &= \pi(1 - e^{-a^2}) \end{aligned}$$

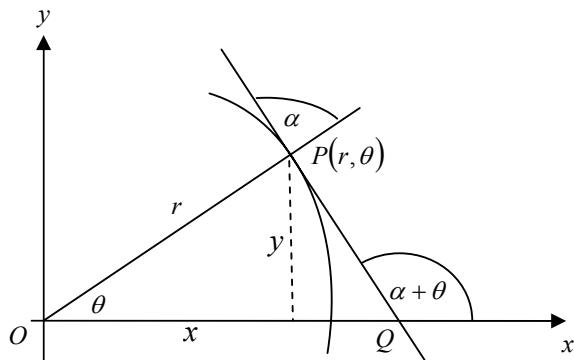
A1

- e. When $a \rightarrow \infty$, $e^{-a^2} = \frac{1}{e^{a^2}} \rightarrow 0$ and so the volume approaches π .

A1

Total 12 marks

Question 3



- a. From $\cos \theta = \frac{x}{r}$, $\sin \theta = \frac{y}{r}$, $x = r \cos \theta$, $y = r \sin \theta$. Given that $r = Ae^{k\theta}$, it follows
 $x = Ae^{k\theta} \cos \theta$, $y = Ae^{k\theta} \sin \theta$.

M1 + A1

b. $\frac{dy}{dx} = \frac{dy}{d\theta} \frac{d\theta}{dx}$

$$\frac{dy}{d\theta} = kAe^{k\theta} \sin \theta + Ae^{k\theta} \cos \theta$$

$$= Ae^{k\theta} (k \sin \theta + \cos \theta)$$

$$\frac{dx}{d\theta} = kAe^{k\theta} \cos \theta - Ae^{k\theta} \sin \theta$$

$$= Ae^{k\theta} (k \cos \theta - \sin \theta)$$

M2

$$\frac{dy}{dx} = \frac{Ae^{k\theta} (k \sin \theta + \cos \theta)}{Ae^{k\theta} (k \cos \theta - \sin \theta)} = \frac{(k \sin \theta + \cos \theta)}{(k \cos \theta - \sin \theta)}.$$

Dividing each term by $\cos \theta$:

M1

$$\frac{dy}{dx} = \frac{\frac{k \sin \theta}{\cos \theta} + \frac{\cos \theta}{\cos \theta}}{\frac{k \cos \theta}{\cos \theta} - \frac{\sin \theta}{\cos \theta}} = \frac{k \tan \theta + 1}{k - \tan \theta}$$

A1

- c. The angle that the tangent makes with the x -axis is the exterior angle of the triangle OPQ, which is equal to the sum of two interior angles $(\alpha + \theta)$. The gradient of the line is $\tan(\alpha + \theta)$.

A1

d. From b. and c.

$$\frac{k \tan \theta + 1}{k - \tan \theta} = \tan(\alpha + \theta).$$

Using the addition formula for tan: $\frac{k \tan \theta + 1}{k - \tan \theta} = \frac{\tan \alpha + \tan \theta}{1 - \tan \alpha \tan \theta}$

M1

Cross multiplying gives:

$$k \tan \theta - k \tan \alpha \tan^2 \theta + 1 - \tan \alpha \tan \theta = k \tan \alpha + k \tan \theta - \tan^2 \theta - \tan \alpha \tan \theta$$

$$k \tan \alpha + k \tan \alpha \tan^2 \theta = 1 + \tan^2 \theta$$

$$k \tan \alpha (1 + \tan^2 \theta) = 1 + \tan^2 \theta$$

M1

$$\tan \alpha = \frac{1}{k}, \quad \alpha = \tan^{-1} \frac{1}{k}$$

A1

Total 10 marks

Question 4

a. The acceleration of the missile is $\mathbf{a}_m = -9.8\mathbf{j}$.

A1

The initial velocity of the missile is $\mathbf{v}_m = -30\mathbf{i} + 3\mathbf{j}$. The velocity at time t will be

$$\mathbf{v}_m = -9.8t\mathbf{j} + (-30\mathbf{i} + 3\mathbf{j}) = -30\mathbf{i} + (3 - 9.8t)\mathbf{j}$$

A1

As the initial position of the missile is $\mathbf{r}_m = 1000\mathbf{i} + 500\mathbf{j}$, the position at time t is

$$\mathbf{r}_m(t) = -30t\mathbf{i} + (-4.9t^2 + 3t)\mathbf{j} + 1000\mathbf{i} + 500\mathbf{j}$$

$$\mathbf{r}_m(t) = (-30t + 1000)\mathbf{i} + (-4.9t^2 + 3t + 500)\mathbf{j}$$

A1

b. The acceleration of the anti-missile is $\mathbf{a}_a = -9.8\mathbf{j}$

The initial velocity is

$$\mathbf{v}_a = 100 \cos \theta \mathbf{i} + 100 \sin \theta \mathbf{j} \text{ and the velocity at time } t \text{ is}$$

A1

$$\mathbf{v}_a = 100 \cos \theta \mathbf{i} + (-9.8t + 100 \sin \theta)\mathbf{j}.$$

M1

The anti-missile is initially at the origin, so its position at time t is obtained by integrating

$$\mathbf{v}_a = 100 \cos \theta \mathbf{i} + (-9.8t + 100 \sin \theta)\mathbf{j} \text{ with respect to } t.$$

$$\text{Therefore } \mathbf{r}_a(t) = 100t \cos \theta \mathbf{i} + (-4.9t^2 + 100t \sin \theta)\mathbf{j}.$$

A1

- c. For an interception, the position vectors must be equal.

M1

$$100t \cos \theta \mathbf{i} + (-4.9t^2 + 100t \sin \theta) \mathbf{j} = (-30t + 1000) \mathbf{i} + (-4.9t^2 + 3t + 500) \mathbf{j}$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components:

$$100t \cos \theta = -30t + 1000 \quad (1)$$

$$-4.9t^2 + 100t \sin \theta = -4.9t^2 + 3t + 500 \quad (2)$$

M1

$$\text{From (1) } t = \frac{1000}{100 \cos \theta + 30}$$

$$\text{Substituting into (2) } \frac{100000 \sin \theta}{100 \cos \theta + 30} = \frac{3000}{100 \cos \theta + 30} + 500$$

M2

A calculator can be used to solve this equation without simplification, or when simplified, it becomes:

$$50 \sin \theta - 25 \cos \theta - 9 = 0 \text{ and } \theta = 35.8^\circ$$

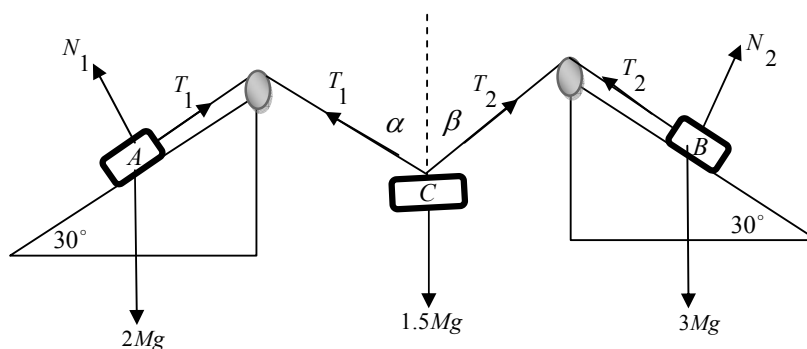
A1

- d. Substituting $\theta = 35.8^\circ$ into $t = \frac{1000}{100 \cos \theta + 30}$ gives 9 seconds.

A1

Total 12 marks

Question 5



- a. Two different tensions and two different normal reaction forces shown

A1

Weights shown

A1

- b.

$$T_1 - 2Mg \sin 30^\circ = 0$$

$$T_1 = Mg$$

$$T_2 - 3Mg \sin 30^\circ = 0$$

$$T_2 = \frac{3Mg}{2}$$

A2

- c. Resolving forces acting on C horizontally:

$$T_1 \sin \alpha = T_2 \sin \beta$$

M1

$$Mg \sin \alpha = \frac{3Mg}{2} \sin \beta$$

$$2 \sin \alpha = 3 \sin \beta$$

A1

Vertically:

$$T_1 \cos \alpha + T_2 \cos \beta - 1.5Mg = 0$$

M1

$$Mg \cos \alpha + \frac{3Mg}{2} \cos \beta - 1.5Mg = 0$$

$$2 \cos \alpha + 3 \cos \beta = 3$$

A1

- d. Equations $2 \sin \alpha = 3 \sin \beta$ and $2 \cos \alpha + 3 \cos \beta = 3$ need to be solved simultaneously.

$$\text{From } \sin \alpha = \frac{3}{2} \sin \beta, \cos \alpha = \sqrt{1 - \frac{9}{4} \sin^2 \beta} = \frac{1}{2} \sqrt{4 - 9 \sin^2 \beta}$$

M1

Substituting $\cos \alpha$ into $2 \cos \alpha + 3 \cos \beta = 3$ gives $\sqrt{4 - 9 \sin^2 \beta} + 3 \cos \beta - 3 = 0$

M1

This equation should be again solved on the calculator by plotting its graph. It gives a solution of $\beta = 38.94244^\circ = 39^\circ$ to the nearest degree.

A1

$$\sin \alpha = \frac{3}{2} \sin 38.94244, \alpha = 70.52877^\circ = 71^\circ \text{ to the nearest degree.}$$

A1

Total 12 marks