

SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 1



2008 Trial Examination

SOLUTIONS

Question 1

a. $\hat{\mathbf{a}} = \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} + \mathbf{k})$ $\hat{\mathbf{b}} = \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} - \mathbf{k})$

A2

b. The angle bisector can be found by adding $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

$$\hat{\mathbf{a}} + \hat{\mathbf{b}} = \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} + \mathbf{k}) + \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} - \mathbf{k})$$

M1

$$= \frac{2}{\sqrt{3}}(\mathbf{i} - \mathbf{j})$$

A1

The magnitude of $\hat{\mathbf{a}} + \hat{\mathbf{b}}$ is $\frac{2\sqrt{2}}{\sqrt{3}}$. The unit vector of the angle bisector is

$$\frac{\sqrt{3}}{2\sqrt{2}} \times \frac{2}{\sqrt{3}}(\mathbf{i} - \mathbf{j}) = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$$

A1

Question 2

a. $z = 1$ $|z| = 1, \text{Arg}z = 0$, thus $z = \text{cis}0$

A1

b. $\sqrt[6]{1} = \text{cis} \frac{0 + 2k\pi}{6} = \text{cis} \frac{k\pi}{3}$, $k = 0, 1, 2, 3, 4, 5$ or any other 6 integers

A1

$$\text{c. } \sqrt[12]{1} = \text{cis} \frac{2k\pi}{12} = \text{cis} \frac{k\pi}{6} \quad \text{A1}$$

$$\sqrt[6]{1} \times \sqrt[12]{1} = \text{cis} \frac{k\pi}{3} \times \text{cis} \frac{k\pi}{6} = \text{cis} \left(\frac{k\pi}{3} + \frac{k\pi}{6} \right) \quad \text{M1}$$

$$= \text{cis} \frac{k\pi}{2} = \text{cis} \frac{2k\pi}{4} \text{ which is } \sqrt[4]{1}. \quad \text{A1}$$

Question 3**a.**

$$\frac{1}{4} \cos 3\theta = \frac{1}{4} \cos(2\theta + \theta) \quad \text{M1}$$

$$= \frac{1}{4} (\cos 2\theta \cos \theta - \sin 2\theta \sin \theta)$$

$$= \frac{1}{4} (2 \cos^3 \theta - \cos \theta - 2 \sin^2 \theta \cos \theta) \quad \text{M1}$$

$$= \frac{1}{4} (2 \cos^3 \theta - \cos \theta - 2 \cos \theta + 2 \cos^3 \theta) \quad \text{A1}$$

$$= \cos^3 \theta - \frac{3}{4} \cos \theta$$

$$\text{b. For } x = \frac{2}{3} \cos \theta \text{ we have } 27 \times \frac{8}{27} \cos^3 \theta - 9 \times \frac{2}{3} \cos \theta = 1$$

$$8 \cos^3 \theta - 6 \cos \theta = 1$$

$$\cos^3 \theta - \frac{3}{4} \cos \theta = \frac{1}{8} \quad \text{M1}$$

$$\text{By substituting the result from a. } \frac{1}{4} \cos 3\theta = \frac{1}{8}$$

$$\cos 3\theta = \frac{1}{2}, \quad \theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9} \quad \text{M1}$$

$$\text{The solutions are } x = \frac{2}{3} \cos \frac{\pi}{9}, \quad \frac{2}{3} \cos \frac{5\pi}{9}, \quad \frac{2}{3} \cos \frac{7\pi}{9} \quad \text{A1}$$

Question 4

From $x = y \ln(xy)$, $\ln(xy) = \frac{x}{y}$.

By differentiating implicitly

$$1 = \frac{dy}{dx} \ln(xy) + \frac{y}{xy} \left(y + x \frac{dy}{dx} \right) \quad \text{Substitute } \ln(xy) = \frac{x}{y} \quad \text{M1}$$

$$1 = \frac{dy}{dx} \frac{x}{y} + \frac{y}{x} + \frac{dy}{dx}$$

$$\frac{dy}{dx} \left(\frac{x}{y} + 1 \right) = 1 - \frac{y}{x} \quad \text{M1}$$

$$\frac{dy}{dx} \left(\frac{x+y}{y} \right) = \frac{x-y}{x}$$

$$\frac{dy}{dx} = \frac{y(x-y)}{x(x+y)} \quad \text{A1}$$

Question 5

$$v \frac{dv}{dx} = -g - 0.1v^2, \quad \frac{dv}{dx} = - \left(\frac{g + 0.1v^2}{v} \right)$$

M1

$$x = - \int \frac{v}{g + 0.1v^2} dv \quad \text{By substituting } u = g + 0.1v^2, \quad \frac{du}{dv} = 0.2v$$

$$\begin{aligned} x &= - \frac{1}{0.2} \int \frac{1}{u} dt = -5 \ln|u| + c \\ &= -5 \ln(g + 0.1v^2) + c \end{aligned}$$

M1

When $x = 0$, $v = 80$, $c = 5 \ln(g + 640)$

$$x = 5 \ln \left(\frac{g + 640}{g + 0.1v^2} \right)$$

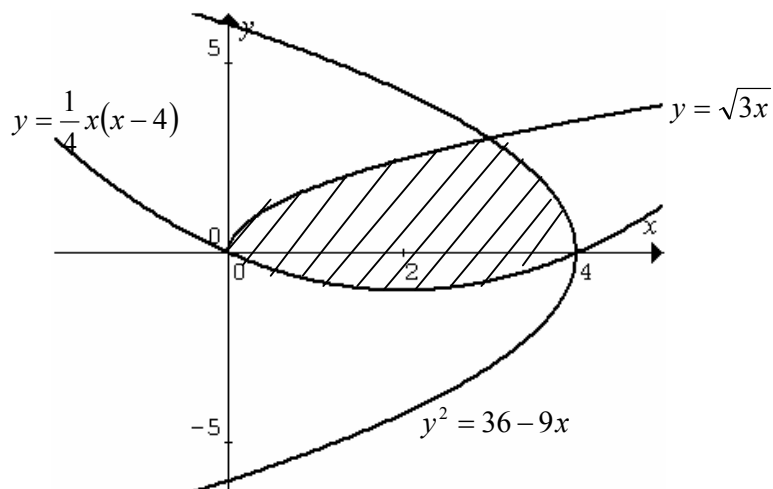
A1

For the maximum height $v = 0$

$$x = 5 \ln \left(\frac{g + 640}{g} \right) = 5 \ln \left(1 + \frac{640}{g} \right)$$

$$a = 5, \quad b = 640$$

A1

Question 6**a.**

A1

$$\text{b. } A = \int_0^3 \sqrt{3x} dx + \int_3^4 \sqrt{36-9x} dx - \int_0^4 \frac{1}{4}x(x-4) dx$$

correct limits

A1

$$\text{or } A = \int_0^3 \left(\sqrt{3x} - \frac{1}{4}x(x-4) \right) dx + \int_3^4 \left(\sqrt{36-9x} - \frac{1}{4}x(x-4) \right) dx$$

correct integrals

A1

c.

$$\begin{aligned} A &= \left[\frac{2}{9}(3x)^{3/2} \right]_0^3 - \left[\frac{2}{27}(36-9x)^{3/2} \right]_3^4 - \left[\frac{x^3}{12} - \frac{x^2}{2} \right]_0^4 \\ &= \frac{2}{9} \times 9^{3/2} + \frac{2}{27} \times 9^{3/2} - \frac{64}{12} + \frac{16}{2} \\ &= \frac{32}{3} \text{ sq units} \end{aligned}$$

M1

A1

Question 7

$$\mathbf{r}(t) = (2 + 3 \cos 2t)\mathbf{i} + (-3 + \sin 2t)\mathbf{j}, \quad t \geq 0.$$

a.

$$x = 2 + 3 \cos 2t \qquad y = -3 + \sin 2t$$

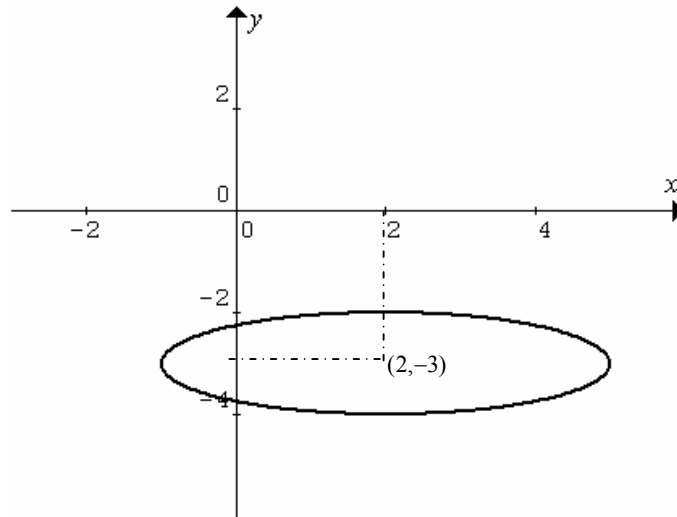
$$\cos 2t = \frac{x-2}{3} \qquad \sin 2t = y+3$$

M1

Substituting into $\sin^2 2t + \cos^2 2t = 1$ gives the ellipse $\frac{(x-2)^2}{9} + (y+3)^2 = 1$

A1

b.



- c. When $t = 0$ $x = 2 + 3 \cos 0 = 5$ $y = -3 + \sin 0 = -3$. Initial position is $(5, -3)$.

A1

The particle is moving anticlockwise as for $t = \frac{\pi}{4}$ its position is $(2, -2)$.

A1

It returns to its initial position after $t = \pi$.

A1

Question 8

$$\int_{-2}^a \frac{3}{(x+2)^2 + 16} dx = \frac{3\pi}{16}$$

$$\frac{3}{4} \left[\tan^{-1} \left(\frac{x+2}{4} \right) \right]_{-2}^a = \frac{3\pi}{16}$$

M1

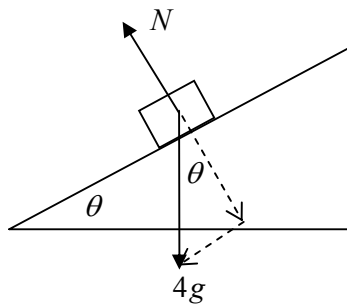
$$\frac{3}{4} \tan^{-1} \left(\frac{a+2}{4} \right) = \frac{3\pi}{16}$$

M1

$$\frac{a+2}{4} = 1$$

$$a = 2$$

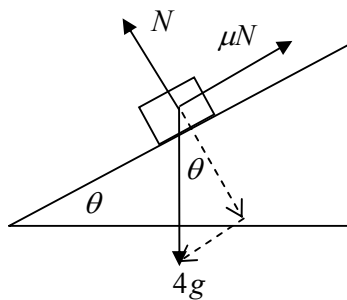
A1

Question 9**a.**

$$4g \sin \theta = 4a$$

$$a = g \sin \theta$$

A1

b.

$$N = 4g \cos \theta, \quad a = \frac{1}{4} g \sin \theta$$

$$4g \sin \theta - \mu \times 4g \cos \theta = 4 \times \frac{1}{4} g \sin \theta$$

$$3 \sin \theta = 4 \mu \cos \theta$$

$$\mu = \frac{3 \sin \theta}{4 \cos \theta} = \frac{3}{4} \tan \theta$$

M2

A1