

The Mathematical Association of Victoria
SPECIALIST MATHEMATICS
2008 Trial written examination 2 – Multiple Choice Solutions

SECTION 1

Answers:

- | | | | | |
|-------|-------|-------|-------|-------|
| 1. B | 2. A | 3. E | 4. C | 5. A |
| 6. D | 7. B | 8. B | 9. C | 10. A |
| 11. C | 12. D | 13. B | 14. D | 15. E |
| 16. E | 17. C | 18. A | 19. B | 20. E |
| 21. D | 22. D | | | |

Worked Solutions:

Question 1

Answer B

Vertical asymptotes occur where $2x^2 + 3x - 5 = 0$

$$(2x + 5)(x - 1) = 0$$

$$x = -\frac{5}{2}, x = 1$$

Horizontal asymptote at $y = 0$

Question 2

Answer A

Graph is a hyperbola of the form $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$

Asymptotes intersect at $(-1, 0) \Rightarrow h = -1, k = 0$

Minimum turning point at $(-1, 2) \Rightarrow a^2 = 1, b^2 = 4$

Equation of hyperbola is $\frac{(y-0)^2}{4} - \frac{(x+1)^2}{1} = 1$

$$\frac{y^2}{4} - (x+1)^2 = 1$$

Question 3**Answer E**

$$\begin{aligned}
& \frac{\sin(3x)}{\sin(x)} + \frac{\cos(3x)}{\cos(x)} \\
&= \frac{\sin(3x)\cos(x) + \cos(3x)\sin(x)}{\sin(x)\cos(x)} \\
&= \frac{\sin(3x+x)}{\sin(x)\cos(x)} \\
&= \frac{2\sin(4x)}{2\sin(x)\cos(x)} \\
&= \frac{2 \times 2\sin(2x)\cos(2x)}{\sin(2x)} \\
&= 4\cos(2x)
\end{aligned}$$

Question 4**Answer C**

$f(x) = \sin^{-1}(x)$ has endpoints $\left(-1, -\frac{\pi}{2}\right)$ and $\left(1, \frac{\pi}{2}\right)$

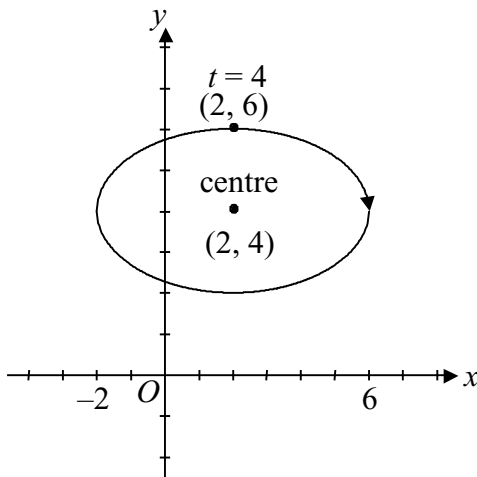
$f(x) = \sin^{-1}(a(x-b))$ has been dilated by a factor of $\frac{1}{a}$ parallel to the x -axis then translated b units right.

The endpoint coordinates will be $\left(b - \frac{1}{a}, -\frac{\pi}{2}\right)$ and $\left(b + \frac{1}{a}, \frac{\pi}{2}\right)$

Hence $\left(\frac{1}{a} + b, \frac{\pi}{2}\right)$

Question 5**Answer A**

Graphing the curve in parametric mode on calculator will show alternatives B, C, D and E are true.



Finding the Cartesian equation

$$x = 2 + 4\sin(\pi t)$$

$$\Rightarrow \frac{x-2}{4} = \sin(\pi t) \dots(1)$$

$$\left(\frac{x-2}{4}\right)^2 + \left(\frac{y-4}{2}\right)^2 = \sin^2(\pi t) + \cos^2(\pi t)$$

$$\frac{(x-2)^2}{16} + \frac{(y-4)^2}{4} = 1$$

The curve follows an elliptical path with centre (2, 4)

$$y = 4 + 2\cos(\pi t)$$

$$\Rightarrow \frac{y-4}{2} = \cos(\pi t) \dots(2)$$

Question 6**Answer D**

Let $z = x + yi$

$$3z - \bar{z} = 12 - 4i$$

$$\Rightarrow 3(x + yi) - (x - yi) = 12 - 4i$$

$$3x + 3yi - x + yi = 12 - 4i$$

$$2x + 4yi = 12 - 4i$$

$$2x = 12, 4y = -4$$

$$\therefore x = 6, y = -1$$

$$z = 6 - i$$

Question 7**Answer B**

Since $2 - i$ and $1 - i$ are solutions of the quadratic equation

$$(z - (2 - i))(z - (1 - i)) \equiv z^2 + az + b$$

Expanding:

$$z^2 - (1 - i)z - (2 - i)z + (2 - i)(1 - i) \equiv z^2 + az + b$$

$$z^2 - (3 - 2i)z + (2 - 3i - 1) \equiv z^2 + az + b$$

$$z^2 - (3 - 2i)z + (1 - 3i) \equiv z^2 + az + b$$

Equating coefficients of powers of z :

$$a = -(3 - 2i), b = 1 - 3i$$

$$a = -3 + 2i$$

Question 8**Answer B**

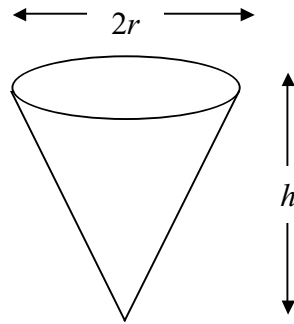
$$|2 - z| = 4$$

$$|2 + 0i - (x + yi)| = 4$$

$$\sqrt{(2 - x)^2 + (0 - y)^2} = 4$$

$$(x - 2)^2 + y^2 = 16$$

Circle with centre $(2, 0)$ and radius 4

Question 9**Answer C**

$$V = \frac{1}{3}\pi r^2 h$$

$$h = 2r \text{ (given)}$$

$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h$$

$$r = \frac{h}{2}$$

$$V = \frac{1}{12}\pi h^3$$

$$\frac{dV}{dh} = \frac{1}{4}\pi h^2$$

Applying the chain rule

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$\frac{dV}{dt} = 10\,000 \text{ cm}^3/\text{min}$$

$$10\,000 = \frac{1}{4}\pi h^2 \times \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{40\,000}{\pi h^2}$$

$$\frac{dh}{dt} = \frac{40\,000}{\pi \times 50^2}$$

$$\frac{dh}{dt} = \frac{16}{\pi} \text{ cm/min}$$

Question 10**Answer A**

$$\int \sin^2(x) \cos^3(x) dx$$

$$= \int \sin^2(x) \cos^2(x) \cos(x) dx$$

$$= \int \sin^2(x) (1 - \sin^2(x)) \cos(x) dx$$

$$\begin{aligned} \text{Let } u &= \sin(x) \\ \frac{du}{dx} &= \cos(x) \\ du &= \cos(x) dx \end{aligned}$$

$$= \int u^2 (1 - u^2) du$$

$$= \int (u^2 - u^4) du$$

Question 11**Answer C**Finding equation of inverse $f^{-1}(x)$

$$x = 2^y - 1$$

$$x + 1 = 2^y$$

$$y = \log_2(x + 1)$$

$$f^{-1}(x) = \log_2(x + 1)$$

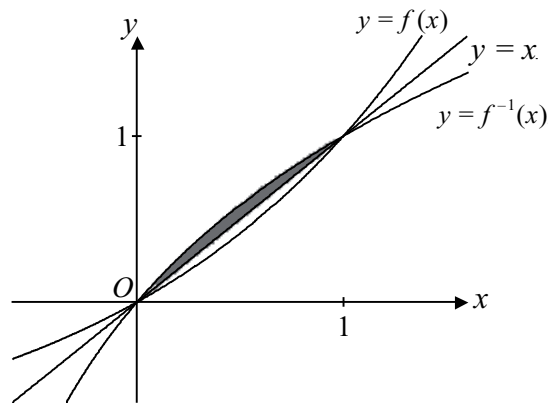
Area enclosed by curves

$$\int_0^1 (f^{-1}(x) - f(x)) dx$$

$$= \int_0^1 (\log_2(x + 1) - (2^x - 1)) dx$$

$$= \int_0^1 (\log_2(x + 1) - 2^x + 1) dx$$

A is true

The graphs are inverses of each other therefore the required area is symmetrical about the line $y = x$.Twice $\int_0^1 (f^{-1}(x) - x) dx$ equals the required area

$$= 2 \int_0^1 (\log_2(x + 1) - x) dx$$

B is true

Twice $\int_0^1 (x - f(x)) dx$ equals the required area

$$= 2 \int_0^1 (x - (2^x - 1)) dx$$

$$= 2 \int_0^1 (x - 2^x + 1) dx$$

However C is $2 \int_0^1 (x - 2^x - 1) dx$, so this is false

(Area of square side length 1) – twice $\int_0^1 f(x) dx$ equals the required area

$$= (1 \times 1) - 2 \int_0^1 (2^x - 1) dx$$

$$= 1 - 2 \int_0^1 (2^x - 1) dx$$

D is true

Twice $\int_0^1 f^{-1}(x) dx$ – (area of square side length 1) equals the required area.

$$= 2 \int_0^1 f^{-1}(x) dx - (1 \times 1)$$

$$= 2 \int_0^1 (\log_2(x+1)) dx - 1$$

E is true

Question 12

Answer D

$$\frac{dy}{dx} = \sqrt{x^4 + 1}$$

$$y = \int \sqrt{x^4 + 1} dx$$

Let $y = f(x) + c \dots(1)$ where $f'(x) = \sqrt{x^4 + 1}$

When $x = 1, y = 4$

$$4 = f(1) + c$$

$$c = f(1) + 4 \dots(2)$$

Substitute (2) into (1)

$$y = f(x) - f(1) + 4$$

When $x = 3, y = f(3) - f(1) + 4$

$$y = \left[f(t) \right]_1^3 + 4$$

$$y = \int_1^3 (\sqrt{t^4 + 1}) dt + 4$$

Question 13**Answer B**

$$\text{Let } f(x, y) = x - y$$

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$\text{Given } x_0 = 2, y_0 = 0, h = 0.5$$

$$y_1 = y_0 + 0.5f(x_0, y_0)$$

$$x_0 = 2, \quad f(x_0, y_0) = x_0 - y_0 = 2 - 0 = 2$$

$$y_1 = 0 + 0.5 \times 2 = 1$$

$$y_2 = y_1 + 0.5f(x_1, y_1)$$

$$x_1 = 2.5, \quad f(x_1, y_1) = x_1 - y_1 = 2.5 - 1 = 1.5$$

$$y_2 = 1 + 0.5 \times 1.5 = 1.75$$

$$y_3 = y_2 + 0.5f(x_2, y_2)$$

$$x_2 = 3, \quad f(x_2, y_2) = x_2 - y_2 = 3 - 1.75 = 1.25$$

$$y_3 = 1.75 + 0.5 \times 1.25 = 2.375$$

$$y_4 = y_3 + 0.5f(x_3, y_3)$$

$$x_3 = 3.5, \quad f(x_3, y_3) = x_3 - y_3 = 3.5 - 2.375 = 1.125$$

$$y_4 = 2.375 + 0.5 \times 1.125 = 2.9375$$

Question 14**Answer D**

The gradients are equal in a horizontal direction, therefore $\frac{dy}{dx} = f(y)$.

Eliminate alternatives A and B.

The field diagram shows a small positive slope at $y = 2$.

At $y = 2$, $\frac{dy}{dx} = \cos(y) = \cos(2) = -0.42$

This slope is negative, however in the field diagram the slope is positive when $y = 2$.

Eliminate E.

At $y = 2$, $\frac{dy}{dx} = y = 2$

A slope of 2 is much steeper than this field diagram shows when $y = 2$.

Eliminate C.

At $y = 2$, $\frac{dy}{dx} = \frac{1}{y^2} = \frac{1}{4}$ This is the only possible option.

The slope in the field diagram looks like the gradient is $\frac{1}{4}$ when $y = 2$.

By solving the differential equation it can be seen that the solution fits the slope field.

$$\frac{dy}{dx} = \frac{1}{y^2}$$

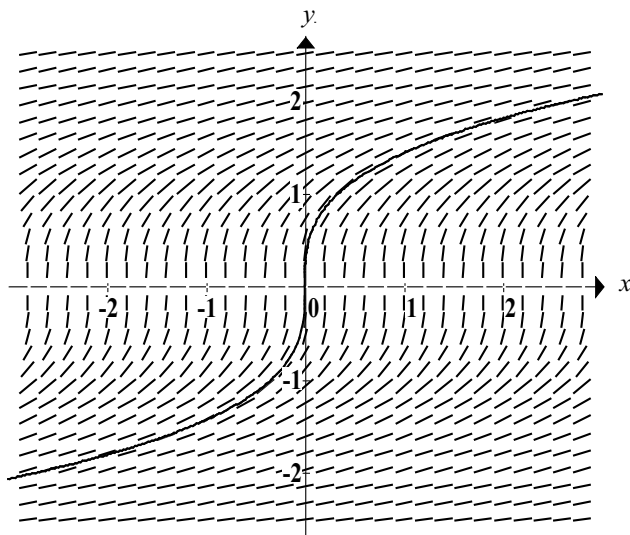
$$\frac{dy}{dx} = y^2$$

$$x = \int y^2 dy + c$$

$$x = \frac{1}{3}y^3 + c \quad \text{Let } c = 0$$

$$y = \sqrt[3]{(3x)}$$

This curve is sketched on the field diagram below.



Question 15**Answer E**

$$a \cdot b = |a| |b| \cos(\theta)$$

$$(2\hat{i} + \hat{j} + 2\hat{k}) \cdot (\hat{i} + m\hat{j} - \hat{k}) = |2\hat{i} + \hat{j} + 2\hat{k}| |\hat{i} + m\hat{j} - \hat{k}| \cos(\theta)$$

$$2 + m - 2 = \sqrt{9} \times \sqrt{m^2 + 2} \cos(\theta)$$

$$m = 3\sqrt{m^2 + 2} \cos(\theta)$$

$$\theta = \cos^{-1} \left(\frac{m}{3\sqrt{m^2 + 2}} \right)$$

Question 16**Answer E**

The dot product of perpendicular vectors is zero.

$$p \cdot e = (\cos(t)\hat{i} + \sin(t)\hat{j} - \hat{k}) \cdot (\sin(t)\hat{i} - \cos(t)\hat{j} + \hat{k})$$

$$p \cdot e = \cos(t)\sin(t) - \sin(t)\cos(t) - 1$$

$$p \cdot e = -1$$

Vectors p and e are not perpendicular.

Question 17**Answer C**

Let T be the time when the particle's velocity changes from negative to positive.

Total area under graph (disregarding sign) is 305.

$$8 \times 15 + \frac{1}{2}(T - 8) \times 15 + \frac{1}{2}(16 - T) \times 25 + 4 \times 25 = 305$$

$$120 + 7.5T - 60 + 200 - 12.5T + 100 = 305$$

$$T = 11$$

The particle changes direction after 11 seconds.

$$\text{Distance travelled with negative velocity} = 8 \times 15 + \frac{1}{2}(11 - 8) \times 15 = 142.5 \text{ metres}$$

$$\text{Distance travelled with positive velocity} = \frac{1}{2}(16 - 11) \times 25 + 4 \times 25 = 162.5 \text{ metres}$$

The particle is $162.5 - 142.5 = 20$ metres from its starting position after 20 seconds.

Question 18**Answer A**

$$a = \frac{dv}{dt} = \frac{1}{2v+1}$$

$$\frac{dt}{dv} = 2v+1$$

$$t = \int (2v+1)dv$$

$$t = v^2 + v$$

$$\text{When } t = 0, v = 0 \Rightarrow c = 0$$

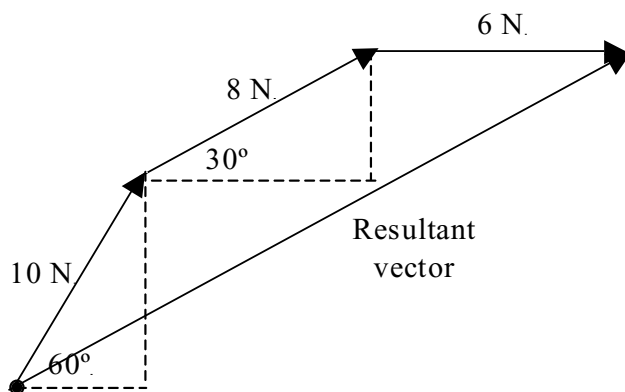
$$t = v^2 + v$$

$$\begin{aligned} \text{When } t = 2, \quad v^2 + v &= 2 \\ v^2 + v - 2 &= 0 \\ (v+2)(v-1) &= 0 \\ v = 1, v \geq 0 \end{aligned}$$

Velocity is 1 m/s after 2 seconds

Question 19**Answer B**

Adding vectors head to tail



Horizontal component of resultant vector

$$= 10\cos(60^\circ) + 8\cos(30^\circ) + 6$$

$$= 5 + 4\sqrt{3} + 6$$

$$= 11 + 4\sqrt{3}$$

Vertical component of resultant vector

$$= 10\sin(60^\circ) + 8\sin(30^\circ)$$

$$= 5\sqrt{3} + 4$$

Magnitude of resultant vector

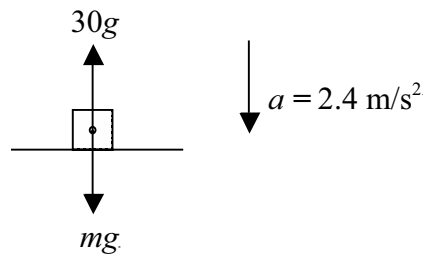
$$= \sqrt{(11 + 4\sqrt{3})^2 + (5\sqrt{3} + 4)^2}$$

$$= 21.9 \text{ newtons}$$

Question 20**Answer E**

Draw diagram showing the forces acting.

Lift is accelerating downwards.



$$mg - 30g = 2.4m$$

$$9.8m - 2.4m = 30 \times 9.8$$

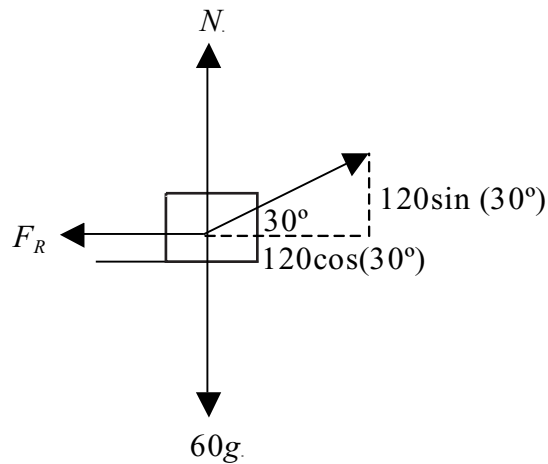
$$7.4m = 294$$

$$m = 39.7 \text{ kg}$$

This is closest to 40 kg

Question 21**Answer D**

Show all forces acting on the diagram.



Resolving vertical forces to find the normal reaction N

$$N + 120\sin(30^\circ) = 60g$$

$$N = 60g - 60$$

Resolving horizontal forces.

$$120\cos(30^\circ) - F_R = ma \quad \text{Mass on the point of moving has a zero acceleration and so } a = 0.$$

$$60\sqrt{3} - \mu N = 0 \quad \text{Friction: } F_R = \mu N$$

$$\mu = \frac{60\sqrt{3}}{N}$$

$$\mu = \frac{60\sqrt{3}}{60g - 60}$$

$$\mu = \frac{\sqrt{3}}{g - 1}$$

Question 22**Answer D**

The truck is towing the car and the tension in the towrope, T , represents this force. The resistance force R_1 acting on the truck is not relevant to the motion of the car.

Mathematical Association of Victoria
SPECIALIST MATHEMATICS
2008 Trial written examination 2 – Extended Analysis Questions Solutions

SECTION 2**Question 1****a.** At $t = 0$

$$r(t) = 30\hat{i} - 20\hat{j} \quad [1M]$$

$$\text{Speed} = \sqrt{30^2 + 20^2} = 36.06 \text{ m/min} \quad [1A]$$

b. $\dot{r}(t) = 30(t+1)\hat{i} + 20(t-1)\hat{j}$

$$r(t) = 30\left(\frac{t^2}{2} + t\right)\hat{i} + 20\left(\frac{t^2}{2} - t\right)\hat{j} + c \quad [1M]$$

$$r(0) = 0\hat{i} + 0\hat{j}$$

$$\therefore c = 0$$

$$\therefore r(t) = 30\left(\frac{t^2}{2} + t\right)\hat{i} + 20\left(\frac{t^2}{2} - t\right)\hat{j} \quad [1A]$$

$$\text{c. } \sqrt{(100-0)^2 + (90-0)^2} = 134.5 \text{ m} \quad [1A]$$

$$\text{d. } \vec{AP} = \vec{AO} + \vec{OP}$$

$$\vec{AP} = -500\hat{i} - 450\hat{j} + 30\left(\frac{t^2}{2} + t\right)\hat{i} + 20\left(\frac{t^2}{2} - t\right)\hat{j}$$

$$\therefore \vec{AP} = (-500 + 15t^2 + 30t)\hat{i} + (-450 + 10t^2 - 20t)\hat{j} \quad [1A]$$

$$\text{e. } \left| \vec{AP} \right| = \sqrt{(-500 + 15t^2 + 30t)^2 + (-450 + 10t^2 - 20t)^2} \quad [1M]$$

Sketch the graph of $\left| \vec{AP} \right|$ in terms of t ,

From the graph of $\left| \vec{AP} \right|$ in terms of t , the minimum point is (5.5, 283.6)

To the nearest minute the airport official is closest to the plane at $t = 6$ minutes [1A]

f. If the plane crosses the path of the vector $\vec{OA}$ then the points O , A and P are collinear and so

there exists a constant b such that $\vec{OP} = b\vec{OA}$

$$\vec{r}(t) = 30\left(\frac{t^2}{2} + t\right)\vec{i} + 20\left(\frac{t^2}{2} - t\right)\vec{j} = 500b\vec{i} + 450b\vec{j}$$

Equating coefficients:

$$30\left(\frac{t^2}{2} + t\right) = 500b \quad 20\left(\frac{t^2}{2} - t\right) = 450b$$

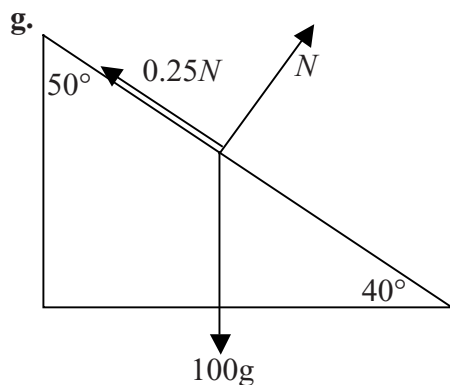
Therefore making b the subject of these equations

$$\frac{30}{500}\left(\frac{t^2}{2} + t\right) = \frac{20}{450}\left(\frac{t^2}{2} - t\right)$$

Solving this equation on a calculator gives no solution (except at the origin O)

Therefore the plane does not cross the path of the vector $\vec{OA}$

[2A]



[1A]

h. $100g\sin 40^\circ - F = ma$ where $F = \mu N$

And

$$N - 100g\cos 40^\circ = 0$$

Giving

$$N = 100g\cos 40^\circ = 750.724$$

$$\therefore F = \mu N = 0.25 \times 750.724 = 187.681$$

[1M]

Using

$$100g\sin 40^\circ - F = ma$$

$$\therefore 629.932 - 187.681 = 100a$$

$$\therefore a = 4.423$$

To 2 decimal places

$$\therefore a = 4.42 \text{ m/s}^2$$

[1A]

i. Using

$$v^2 = u^2 + 2as$$

$$v^2 = 0^2 + 2 \times 4.423 \times 6$$

$$\therefore v = 7.29$$

To 2 significant figures

$$v = 7.3 \text{ m/s}$$

[1A]

Question 2

a. $P(1+i) = (1+i)^4 + 2(1+i)^2 - 4(1+i) + 8$ [1M]

$$(1+i)^2 = 1 + 2i + i^2 = 1 + 2i - 1 = 2i$$

$$\therefore (1+i)^4 = (2i)^2 = 4i^2 = -4$$

$$\therefore P(1+i) = (2i)^2 + 2(2i) - 4 - 4i + 8$$

$$= -4 + 4i - 4 - 4i + 8 = 0$$

[1H]

b. Using conjugate factors

$$P(z) = (z-1-i)(z-1+i)(z+1+i\sqrt{3})(z+1-i\sqrt{3})$$
 [1M]

Solutions are

$$z = 1 \pm i, -1 \pm i\sqrt{3}$$

In polar form

$$z = \sqrt{2} \operatorname{cis} \left(\pm \frac{\pi}{4} \right), 2 \operatorname{cis} \left(\pm \frac{2\pi}{3} \right)$$
 [1A]

c. $z_1 z_2 = \sqrt{2} \operatorname{cis} \frac{\pi}{4} \times 2 \operatorname{cis} \left(-\frac{2\pi}{3} \right)$ [1M]

$$= 2\sqrt{2} \operatorname{cis} \left(\frac{\pi}{4} + \frac{-2\pi}{3} \right)$$

$$\therefore z_1 z_2 = 2\sqrt{2} \operatorname{cis} \left(-\frac{5\pi}{12} \right)$$
 [1A]

d. $z_1 = 1+i$ and $z_2 = -1-i\sqrt{3}$

$$\frac{z_1}{z_2} = \frac{1+i}{-1-i\sqrt{3}} \times \frac{-1+i\sqrt{3}}{-1+i\sqrt{3}}$$

$$\therefore \frac{z_1}{z_2} = \frac{(-1-\sqrt{3}) + (\sqrt{3}-1)i}{4}$$

$$\text{Using } \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\sqrt{3}-1}{-1-\sqrt{3}}$$

$$\therefore \tan \theta = \frac{\sqrt{3}-1}{-1-\sqrt{3}} \times \frac{-1+\sqrt{3}}{-1+\sqrt{3}} = \frac{4-2\sqrt{3}}{-2}$$

$$\text{Giving } \tan \theta = -2 + \sqrt{3} \text{ as required}$$
 [1A]

$$\text{Giving } \theta = \pi - \tan^{-1}(2 - \sqrt{3})$$

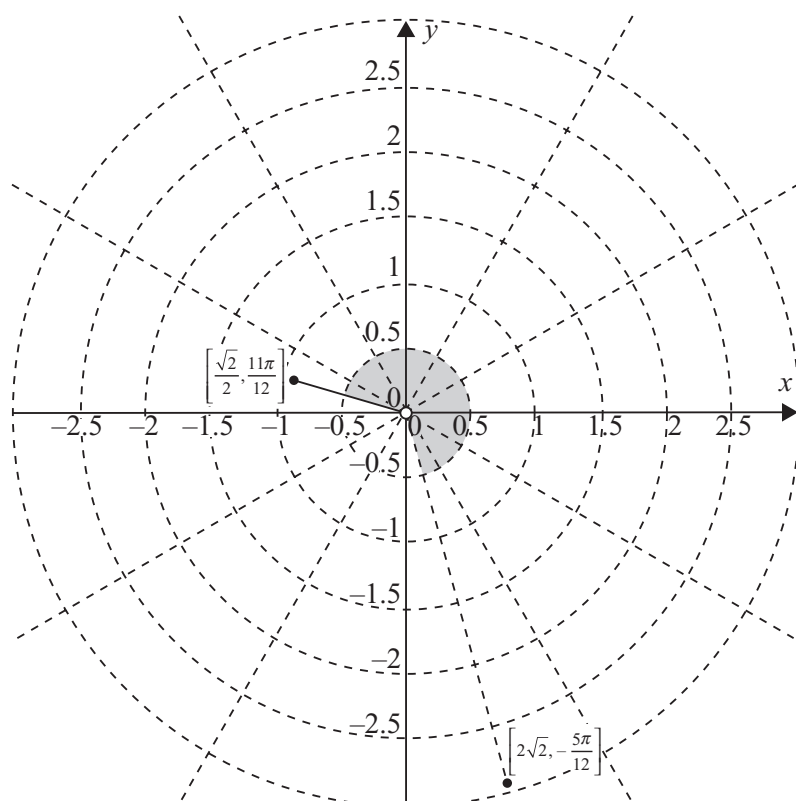
$$\operatorname{Arg} \frac{z_1}{z_2} = \theta = \pi - \frac{\pi}{12} = \frac{11\pi}{12}$$

$$\therefore \frac{z_1}{z_2} = \frac{\sqrt{2}}{2} \operatorname{cis} \frac{11\pi}{12}$$
 [1H]

e. See solution f.

First point [1A]
Second point [1A]

f.



Shading [1A]
Boundaries [1A]

Question 3

a. $\frac{dy}{dt} = 80k(80 - y)$

$$\therefore \frac{dt}{dy} = \frac{1}{80k(80 - y)}$$

$$\therefore t = \frac{1}{80k} \int \frac{1}{80 - y} dy$$

$$\therefore t = -\frac{1}{80k} \int \frac{-1}{80 - y} dy \quad [1M]$$

$$\therefore t = -\frac{1}{80k} \ln|80 - y| + c$$

$$\therefore -80k(t - c) = \ln|80 - y|$$

For $y < 80$

$$\Leftrightarrow e^{-80k(t-c)} = 80 - y \quad [1A]$$

$$\therefore y = 80 - e^{-80k(t-c)}$$

b. $y = 80 - e^{-80k(t-c)}$

$$t = 0, y = 1$$

$$\therefore 1 = 80 - e^{80kc}$$

$$\therefore e^{80kc} = 80 - 1 \quad [1M][1H]$$

$$\therefore \ln 79 = 80kc$$

$$\therefore c = \frac{1}{80k} \ln 79$$

c. $y = 80 - e^{-80k(t-c)}$ where $c = \frac{1}{80k} \ln 79$

$$y = 80 - e^{-80k(t - \frac{1}{80k} \ln 79)}$$

$$\therefore y = 80 - e^{-80kt + \ln 79}$$

$$\therefore y = 80 - e^{-80kt} e^{\ln 79} \quad [1M]$$

$$\therefore y = 80 - e^{-80kt} (79)$$

$$\therefore y = 80 - 79e^{-80kt}$$

$$t = 10, y = 44$$

$$\therefore 44 = 80 - 79e^{-800k} \quad [1A]$$

$$\therefore 79e^{-800k} = 36$$

d. $44 = 80 - 79e^{-800k}$

$$\therefore 79e^{-800k} = 36$$

$$\Leftrightarrow \ln \frac{36}{79} = -800k$$

$$\therefore k = 0.0009824$$

$$\therefore k = 0.0010$$

[1A]

k to 4 decimal places.

e. Using $c = \frac{1}{80k} \ln 79$

with $k = 0.0010$, $c = 54.618$

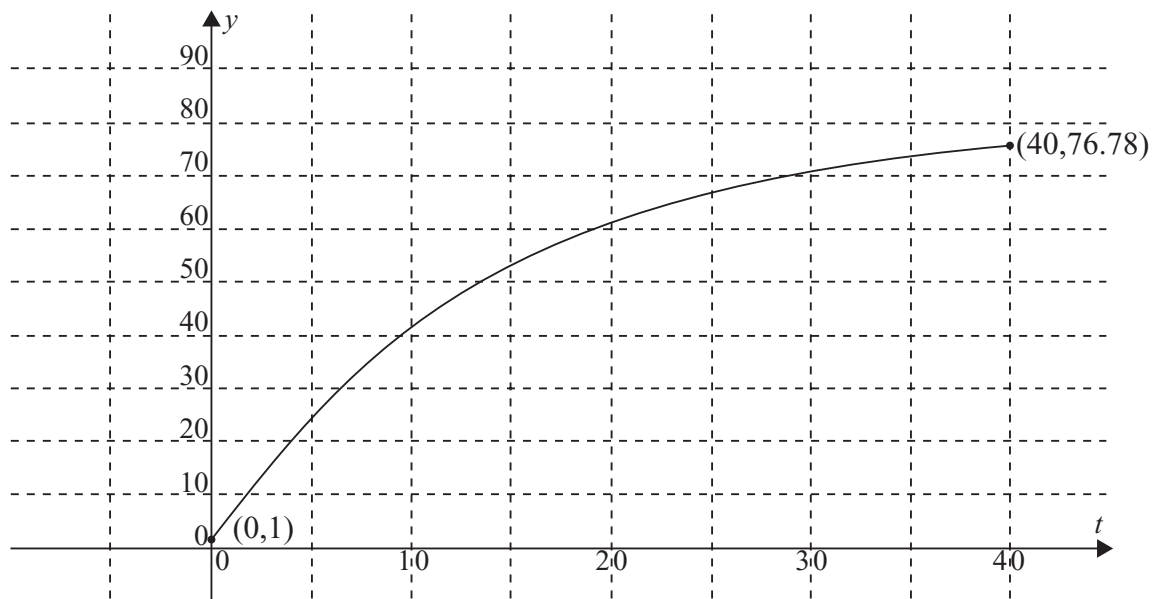
equation is $y = 80 - e^{-0.08(t - 54.618)}$

$$y = 80 - e^{-0.08t} e^{4.369}$$

$$\therefore y = 80 - 78.999e^{-0.08t}$$

Approximating to $y = 80 - 79e^{-0.08t}$

Check initial condition $t = 0$, $y = 1$, giving $y = 80 - 79e^0 = 1$, which is correct



Shape [1A]

Correct points [1A]

f. $y = 64\%$ uptake after 20 months to the nearest %.

[1A]

g. $y = 80$ represents the upper limit. 80% is the upper limit that the uptake of a certain mobile network approaches and levels out at.

Horizontal asymptote at $y = 80$

[1A]

Question 4

- a. Convert 100 km/hr to $\frac{100}{3.6} = 27\frac{7}{9}$ m/s.

Using

$$s = \frac{u + v}{2}t$$

$$\therefore s = \frac{0 + 27\frac{7}{9}}{2} \times 6.8$$

$$s = 94\frac{4}{9}$$

[1M]

To 2 decimal places $s = 94.44$ m.

Also

$$v = u + at$$

$$\therefore 27\frac{7}{9} = 0 + 6.8a$$

$$a = 4.08497$$

To 2 decimal places $a = 4.08$ m/s²

[1A]

- b. $R = ma$

[1A]

$$-1000 = 200a$$

$$\therefore a = -5 \text{ m/s}^2$$

Using constant acceleration

$$v^2 = u^2 + 2as$$

$$\therefore 0 = (27\frac{7}{9})^2 - 10s$$

$$\therefore s = 77.1605$$

To 2 decimal places $s = 77.16$ m

[1A]

$$\text{c. } a = \frac{1}{50}(v+1)^2$$

$$\therefore v \frac{dv}{dx} = \frac{1}{50}(v+1)^2$$

$$\therefore \frac{dv}{dx} = \frac{1}{50} \frac{(v+1)^2}{v}$$

$$\therefore \frac{dx}{dv} = \frac{50v}{(v+1)^2}$$

$$x = 50 \int \frac{v}{(v+1)^2} dv \quad [1M]$$

Let

$$u = v + 1, v = u - 1$$

$$\therefore du = dv$$

$$\therefore x = 50 \int \frac{u-1}{u^2} du$$

$$\therefore x = 50 \int \frac{1}{u} - \frac{1}{u^2} dv$$

$$\therefore x = 50 \left(\ln|u| + \frac{1}{u} + c \right) \quad [1H]$$

$$\therefore x = 50 \left(\ln|v+1| + \frac{1}{v+1} + c \right)$$

At

$$x = 0, v = 0, \therefore c = -1$$

$$x = 50 \left(\ln|v+1| + \frac{1}{v+1} - 1 \right) \quad [1H]$$

(As v is positive, students do not need to include the modulus signs)

$$\text{d. } x = 50 \left(\ln|v+1| + \frac{1}{v+1} - 1 \right)$$

Let

$$x = 100$$

$$v = 18.1 \text{ m/s (found on calculator)} \quad [1A]$$

$$\text{e. } x = 50 \left(\ln|v+1| + \frac{1}{v+1} - 1 \right)$$

Let

$$v = 50$$

$$x = 147.6 \text{ m (found on calculator)} \quad [1A]$$

Question 5

a. $-1 \leq 2x + 1 \leq 1$

$$\therefore -2 \leq 2x \leq 0$$

Therefore

$$-1 \leq x \leq 0$$

$$\therefore a = -1, b = 0$$

[1A]

b. $f(x) = 0.5\cos^{-1}(2x+1)$

$$\therefore f'(x) = 0.5 \frac{-1}{\sqrt{1-(2x+1)^2}} \times 2$$

$$\therefore f'(x) = \frac{-1}{\sqrt{1-(2x+1)^2}}$$

For a stationary point solve

$$\frac{-1}{\sqrt{1-(2x+1)^2}} = 0$$

No solution found

No solution hence no stationary point

[1A]

c. $\frac{d}{dx}(x \cos^{-1}(2x+1))$

Using the Product Rule:

$$= \cos^{-1}(2x+1) \cdot 1 + x \cdot \frac{-2}{\sqrt{1-(2x+1)^2}}$$

$$= \cos^{-1}(2x+1) - \frac{2x}{\sqrt{1-(2x+1)^2}}$$

[1A]

d. $\int_{-\frac{1}{2}}^0 \left(\cos^{-1}(2x+1) - \frac{2x}{\sqrt{1-(2x+1)^2}} \right) dx = \left[x \cos^{-1}(2x+1) \right]_{-\frac{1}{2}}^0$

$$\therefore \int_{-\frac{1}{2}}^0 \cos^{-1}(2x+1) dx = \int_{-\frac{1}{2}}^0 \frac{2x}{\sqrt{1-(2x+1)^2}} dx + \left[x \cos^{-1}(2x+1) \right]_{-\frac{1}{2}}^0$$

[2A]

e. Working below, using answer for d.

$$\begin{aligned}
 0.5 \int_{-\frac{1}{2}}^0 \cos^{-1}(2x+1) dx &= 0.5 \int_{-\frac{1}{2}}^0 \frac{2x}{\sqrt{1-(2x+1)^2}} dx + 0.5 \left[x \cos^{-1}(2x+1) \right]_{-\frac{1}{2}}^0 \\
 \therefore 0.5 \int_{-\frac{1}{2}}^0 \cos^{-1}(2x+1) dx &= 0.5 \int_{-\frac{1}{2}}^0 \frac{2x}{\sqrt{1-(2x+1)^2}} dx + 0.5 \left[0 - \frac{-1}{2} \cos^{-1}(0) \right] \\
 &= 0.5 \left(\frac{1}{2} - \frac{\pi}{4} \right) + 0.5 \frac{1}{2} \cos^{-1} 0 \\
 &= \frac{1}{4} - \frac{\pi}{8} + \frac{\pi}{8} \\
 &= \frac{1}{4}
 \end{aligned}$$

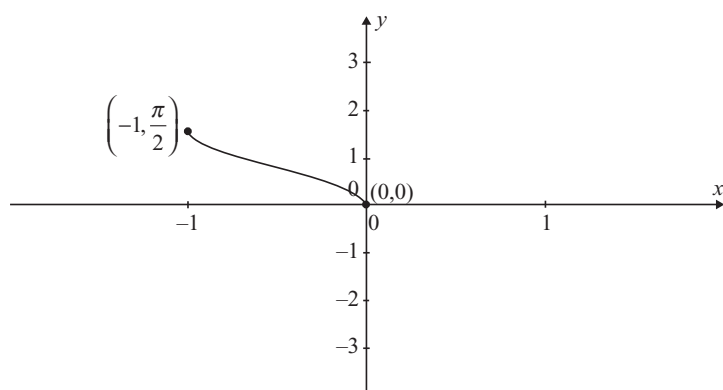
[1M][1A]

Check:

$$0.5 \int_{-\frac{1}{2}}^0 \cos^{-1}(2x+1) dx = 0.25$$

found on calculator

f.



[1A]

g. Volume = $\pi \int_0^h x^2 dy$
Using

$$y = 0.5 \cos^{-1}(2x+1)$$

$$\therefore 2y = \cos^{-1}(2x+1)$$

$$\therefore \cos(2y) = 2x+1$$

$$\therefore x = \frac{\cos(2y) - 1}{2}$$

$$\therefore x^2 = \left(\frac{\cos(2y) - 1}{2} \right)^2$$

$$\text{Volume} = \frac{\pi}{4} \int_0^h (\cos(2y) - 1)^2 dy \quad \text{for } 0 < h < \frac{\pi}{2}$$

[1A]

$$\mathbf{h. Volume} = \frac{\pi}{4} \int_0^h (\cos(2y) - 1)^2 dy$$

$$\frac{\pi}{4} \int_0^h (\cos(2y) - 1)^2 dy$$

Using double angle formula

$$\cos(4y) = 2\cos^2(2y) - 1$$

$$\therefore \cos^2(2y) = \frac{\cos(4y) + 1}{2}$$

$$= \frac{\pi}{4} \int_0^h (\cos^2(2y) - 2\cos(2y) + 1) dy$$

$$= \frac{\pi}{4} \int_0^h \left(\frac{\cos(4y) + 1}{2} - 2\cos(2y) + 1 \right) dy$$

$$= \frac{\pi}{4} \left[\frac{1}{8} \sin(4y) + \frac{y}{2} - \sin(2y) + y \right]_0^h \quad [1M]$$

$$= \frac{\pi}{4} \left(\frac{1}{8} \sin(4h) + \frac{3h}{2} - \sin(2h) \right)$$

$$\text{For } h = \frac{\pi}{4} \quad [1M]$$

$$\text{Volume} = \frac{\pi}{4} \left(\frac{1}{8} \sin(\pi) + \frac{3\pi}{8} - \sin\left(\frac{\pi}{2}\right) \right)$$

$$= \frac{\pi}{4} \left(\frac{3\pi}{8} - 1 \right) \text{ cm}^3 \quad [1A]$$