

The Mathematical Association of Victoria
SPECIALIST MATHEMATICS
2008 Trial written examination 1 – Worked Solutions

Question 1

a. $z = 6\text{cis}\left(\frac{5\pi}{6}\right)$

$$z^3 = \left(6\text{cis}\left(\frac{5\pi}{6}\right)\right)^3$$

$$z^3 = 6^3 \text{cis}\left(\frac{5\pi}{6} \times 3\right)$$

$$z^3 = 216\text{cis}\left(\frac{5\pi}{2}\right)$$

$$z^3 = 216\left(\cos\left(\frac{5\pi}{2}\right) + i\sin\left(\frac{5\pi}{2}\right)\right)$$

$$z^3 = 216(0 + i)$$

$$z^3 = 216i$$

$$\therefore m = 216(0 + i)$$

[A1]

- b.** There are three solutions of the equation $z^3 = 216i$. They are equally spaced around the circumference of a circle of radius 6. The angle between each solution is

$$\frac{2\pi}{3}.$$

The remaining two solutions are:

$$z = 6\text{cis}\left(\frac{5\pi}{6} - \frac{2\pi}{3}\right) \text{ and } z = 6\text{cis}\left(\frac{5\pi}{6} + \frac{2\pi}{3}\right)$$

$$z = 6\text{cis}\left(\frac{\pi}{6}\right) \qquad z = 6\text{cis}\left(-\frac{\pi}{2}\right)$$

[A1]

Alternative method of solution

$$z^3 = 216i$$

$$z^3 = 216 \operatorname{cis} \left(\frac{\pi}{2} \right)$$

$$z = \left(216 \operatorname{cis} \left(\frac{\pi}{2} + 2k\pi \right) \right)^{\frac{1}{3}} \text{ where } k = 0, \pm 1$$

$$z = 216^{\frac{1}{3}} \operatorname{cis} \frac{1}{3} \left(\frac{\pi}{2} + 2k\pi \right) \text{ by De Moivre's Theorem}$$

$$z = 6 \operatorname{cis} \left(\frac{\pi}{6} + \frac{2k\pi}{3} \right)$$

$$k = 0, \quad z = 6 \operatorname{cis} \left(\frac{\pi}{6} \right)$$

$$k = 1, \quad z = 6 \operatorname{cis} \left(\frac{\pi}{6} + \frac{2\pi}{3} \right) = 6 \operatorname{cis} \left(\frac{5\pi}{6} \right) \text{ (solution given in part a.)}$$

$$k = 1, \quad z = 6 \operatorname{cis} \left(\frac{\pi}{6} - \frac{2\pi}{3} \right) = 6 \operatorname{cis} \left(-\frac{\pi}{2} \right)$$

$$\text{The two other solutions are: } 6 \operatorname{cis} \left(\frac{\pi}{6} \right)$$

$$\text{and } 6 \operatorname{cis} \left(-\frac{\pi}{2} \right)$$

Question 2

a. $u = 1 + i$

$$u = r \operatorname{cis}(\theta) \quad \text{where } r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\text{and } \theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$u = \sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right) \quad [\text{A1}]$$

b. $uv = \sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right) \times 2 \operatorname{cis}\left(-\frac{\pi}{6}\right)$

$$uv = 2\sqrt{2} \operatorname{cis}\left(\frac{\pi}{4} + \left(-\frac{\pi}{6}\right)\right)$$

$$uv = 2\sqrt{2} \operatorname{cis}\left(\frac{\pi}{12}\right) \quad [\text{A1}]$$

c. $v = 2 \operatorname{cis}\left(-\frac{\pi}{6}\right)$

$$v = 2 \cos\left(-\frac{\pi}{6}\right) + 2 \sin\left(-\frac{\pi}{6}\right) i$$

$$v = \sqrt{3} - i \quad [\text{A1}]$$

d. $uv = (1 + i)(\sqrt{3} - i)$

$$uv = \sqrt{3} - i + i\sqrt{3} + 1$$

$$uv = (\sqrt{3} + 1) + (\sqrt{3} - 1) i \quad [\text{A1}]$$

e. From **b.** and **d.**

$$uv = 2\sqrt{2} \operatorname{cis}\left(\frac{\pi}{12}\right) = (\sqrt{3} + 1) + (\sqrt{3} - 1) i$$

$$uv = 2\sqrt{2} \cos\left(\frac{\pi}{12}\right) + 2\sqrt{2} \sin\left(\frac{\pi}{12}\right) i = (\sqrt{3} + 1) + (\sqrt{3} - 1) i$$

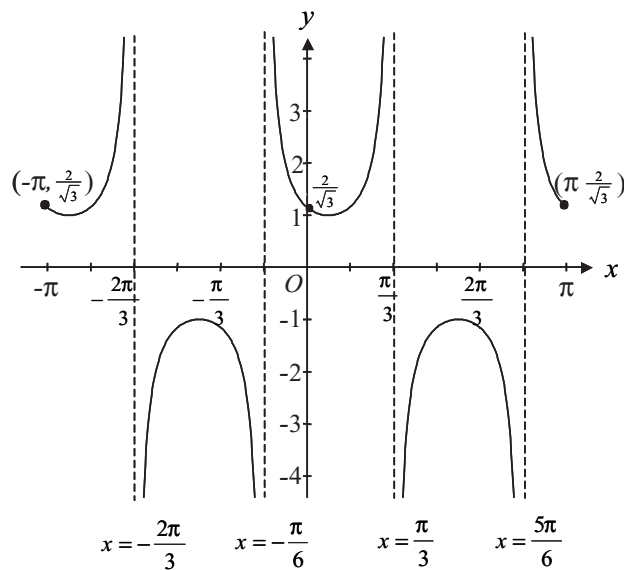
Equating imaginary components:

$$2\sqrt{2} \sin\left(\frac{\pi}{12}\right) = (\sqrt{3} - 1)$$

$$\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{3} - 1}{2\sqrt{2}} \quad [\text{A1}]$$

$$\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{2}}{4}(\sqrt{3} - 1) \quad \text{or} \quad \frac{1}{4}(\sqrt{6} - \sqrt{2}) \quad (\text{rationalised})$$

Question 3



Shape [A1]
Asymptotes [A1]
Endpoints [A1]

Endpoint coordinates:

$$x = \pi \quad f(\pi) = \operatorname{cosec}\left(2\pi + \frac{\pi}{3}\right) = \frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{2}{\sqrt{3}} \quad \left(\pi, \frac{2}{\sqrt{3}}\right)$$

$$\therefore x = -\pi, \quad f(-\pi) = \frac{2}{\sqrt{3}} \quad \left(-\pi, \frac{2}{\sqrt{3}}\right)$$

y-intercept:

$$x = 0, \quad f(0) = \operatorname{cosec}\left(0 + \frac{\pi}{3}\right) = \frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{2}{\sqrt{3}} \quad \left(0, \frac{2}{\sqrt{3}}\right)$$

Asymptotes:

$$2x + \frac{\pi}{3} = -\pi, \quad 0, \quad \pi, \quad 2\pi$$

$$2x = -\frac{4\pi}{3}, \quad -\frac{\pi}{3}, \quad \frac{2\pi}{3}, \quad \frac{5\pi}{3}$$

$$x = -\frac{2\pi}{3}, \quad -\frac{\pi}{6}, \quad \frac{\pi}{3}, \quad \frac{5\pi}{6}$$

Question 4

Using implicit differentiation:

$$x \sin(y) = 1$$

$$1 \cdot \sin(y) + x \cos(y) \frac{dy}{dx} = 0 \quad [\text{M1}]$$

$$\frac{dy}{dx} = -\frac{\sin(y)}{x \cos(y)}$$

$$\frac{dy}{dx} = -\frac{\tan(y)}{x} \quad [\text{A1}]$$

$$\text{When } y = \frac{\pi}{6}, x = \frac{1}{\sin\left(\frac{\pi}{6}\right)} = 2$$

$$\frac{dy}{dx} = -\frac{\tan\left(\frac{\pi}{6}\right)}{2} = -\frac{1}{2\sqrt{3}}$$

$$\frac{dy}{dx} = -\frac{\sqrt{3}}{6} \quad [\text{A1}]$$

Question 5**a. Finding x -intercepts:**

$$0 = x - 2\sqrt{\frac{3}{8-x^2}}$$

$$2\sqrt{\frac{3}{8-x^2}} = x$$

$$4\left(\frac{3}{8-x^2}\right) = x^2$$

[M1]

$$12 = x^2(8-x^2)$$

$$x^4 - 8x^2 + 12 = 0$$

$$(x^2 - 2)(x^2 - 6) = 0$$

[A1]

$$x = \pm \sqrt{2} \text{ or } x = \pm \sqrt{6}$$

$$\therefore m = \sqrt{2} \text{ and } n = \sqrt{6}$$

$$\text{b. Shaded area} = \int_{\sqrt{2}}^{\sqrt{6}} \left(x - 2\sqrt{\frac{3}{8-x^2}} \right) dx$$

$$\int_{\sqrt{2}}^{\sqrt{6}} x \, dx - 2\sqrt{3} \int_{\sqrt{2}}^{\sqrt{6}} \frac{1}{\sqrt{8-x^2}} \, dx$$

$$= \left[\frac{1}{2}x^2 - 2\sqrt{3} \sin^{-1}\left(\frac{x}{\sqrt{8}}\right) \right]_{\sqrt{2}}^{\sqrt{6}}$$

[A1]

$$= \left[\frac{1}{2}(\sqrt{6})^2 - 2\sqrt{3} \sin^{-1}\left(\frac{\sqrt{6}}{\sqrt{8}}\right) \right] - \left[\frac{1}{2}(\sqrt{2})^2 - 2\sqrt{3} \sin^{-1}\left(\frac{\sqrt{2}}{\sqrt{8}}\right) \right]$$

$$= \left[3 - 2\sqrt{3} \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \right] - \left[1 - 2\sqrt{3} \sin^{-1}\left(\frac{1}{2}\right) \right]$$

[A1]

$$= \left[3 - 2\sqrt{3} \times \frac{\pi}{3} \right] - \left[1 - 2\sqrt{3} \times \frac{\pi}{6} \right]$$

$$= 2 - \frac{\pi\sqrt{3}}{3} \text{ square units}$$

[A1]

Question 6

$$y = \int \frac{1+x}{(1-x)^2} dx$$

$$\text{Let } u = 1 - x \quad x = 1 - u$$

$$\frac{du}{dx} = -1 \quad 1 + x = 2 - u \quad [\text{A1}]$$

$$dx = -du$$

$$y = \int \frac{2-u}{u^2} (-du) \quad [\text{A1}]$$

$$y = \int \left(\frac{1}{u} - \frac{2}{u^2} \right) du$$

$$y = \log_e |u| + \frac{2}{u} + c$$

$$y = \log_e |1-x| + \frac{2}{1-x} + c, \quad x \neq 1 \quad [\text{A1}]$$

When $y = 0, x = 0$

$$0 = \log_e |1-0| + \frac{2}{1-0} + c$$

$$c = -2$$

$$\therefore y = \log_e |1-x| + \frac{2}{1-x} - 2, \quad x \neq 1 \quad [\text{A1}]$$

Since we are dealing with the part of the function where $x = 0$, we only need consider $x < 1$.

$$\text{Hence solution is } y = \log_e (1-x) + \frac{2}{1-x} - 2, \quad x \neq 1$$

Alternative method of solution using partial fractions:

$$\frac{1+x}{(1-x)^2} = \frac{A}{1-x} + \frac{B}{(1-x)^2}, \quad x \neq 1$$

$$1+x = A(1-x) + B$$

$$\text{Let } x = 1, B = 2$$

$$\text{Let } x = 0, A + B = 1 \text{ and so } A = -1$$

$$\int \frac{1+x}{(1-x)^2} dx = \int \left(\frac{-1}{1-x} + \frac{2}{(1-x)^2} \right) dx$$

$$= \log_e |1-x| + \frac{2}{1-x} - c, \quad x \neq 1$$

Continued as shown above to find constant, c , etc.

Question 7

Let u be the vector resolute of a parallel to b .

$$u = \left(a \cdot \hat{b} \right) \cdot \hat{b} \quad \text{where } \hat{b} = \frac{\begin{pmatrix} -i + 2j - k \end{pmatrix}}{\sqrt{(-1)^2 + 2^2 + (-1)^2}} \\ = \frac{1}{\sqrt{6}} \begin{pmatrix} -i + 2j - k \end{pmatrix}$$

$$u = \left(\begin{pmatrix} i - j + k \end{pmatrix} \cdot \frac{1}{\sqrt{6}} \begin{pmatrix} -i + 2j - k \end{pmatrix} \right) \frac{1}{\sqrt{6}} \begin{pmatrix} -i + 2j - k \end{pmatrix}$$

[A1]

$$u = \frac{1}{6} (-1 - 2 - 1) \begin{pmatrix} -i + 2j - k \end{pmatrix}$$

$$u = -\frac{4}{6} \begin{pmatrix} -i + 2j - k \end{pmatrix}$$

$$u = \frac{2}{3} \begin{pmatrix} i - 2j + k \end{pmatrix}$$

[A1]

Let v be the vector resolute of a perpendicular to b .

$$v = a - u$$

$$v = \begin{pmatrix} i - j + k \end{pmatrix} - \frac{2}{3} \begin{pmatrix} i - 2j + k \end{pmatrix}$$

$$v = \frac{1}{3}i + \frac{1}{3}j + \frac{1}{3}k$$

$$v = \frac{1}{3} \begin{pmatrix} i + j + k \end{pmatrix}$$

[A1]

Question 8

Since $OABC$ is a parallelogram, $\vec{CB} = \vec{a}$ and $\vec{CM} = \frac{1}{2}\vec{a}$

$$\vec{OM} = \vec{OC} + \vec{CM} = \vec{c} + \frac{1}{2}\vec{a}$$

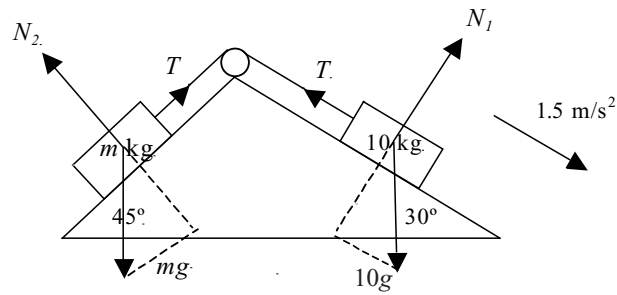
$$\vec{OP} = \frac{2}{3}\vec{OM} = \frac{2}{3}\left(\vec{c} + \frac{1}{2}\vec{a}\right) = \frac{2}{3}\vec{c} + \frac{1}{3}\vec{a} \quad [\text{M1}]$$

$$\vec{PC} = \vec{PO} + \vec{OC} = -\left(\frac{2}{3}\vec{c} + \frac{1}{3}\vec{a}\right) + \vec{c} = -\frac{1}{3}\vec{a} + \frac{1}{3}\vec{c} \quad [\text{A1}]$$

$$\vec{AP} = \vec{AO} + \vec{OP} = -\vec{a} + \left(\frac{2}{3}\vec{c} + \frac{1}{3}\vec{a}\right) = -\frac{2}{3}\vec{a} + \frac{2}{3}\vec{c} = 2\left(-\frac{1}{3}\vec{a} + \frac{1}{3}\vec{c}\right) = 2\vec{PC} \quad [\text{A1}]$$

$\therefore \vec{AP} = 2\vec{PC}$ as required.

Question 9



- a. The 10kg mass is moving down the plane.

The component of the weight force parallel to the plane is $10g \sin(30^\circ)$.

Resolving forces parallel to the plane:

$$10g \sin(30^\circ) - T = 10a$$

[A1]

$$T = 10 \times 9.8 \sin(30^\circ) - 10 \times 1.5$$

$$T = 34 \text{ newtons}$$

[A1]

- b. The m kg mass is moving up the plane.

The component of the weight force parallel to the plane is $mg \sin(45^\circ)$.

Resolving forces parallel to the plane:

$$T - mg \sin(45^\circ) = ma$$

[A1]

$$34 - mg \times \frac{\sqrt{2}}{2} = m \times 1.5$$

$$68 - mg\sqrt{2} = 3m$$

$$m(g\sqrt{2} + 3) = 68$$

$$m = \frac{68}{g\sqrt{2} + 3}$$

[A1]

$$\therefore a = 68, b = 2, c = 3$$

Question 10

- a. Differentiating the parametric equations.

$$\frac{dx}{dt} = \frac{1(t^2 + 1) - t(2t)}{(t^2 + 1)^2} \qquad \frac{dy}{dt} = \frac{-2t}{(t^2 + 1)^2}$$

$$\frac{dx}{dt} = \frac{1 - t^2}{(t^2 + 1)^2} \qquad \qquad \qquad \text{[A1]}$$

Finding $\frac{dy}{dx}$ using the chain rule.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} \\ \frac{dy}{dx} &= \frac{-2t}{(t^2 + 1)^2} \times \frac{(t^2 + 1)^2}{1 - t^2} \end{aligned} \qquad \qquad \qquad \text{[M1]}$$

$$\frac{dy}{dx} = \frac{2t}{t^2 - 1}$$

$$\text{When } t = 2, \quad \frac{dy}{dx} = \frac{2 \times 2}{2^2 - 1} = \frac{4}{3} \qquad \qquad \qquad \text{[A1]}$$

b. Finding the Cartesian equation by eliminating the parameter.

$$x = \frac{t}{t^2 + 1} \quad y = \frac{1}{t^2 + 1}$$

$$x = t \left(\frac{1}{t^2 + 1} \right)$$

$$x = t \times y$$

$$\frac{x}{y} = t$$

$$\frac{x^2}{y^2} + 1 = t^2 + 1$$

[M1]

$$\frac{y^2}{x^2 + y^2} = \frac{1}{t^2 + 1}$$

$$\frac{y^2}{x^2 + y^2} = y$$

$$y = x^2 + y^2$$

[A1]

$$x^2 + y^2 - y + \frac{1}{4} = \frac{1}{4}$$

$$x^2 + \left(y - \frac{1}{2} \right)^2 = \left(\frac{1}{2} \right)^2$$

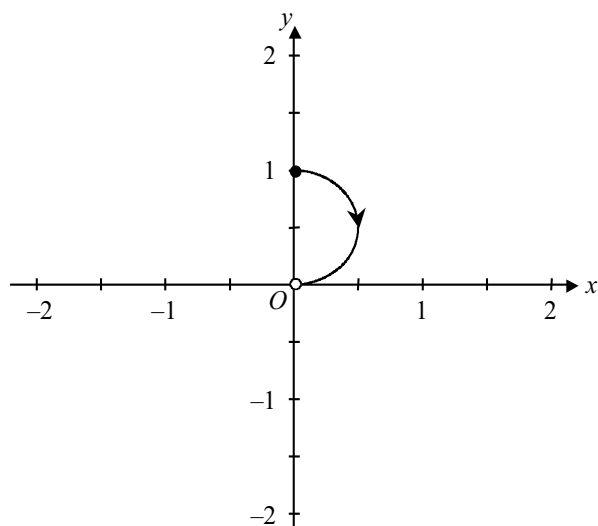
Curve is a circle with centre $\left(0, \frac{1}{2} \right)$ and radius $\frac{1}{2}$

c. Sketching circle centre $\left(0, \frac{1}{2}\right)$ and radius $\frac{1}{2}$ for $t \geq 0$

When $t = 0$ $x = 0$, $y = 1$

When $t = 1$ $x = \frac{1}{2}$, $y = \frac{1}{2}$

As $t \rightarrow \infty$, $x \rightarrow 0$, $y \rightarrow 0$



Shape and position [A1]

End point (0, 1) included, end point (0, 0) excluded [A1]

Direction of motion not required