

Year 2008
VCE
Specialist Mathematics
Solutions
Trial Examination 2



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SECTION 1

ANSWERS

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E
12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E
21	A	B	C	D	E
22	A	B	C	D	E

SECTION 1

Question 1

Answer D

The x -axis is a horizontal asymptote, option **A.** is true.

When $x = 0$ $y = \frac{1}{c}$ as the y -intercept, option **B.** is true.

When $-4 - 2x = 0 \Rightarrow x = -2$ is a minimum turning point. option **C.** is true.

The quadratic in the denominator $c - 4x - x^2$ has a discriminant of

$$\Delta = (-4)^2 - 4 \times 1 \times c = 16 + 4c = 4(4 + c) \text{ so}$$

If $\Delta > 0$ $c > -4$ the quadratic has two real solutions, and hence $f(x)$ has two vertical asymptotes, so option **D.** is false.

If $\Delta = 0$ $c = -4$ the quadratic has one (repeated) real solution, and hence $f(x)$ has one vertical asymptote **E.** is true.

Question 2

Answer B

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\text{crosses the } x\text{-axis when } y = 0 \quad \frac{(x-h)^2}{a^2} - \frac{(-k)^2}{b^2} = 1 \Rightarrow \frac{(x-h)^2}{a^2} = \frac{k^2}{b^2} + 1 = \frac{k^2 + b^2}{b^2}$$

$$(x-h)^2 = \frac{a^2(k^2 + b^2)}{b^2} \Rightarrow x = h \pm \frac{a}{b} \sqrt{k^2 + b^2} \text{ option A. is true.}$$

$$\text{crosses the } y\text{-axis when } x = 0 \quad \frac{(-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \Rightarrow \frac{(y-k)^2}{b^2} = \frac{h^2}{a^2} - 1 = \frac{h^2 - a^2}{a^2}$$

$$(y-k)^2 = \frac{b^2(h^2 - a^2)}{a^2} \Rightarrow y = k \pm \frac{b}{a} \sqrt{h^2 - a^2} \text{ option B. is false.}$$

All the other options **C.** **D.** and **E.** are true

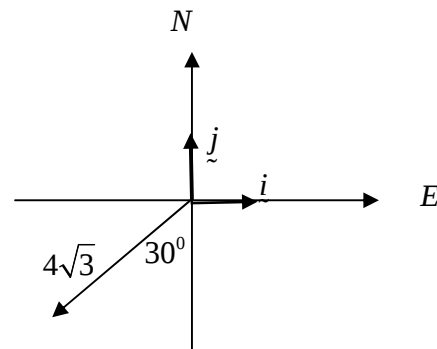
Question 3

Answer C

$$\vec{F} = -4\sqrt{3} \sin(30^\circ) \hat{i} - 4\sqrt{3} \cos(30^\circ) \hat{j}$$

$$\vec{F} = -4\sqrt{3} \left(\frac{1}{2} \hat{i} + \frac{\sqrt{3}}{2} \hat{j} \right)$$

$$\vec{F} = -2\sqrt{3} \hat{i} - 6 \hat{j}$$



Question 4**Answer B**

$$\underline{r}(t) = (\cos(t) - \sin(t))\underline{i} + (\cos(t) + \sin(t))\underline{j}$$

$$x = \cos(t) - \sin(t) \quad \text{and} \quad y = \cos(t) + \sin(t)$$

$$\frac{x+y}{2} = \cos(t) \quad \frac{y-x}{2} = \sin(t) \quad \text{since} \quad \cos^2(t) + \sin^2(t) = 1$$

$$\left(\frac{x+y}{2}\right)^2 + \left(\frac{y-x}{2}\right)^2 = 1 \Rightarrow x^2 + 2xy + y^2 + y^2 - 2xy + x^2 = 4$$

$x^2 + y^2 = 2$ is the Cartesian equation, which is a circle.

Question 5**Answer A**

$$y = \frac{ax^3 + b}{x^2} = ax + \frac{b}{x^2}$$

A sketch graph with $a = 1$ $b = 2$

shows a minimum turning point.

All the other options **C**, **D**, and **E**, are true

If $a < 0$ and $b > 0$ the graph has a

minimum turning point, option **B**, is true

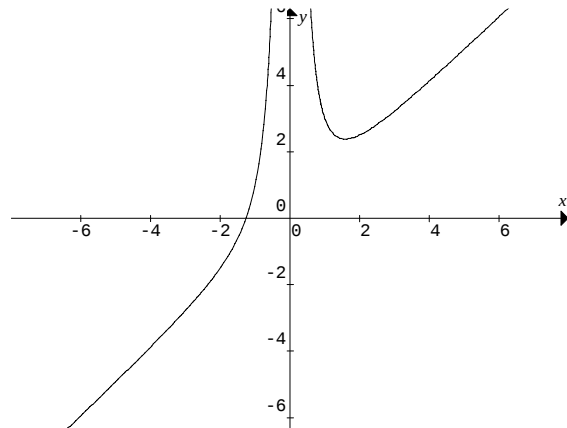
The graph has $x = 0$ as a horizontal

asymptote, option **C**, is true

The graph has the line $y = ax$ as an oblique

asymptote,

option **D**, is true



The graph has an x-intercept when $y = 0$ when $ax^3 + b = 0 \Rightarrow ax^3 = -b$ at $x = \sqrt[3]{-\frac{b}{a}}$,

option **E**, is true.

Question 6**Answer D**

$$z^4 + a^4 i = 0$$

$$z^4 = -a^4 i = a^4 \operatorname{cis}\left(-\frac{\pi}{2} + 2k\pi\right) \quad k \in \mathbb{Z}$$

$$z = a \operatorname{cis}\left(-\frac{\pi}{8} + \frac{k\pi}{2}\right)$$

so one root when $k = 0$ is $z = a \operatorname{cis}\left(-\frac{\pi}{8}\right)$, which lies on the circle of radius a at an

angle of -22.5° from the positive real axis, all the other roots must be equally spaced and on the circle of radius a , the roots do not occur in conjugate pairs, so option **D**, is correct.

Question 7**Answer B**

$$\text{let } z = a + bi \quad \text{Arg}(z) = \theta$$

$$a > 0 \quad b > 0 \Rightarrow 0 < \theta < \frac{\pi}{2}$$

$$i^3 z = -i(a + bi)$$

$$= -ai - bi^2$$

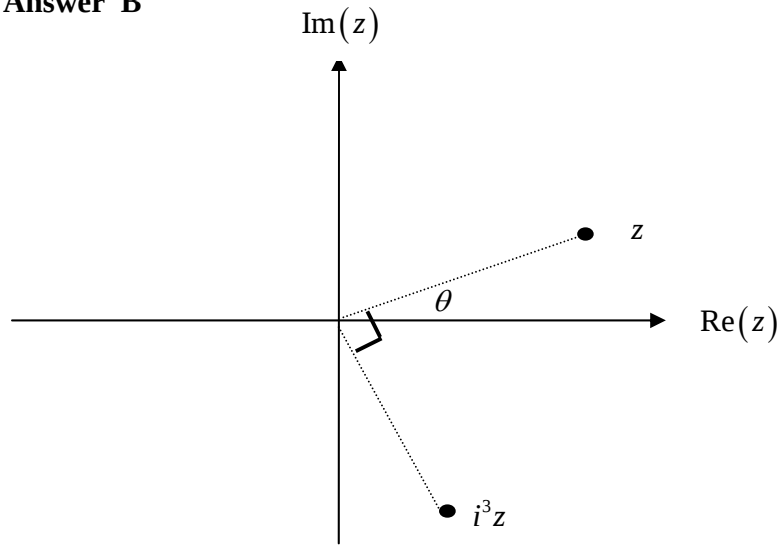
$$= b - ai$$

Since i is a rotation 90° anti-clockwise, i^3 is a rotation 270° anti-clockwise

$$\arg(b - ai) = \theta + \frac{3\pi}{2}$$

$$\text{Arg}(b - ai) = \theta - \frac{\pi}{2}$$

Since $\text{Arg}(z)$ must be in $(-\pi, \pi]$

**Question 8****Answer E**

$$P(z) = z^3 + bz^2 + cz + d$$

Since $P(ki) = 0$ and b, c, d and k are all real, by the conjugate root theorem

$$P(-ki) = 0$$

option **A.** is true, so $(z - ki)(z - (-ki)) = (z - ki)(z + ki) = z^2 - k^2 i^2 = z^2 + k^2$ is a factor

option **E.** is false

All the other options **B.** **C.** and **D.** are true, since

$$z^3 + bz^2 + cz + d = (z^2 + k^2) \left(z + \frac{d}{k^2} \right) = z^3 + \frac{d}{k^2} z^2 + k^2 z + d$$

From the coefficient of z : $k^2 = c$ option **B.** is true.

From the coefficient of z^2 : $b = \frac{d}{k^2} = \frac{d}{c} \Rightarrow bc = d$ option **C.** is true.

$\left(z + \frac{d}{k^2} \right)$ is a factor of $P(z)$ option **D.** is true.

Question 9**Answer E**

$$\cos\left(\pi - \cos^{-1}\left(\frac{1}{3}\right)\right) = \cos(\pi)\cos\left(\cos^{-1}\left(\frac{1}{3}\right)\right) - \sin(\pi)\sin\left(\cos^{-1}\left(\frac{1}{3}\right)\right) = -\frac{1}{3}$$

$$\cos\left(\pi - \cos^{-1}\left(\frac{1}{3}\right)\right) = -\frac{1}{3} = \frac{y}{\sqrt{x^2 + y^2 + 4}} \quad \text{we require } y < 0$$

only $x = \pm 2$ and $y = -1$ satisfy this.

Question 10**Answer E**

All the other options **A. B. C.** and **D.** are true, option **E.** is false

The graph of $f(x) = \frac{ab}{b^2x^2 + 1}$ has a turning point when

$$f'(x) = -\frac{2ab^3x}{(b^2x^2 + 1)^2} = 0 \Rightarrow x = 0 \quad \text{and} \quad f(0) = ab, \text{ the range is } (0, ab]$$

Question 11**Answer D**

Option **A.** Let $s = 2x + 9$ $x = 8$ $s = 25$ and $x = 0$ $s = 9$ $x = \frac{1}{2}(s - 9)$ $dx = \frac{1}{2}ds$

$$\int_0^8 \frac{x}{\sqrt{2x+9}} dx = \frac{1}{4} \int_9^{25} \frac{s-9}{\sqrt{s}} ds \quad \text{is true.}$$

Option **B.** Let $t = x$ $\int_0^8 \frac{x}{\sqrt{2x+9}} dx = \int_4^8 \frac{t}{\sqrt{2t+9}} dt$ is true.

Option **C.** Let $u = 2x$ $x = 8$ $u = 16$ and $x = 0$ $u = 0$ $x = \frac{u}{2}$ $dx = \frac{1}{2}du$

$$\int_0^8 \frac{x}{\sqrt{2x+9}} dx = \frac{1}{4} \int_0^{16} \frac{u}{\sqrt{u+9}} du \quad \text{is true.}$$

Option **D.** Let $2v = 2x + 9$ $x = 8$ $v = \frac{25}{2}$ and $x = 0$ $v = \frac{9}{2}$ $x = \frac{(2v-9)}{2}$ $dx = dv$

$$\int_0^8 \frac{x}{\sqrt{2x+9}} dx \neq \frac{1}{2} \int_9^{25} \frac{2v-9}{\sqrt{2v}} dv \quad \text{D. is false}$$

Option **E.** Let $z^2 = 2x + 9$

$$x = 8 \quad z = 5 \quad \text{and} \quad x = 0 \quad z = 3 \quad x = \frac{1}{2}(z^2 - 9) \quad 2dx = 2z dz$$

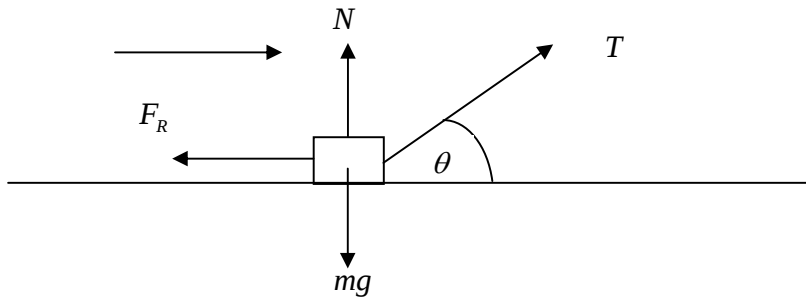
$$\int_0^8 \frac{x}{\sqrt{2x+9}} dx = \frac{1}{2} \int_3^5 (z^2 - 9) dz \quad \text{is true.}$$

Question 12**Answer D**

The period $T = \frac{2\pi}{n} = 2a \Rightarrow n = \frac{\pi}{a}$ this is true for graphs of $\sin$, $\cos$, $\sec$ and $\csc$,

however the period of $\tan$ and $\cot$ graphs are $T = \frac{\pi}{n} = 2a \Rightarrow n = \frac{\pi}{2a}$, options **A. B. C.**

and **E.** are correct, option **D.** is false, its period is a

Question 13**Answer A**

$$T = 10g \text{ newtons} \quad \theta = 30^\circ \quad m = 20 \text{ kg} \quad \mu = ?$$

$$\text{resolving parallel to the plane} \quad (1) \quad T \cos(\theta) - F_R = 0$$

$$\text{resolving perpendicular to the plane} \quad (2) \quad T \sin(\theta) + N - mg = 0$$

$$\text{from (1)} \quad F_R = T \cos(\theta) = 10g \times \frac{\sqrt{3}}{2} = 5\sqrt{3}g$$

$$\text{from (2)} \quad N = mg - T \sin(\theta) = 20g - 10g \sin(30^\circ) = 15g$$

$$\text{for equilibrium to be maintained, } F_R \leq \mu N \text{ so that } 5\sqrt{3}g \leq 15\mu g \Rightarrow \mu \geq \frac{\sqrt{3}}{3}$$

Question 14**Answer A**

resolving horizontally gives

$$T_1 \sin(46^\circ) = T_2 \sin(44^\circ) \Rightarrow \frac{T_1}{T_2} = \frac{\sin(44^\circ)}{\sin(46^\circ)} = \frac{\sin(44^\circ)}{\cos(44^\circ)} = \tan(44^\circ)$$

Question 15**Answer A**

$$\underline{a} = x\underline{i} + y\underline{j} - 2\underline{k} \text{ and } \underline{b} = -3\underline{i} + 2\underline{j} + \underline{k} \text{ A. is false}$$

If $x = 9$ and $y = -6$ then vectors $\underline{a}$ and $\underline{b}$ are collinear is falseIf $x = 6$ and $y = -4$ the vector $\underline{a}$ is parallel to the vector $\underline{b}$, is true, $\underline{a} = -2\underline{b}$, option **B.** is true.If $x = y = \sqrt{5}$ then $|\underline{a}| = |\underline{b}|$, is true, $|\underline{a}| = \sqrt{x^2 + y^2 + 4} = \sqrt{14}$ and $|\underline{b}| = \sqrt{9 + 4 + 1} = \sqrt{14}$, option **C.** is trueIf $x = -4$ and $y = -5$ the vector $\underline{a}$ is perpendicular to the vector $\underline{b}$, is true, since $\underline{a} \cdot \underline{b} = -3x + 2y - 2 = 12 - 10 - 2 = 0$, option **D.** is true.

$$\text{If } x = 0 \text{ and } y = 2 \text{ then the scalar resolute of } \underline{a} \text{ in the direction of } \underline{b} \text{ is } \underline{a} \cdot \hat{\underline{b}} = \frac{\underline{a} \cdot \underline{b}}{|\underline{b}|} = \frac{2}{\sqrt{14}}$$

Option **E.** is true.

Question 16**Answer C**

When $x = 2$ $y_2 = \cos^{-1}(1) = \frac{\pi}{2}$, let $y_1 = \cos^{-1}\left(\frac{x}{2}\right)$. The volume required is

$V_x = \pi \int_a^b (y_2^2 - y_1^2) dx$ where y_1 and y_2 are the inner and outer radii respectively,

$$V = \pi \int_0^2 \left(\frac{\pi^2}{4} - \left(\cos^{-1}\left(\frac{x}{2}\right) \right)^2 \right) dx$$

Question 17**Answer A**

$\frac{dQ}{dt} = \text{inflow} - \text{outflow} = bg - \frac{fQ}{V}$ and the volume $V = V(t) = V_0 + (g - f)t$

$$\frac{dQ}{dt} = bg - \frac{fQ}{V_0 + (g - f)t} \quad Q(0) = a$$

Question 18**Answer C**

Using constant acceleration formulae $a = -9.8$ $u = 0$ $t = 8$ $s = ?$

$$s = ut + \frac{1}{2}at^2 \quad s = 0 - \frac{1}{2} \times 9.8 \times 8^2 = -313.6$$

The height h is 313 metres. The stone hits the ground 314 metres below the initial point.

Question 19**Answer C**

$\dot{\underline{r}}(t) = 4 \sin(2t) \underline{i} + 6e^{-\frac{t}{3}} \underline{j}$ integrating

$$\underline{r}(t) = \int 4 \sin(2t) dt \underline{i} + \int 6e^{-\frac{t}{3}} dt \underline{j}$$

$$\underline{r}(t) = -2 \cos(2t) \underline{i} - 18e^{-\frac{t}{3}} \underline{j} + \underline{c} \quad \text{to find } \underline{c}$$

$$\underline{r}(0) = \underline{0} = -2\underline{i} - 18\underline{j} + \underline{c} \Rightarrow \underline{c} = 2\underline{i} + 18\underline{j}$$

$$\underline{r}(t) = 2(1 - \cos(2t)) \underline{i} + 18 \left(1 - e^{-\frac{t}{3}} \right) \underline{j} \quad \text{since } \sin^2(A) = \frac{1}{2}(1 - \cos(2A))$$

$$\underline{r}(t) = 4 \sin^2(t) \underline{i} + 18 \left(1 - e^{-\frac{t}{3}} \right) \underline{j}$$

Question 20**Answer E**

$$\frac{dx}{dt} = e^{t^2}$$

$$x = \int_0^t e^{u^2} du + C \quad \text{now to find } C, x=1 \text{ when } t=0,$$

$$1 = \int_0^0 e^{u^2} du + C \Rightarrow C = 1$$

$$x = \int_0^t e^{u^2} du + 1 \quad \text{now when } t=2 \quad x = \int_0^2 e^{u^2} du + 1$$

Question 21**Answer D**

From the velocity time graph, using a calculator, the graph crosses the t -axis at $t = 0.77$.

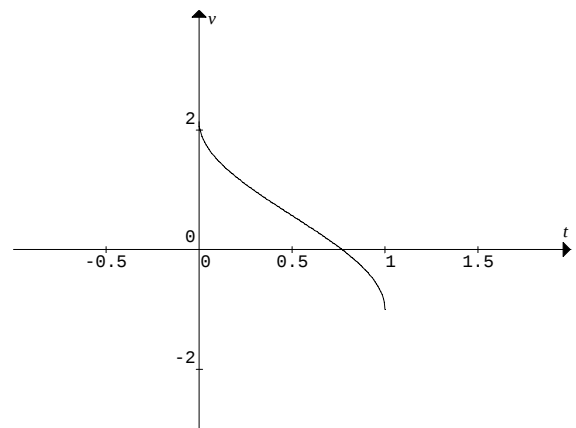
The distance travelled over $t \in [0, 1]$ is

the total area bounded by the graph and the t -axes. The area from

$t = 0.77$ to $t = 1$ is below the axes and negative,

the total area, or distance travelled is

$$\int_0^{0.77} (\cos^{-1}(2t-1) - 1) dt - \int_{0.77}^1 (\cos^{-1}(2t-1) - 1) dt$$

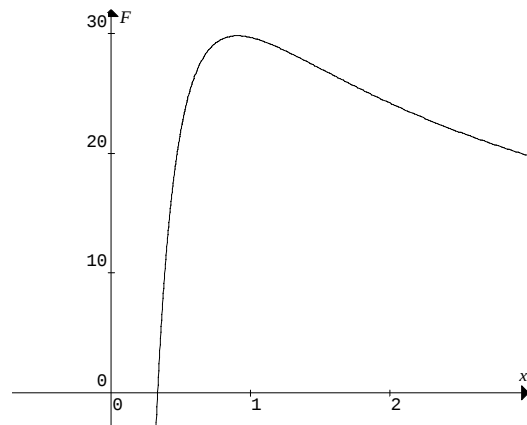
**Question 22****Answer B**

$$m = 3 \text{ kg} \quad v = 3 \log_e(3x) \Rightarrow \frac{dv}{dx} = \frac{3}{x}$$

$$F = ma = mv \frac{dv}{dx} = 3(3 \log_e(3x)) \frac{3}{x} = \frac{27 \log_e(3x)}{x}$$

Using a calculator, this graph has a maximum at $x = 0.87$ $F_{\max} = 29.6$,

the maximum force is closest to 30.

**END OF SECTION 1 SUGGESTED ANSWERS**

SECTION 2

Question 1

a. $u = 1 + 3i$ $\bar{u} = 1 - 3i$ $u + \bar{u} = 2$ $u\bar{u} = 1 - 9i^2 = 10$

so that $z^2 - 2z + 10$ is a factor of $z^3 - 4z^2 + bz + c = 0$

M1

$$z^3 - 4z^2 + bz + c = (z^2 - 2z + 10)(z + d)$$

coefficient of z^2 : $d - 2 = -4 \Rightarrow d = -2$

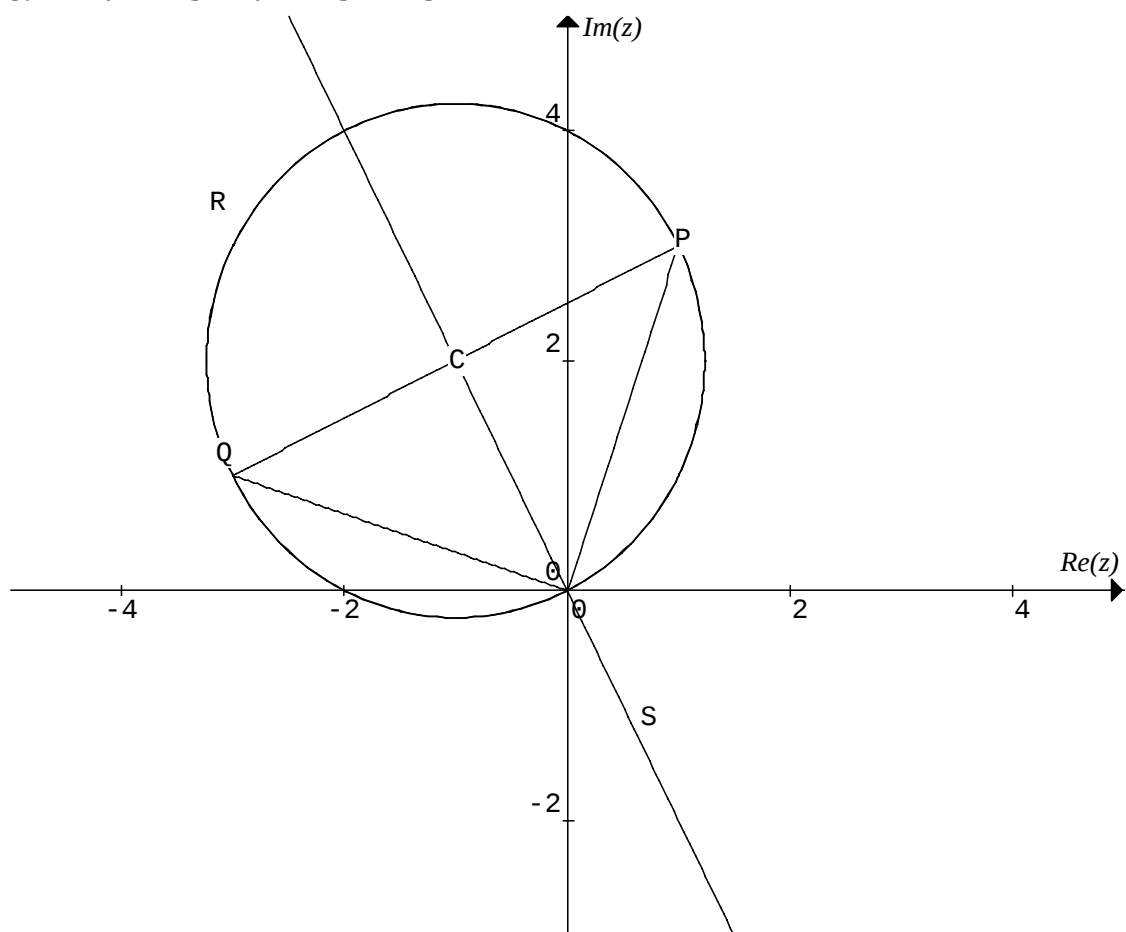
coefficient of z^1 : $10 - 2d = b \Rightarrow b = 14$

A1

coefficient of z^0 : $10d = c \Rightarrow c = -20$

A1

b. $u = 1 + 3i$ $iu = i + 3i^2 = -3 + i$



diagram

A1

c. $\overrightarrow{OP} = \underline{i} + 3\underline{j}$ $\overrightarrow{OQ} = -3\underline{i} + \underline{j}$

$\overrightarrow{OP} \cdot \overrightarrow{OQ} = -3 + 3 = 0$ so that $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{OQ}$

A1

d. $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = -4\underline{i} - 2\underline{j}$ $|\overrightarrow{PQ}| = \sqrt{16 + 4} = 2\sqrt{5}$

A1

- e. $p = 1 + 3i$ $q = -3 + i$
 $S = \{z : |z - p| = |z - q|\}$
 $|z - (1 + 3i)| = |z - (-3 + i)|$ let $z = x + yi$ M1
 $\sqrt{(x-1)^2 + (y-3)^2} = \sqrt{(x+3)^2 + (y-1)^2}$ square both sides and expand
 $x^2 - 2x + 1 + y^2 - 6y + 9 = x^2 + 6x + 9 + y^2 - 2y + 9$
 $y = -2x$ A1
- f. PQ is the diameter of the circle
the centre of the circle is at $C(-1, 2)$ the mid-point of PQ ,
so that $c = -1 + 2i$ A1
and the radius of the circle is $r = \frac{1}{2}|PQ| = |c| = \sqrt{5}$ A1
 $R = \{z : |z - c| = r\}$
 $|z - (-1 + 2i)| = \sqrt{5}$ let $z = x + yi$
 $\sqrt{(x+1)^2 + (y-2)^2} = \sqrt{5}$ square both sides and expand
 $x^2 + 2x + 1 + y^2 - 4y + 4 = 5$
 $x^2 + 2x + y^2 - 4y = 0$ or $(x+1)^2 + (y-2)^2 = 5$ A1

Question 2

- a. $O(0,0)$, $A(a,b)$, $B(-a,b)$, $D(0,-r)$
 $\vec{OA} = a\vec{i} + b\vec{j}$ $\vec{OB} = -a\vec{i} + b\vec{j}$
 $|\vec{OA}| = \sqrt{a^2 + b^2}$ and $|\vec{OB}| = \sqrt{(-a)^2 + b^2} = \sqrt{a^2 + b^2}$
since $|\vec{OB}| = |\vec{OA}| = \sqrt{a^2 + b^2} = r$ both are radii of the circle A1
 $\cos(\theta) = \frac{\vec{OA} \cdot \vec{OB}}{|\vec{OA}| |\vec{OB}|} = \frac{-a^2 + b^2}{r^2}$ A1
 $\cos(\theta) = \frac{b^2 - a^2}{a^2 + b^2}$ A1
- b. $\vec{OD} = -r\vec{j}$ $|\vec{OD}| = r$
 $\vec{DB} = \vec{DO} + \vec{OB} = \vec{OB} - \vec{OD} = -a\vec{i} + (b+r)\vec{j}$ $|\vec{DB}| = \sqrt{a^2 + (b+r)^2}$
 $\vec{DA} = \vec{DO} + \vec{OA} = \vec{OA} - \vec{OD} = a\vec{i} + (b+r)\vec{j}$ $|\vec{DA}| = \sqrt{a^2 + (b+r)^2}$ A1

$$\cos(\alpha) = \frac{\overrightarrow{DB} \cdot \overrightarrow{DA}}{|\overrightarrow{DB}| |\overrightarrow{DA}|} = \frac{-a^2 + (b+r)^2}{a^2 + (b+r)^2} \quad \text{M1}$$

$$\cos(\alpha) = \frac{b^2 + 2br + r^2 - a^2}{a^2 + b^2 + 2br + r^2} \quad \text{but } r^2 = a^2 + b^2$$

$$\cos(\alpha) = \frac{b^2 + 2b\sqrt{a^2 + b^2} + a^2 + b^2 - a^2}{a^2 + b^2 + 2b\sqrt{a^2 + b^2} + a^2 + b^2} \quad \text{M1}$$

$$\cos(\alpha) = \frac{2b^2 + 2b\sqrt{a^2 + b^2}}{2a^2 + 2b^2 + 2b\sqrt{a^2 + b^2}}$$

$$\cos(\alpha) = \frac{2b(b + \sqrt{a^2 + b^2})}{2\sqrt{a^2 + b^2}(b + \sqrt{a^2 + b^2})}$$

$$\cos(\alpha) = \frac{b}{\sqrt{a^2 + b^2}} \quad \text{A1}$$

c. to show that $2\alpha = \theta$ or $\cos(2\alpha) = \cos(\theta)$

$$\cos(2\alpha) = 2\cos^2(\alpha) - 1$$

$$\cos(2\alpha) = 2\left(\frac{b}{\sqrt{a^2 + b^2}}\right)^2 - 1 \quad \text{M1}$$

$$\cos(2\alpha) = \frac{2b^2}{a^2 + b^2} - 1$$

$$\cos(2\alpha) = \frac{2b^2 - (a^2 + b^2)}{a^2 + b^2} \quad \text{M1}$$

$$\cos(2\alpha) = \frac{b^2 - a^2}{a^2 + b^2} = \cos(\theta) \quad \text{shown} \quad \text{A1}$$

Question 3

a. $\int_{2\sqrt{3}}^5 \frac{x}{\sqrt{x^2 - 9}} dx \quad \text{let } u = x^2 - 9 \quad \frac{du}{dx} = 2x$

terminals when $x = 5 \quad u = 16$ and $x = 2\sqrt{3} \quad u = 3$ M1

$$\int_{2\sqrt{3}}^5 \frac{x}{\sqrt{x^2 - 9}} dx = \frac{1}{2} \int_3^{16} u^{-\frac{1}{2}} du = \left[\sqrt{u} \right]_3^{16} = 4 - \sqrt{3} \quad \text{A1}$$

b. $x = 3\cos(t)$ $dx = -3\sin(t)dt$

$x^2 = 9\cos^2(t)$ terminals

when $x = 3$ $\cos(t) = 1$ $t = 0$ and when $x = 0$ $\cos(t) = 0$ $t = \frac{\pi}{2}$ M1

$\sqrt{9-x^2} = \sqrt{9-9\cos^2(t)} = \sqrt{9(1-\cos^2(t))} = |3\sin(t)| = 3\sin(t)$ since $0 \leq t \leq \frac{\pi}{2}$

$\int_0^3 \sqrt{9-x^2} dx = -9 \int_0^{\frac{\pi}{2}} \sin^2(t) dt = \frac{9}{2} \int_0^{\frac{\pi}{2}} (1 - \cos(2t)) dt$ A1

$= \frac{9}{2} \left[t - \frac{1}{2} \sin(2t) \right]_0^{\frac{\pi}{2}} = \frac{9}{2} \left[\left(\frac{\pi}{2} - \frac{1}{2} \sin(\pi) \right) - \left(0 - \frac{1}{2} \sin(0) \right) \right]$ A1

$= \frac{9\pi}{4}$

c. $\frac{x^2}{9} + \frac{y^2}{4} = 1$ A1

d. $LHS = \frac{9}{x^2} + \frac{4}{y^2} = \frac{9}{9\sec^2 \theta} + \frac{4}{4\operatorname{cosec}^2(\theta)} = \frac{1}{\sec^2(\theta)} + \frac{1}{\operatorname{cosec}^2(\theta)}$
 $= \cos^2(\theta) + \sin^2(\theta) = 1 = RHS$ A1

e. when $y = 4$ $\frac{9}{x^2} + \frac{4}{16} = 1$

$\frac{9}{x^2} = 1 - \frac{1}{4} = \frac{3}{4}$ $x^2 = 12 \Rightarrow x = \pm 2\sqrt{3}$ but in the north-west $x < 0$

so that $x = -2\sqrt{3}$ the point is $P(-2\sqrt{3}, 4)$ A1

$9x^{-2} + 4y^{-2} = 1$ using implicit differentiation

(alternate methods are to transpose and use the chain rule to find $\frac{dy}{dx}$)

$-18x^{-3} - 8y^{-3} \frac{dy}{dx} = 0$ $\frac{dy}{dx} = -\frac{18x^{-3}}{8y^{-3}}$ M1

$\frac{dy}{dx} = -\frac{9y^3}{4x^3}$ at $P(-2\sqrt{3}, 4)$

$\frac{dy}{dx} \Big|_{(-2\sqrt{3}, 4)} = -\frac{9(4)^3}{4(-2\sqrt{3})^3} = 2\sqrt{3}$ A1

$$\text{f.} \quad \frac{dy}{dx} = -\frac{9y^3}{4x^3} = -\frac{9 \times 8 \operatorname{cosec}^3(\theta)}{4 \times 27 \sec^3(\theta)} = -\frac{2 \cos^3(\theta)}{3 \sin^3(\theta)} = -\frac{2}{3} \cot^3(\theta) \quad \text{M1}$$

$$b = -\frac{2}{3} \quad \text{and} \quad n = 3 \quad \text{A1}$$

There are of course many other methods available to produce these results as in **g**.

$$\text{g.} \quad \text{Since } \frac{x^2}{9} + \frac{y^2}{4} = 1 \quad \text{re-arranged becomes } \frac{y^2}{4} = 1 - \frac{x^2}{9} = \frac{9-x^2}{9}$$

$$y = \pm \frac{2}{3} \sqrt{9-x^2} \quad \text{and}$$

$$\text{since } \frac{9}{x^2} + \frac{4}{y^2} = 1 \quad \text{re-arranged becomes } \frac{4}{y^2} = 1 - \frac{9}{x^2} = \frac{x^2-9}{x^2}$$

$$y = \pm \frac{2x}{\sqrt{x^2-9}} \quad \text{M1}$$

by the symmetry of the required area region and when $y = 4$ $x = \pm 2\sqrt{3}$
the required area is the 4 times area of the rectangle plus the bit of area from

$2\sqrt{3}$ to 5 for $y = \frac{2x}{\sqrt{x^2-9}}$ minus the area of the elliptical roundabout

$$A = 4 \left[8\sqrt{3} + \int_{2\sqrt{3}}^5 \frac{2x}{\sqrt{x^2-9}} dx - \frac{2}{3} \int_0^3 \sqrt{9-x^2} dx \right] \quad \text{A1}$$

$$\text{h.} \quad A = 4 \left[8\sqrt{3} + 2(4-\sqrt{3}) - \frac{2}{3} \left(\frac{9\pi}{4} \right) \right] \quad \text{from a.}$$

$$A = 24\sqrt{3} + 32 - 6\pi \quad \text{A1}$$

Question 4

$$\text{a.i.} \quad \text{now } P = 200,000 \text{ W} \quad R = \frac{\sqrt{v}}{2} \quad m = 1800 \text{ kg}$$

$$ma = \frac{P}{v} - R$$

$$1800a = \frac{200,000}{v} - \frac{\sqrt{v}}{2}$$

$$a = \frac{400,000 - v^{\frac{3}{2}}}{3600v} \quad \text{A1}$$

ii. Use $a = v \frac{dv}{dx}$ $v \frac{dv}{dx} = \frac{400,000 - v^{\frac{3}{2}}}{3600v}$

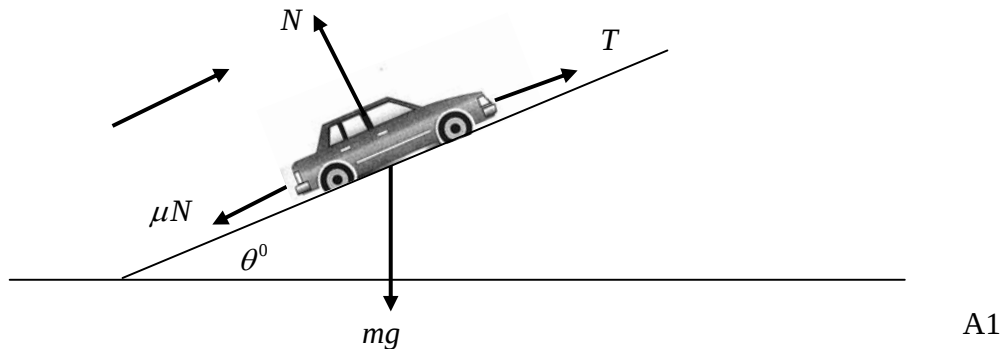
the distance travelled from rest to 10 m/s, is the definite integral

$$x = \int_0^{10} \frac{3600v^2}{400,000 - v^{\frac{3}{2}}} dv \quad \text{A1}$$

iii. using a graphics calculator the distance is 3 m A1

b. $m = 1800 \text{ kg}$ $\mu = 0.2$ $g = 9.8$ $\theta = 50^\circ$

i.



ii. resolving up and parallel to the roadway (1) $T - \mu N - mg \sin(\theta) = 0$

resolving perpendicular to the roadway (2) $N - mg \cos(\theta) = 0$ A1

to find T we need to eliminate N

from (2) $N = mg \cos(\theta)$ substituting into (1) gives

$$T - \mu mg \cos(\theta) - mg \sin(\theta) = 0 \quad \text{M1}$$

$$T = mg(\sin(\theta) + \mu \cos(\theta)) \quad \text{A1}$$

$$T_{\max} = 1800 \times 9.8 (\sin(50^\circ) + 0.2 \cos(50^\circ)) = 15,780.78 \text{ newtons} \quad \text{A1}$$

this is the maximum tension, the minimum tension is obtained when the car is on the point of moving down the roadway, reversing the sign of μ

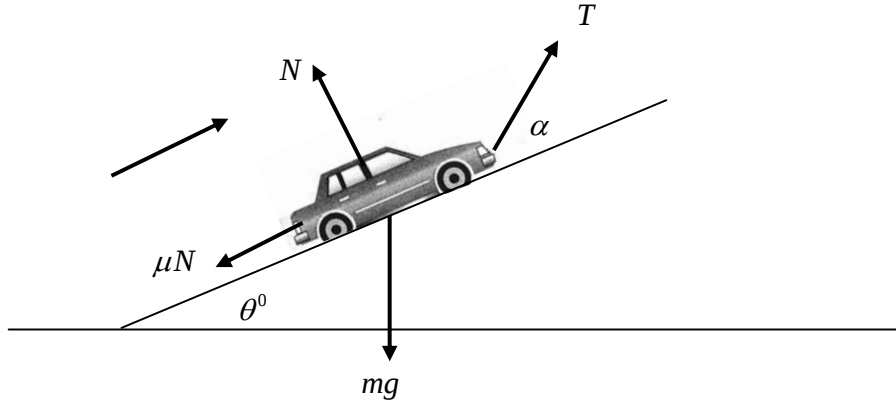
so that the frictional force μN is up and parallel to the plane.

$$T_{\min} = 1800 \times 9.8 (\sin(50^\circ) - 0.2 \cos(50^\circ)) = 11,245.27 \text{ newtons so}$$

$$11,245.27 = T_{\min} \leq T \leq T_{\max} = 15,780.78 \quad \text{A1}$$

c. $m = 1800 \text{ kg}$ $\mu = 0.2$ $g = 9.8$ $\theta = 50^\circ$ $T = 16,000 \text{ newtons}$ $a = 0.1 \text{ m/s}^2$ $\alpha = ?$

i.



A1

ii. resolving up and parallel to the roadway

$$(1) T \cos(\alpha) - \mu N - mg \sin(\theta) = ma$$

resolving perpendicular to the roadway

$$(2) T \sin(\alpha) + N - mg \cos(\theta) = 0$$

M1

to find α we need to eliminate N

from (2) $N = mg \cos(\theta) - T \sin(\alpha)$ substituting into (1) gives

$$T \cos(\alpha) - \mu (mg \cos(\theta) - T \sin(\alpha)) - mg \sin(\theta) = ma$$

A1

$$T (\cos(\alpha) + \mu \sin(\alpha)) = ma + mg (\sin(\theta) + \mu \cos(\theta))$$

$$\cos(\alpha) + \mu \sin(\alpha) = \frac{m(a + g(\sin(\theta) + \mu \cos(\theta)))}{T}$$

A1

$$\cos(\alpha) + 0.2 \sin(\alpha) = \frac{1800(0.1 + 9.8(\sin(50^\circ) + 0.2 \cos(50^\circ)))}{16,000} = 0.9975$$

solving on calculator gives $\alpha = 23.3^\circ$

A1

Question 5

a. $\dot{\mathbf{r}}(t) = 8\hat{i} + 26\hat{j} + \frac{5\pi\sqrt{2}}{4} \cos\left(\frac{\pi t}{2}\right)\hat{k}$ A1

$$\dot{\mathbf{r}}(0) = 8\hat{i} + 26\hat{j} + \frac{5\pi\sqrt{2}}{4} \cos(0)\hat{k} = 8\hat{i} + 26\hat{j} + \frac{5\pi\sqrt{2}}{4}\hat{k}$$

initial speed is $|\dot{\mathbf{r}}(0)| = \sqrt{8^2 + 26^2 + \left(\frac{5\pi\sqrt{2}}{4}\right)^2} = 27.76 \text{ m/s}$ A1

b. $\tan(\theta) = \frac{\frac{5\pi\sqrt{2}}{4}}{\sqrt{8^2 + 26^2}}$
 $\theta = \tan^{-1}(0.2042) = 11.5^\circ$ A1

c. at maximum height $\sin\left(\frac{\pi t}{2}\right) = 1$ $\frac{\pi t}{2} = \sin^{-1}(1) = \frac{\pi}{2} \Rightarrow t = 1$
the soccer ball takes one second to get to a maximum height
and its position vector at this time is $\mathbf{r}(1) = 8\hat{i} + 26\hat{j} + \frac{5\sqrt{2}}{2}\hat{k}$ A1

d. he heads the ball when $\frac{5\sqrt{2}}{2} \sin\left(\frac{\pi t}{2}\right) = \frac{5}{2}$ or $\sin\left(\frac{\pi t}{2}\right) = \frac{1}{\sqrt{2}}$
 $\frac{\pi t}{2} = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}, \frac{3\pi}{4}$ take $\frac{3\pi}{4}$ since the ball has reached its maximum
height and is on its downwards trajectory, so that $t = \frac{3}{2} = 1.5$ seconds A1

e. $\mathbf{r}\left(\frac{3}{2}\right) = 12\hat{i} + 39\hat{j} + \frac{5}{2}\hat{k}$ A1
the required distance is $\sqrt{12^2 + 39^2} = 41$ metres A1

f. $\dot{\mathbf{r}}\left(\frac{3}{2}\right) = 8\hat{i} + 26\hat{j} + \frac{5\pi\sqrt{2}}{4} \cos\left(\frac{3\pi}{4}\right)\hat{k} = 8\hat{i} + 26\hat{j} - \frac{5\pi}{4}\hat{k}$ A1
 $\left|\dot{\mathbf{r}}\left(\frac{3}{2}\right)\right| = \sqrt{8^2 + 26^2 + \left(-\frac{5\pi}{4}\right)^2} = 27.485$ and $m = 0.43 \text{ kg}$
magnitude of momentum $|\mathbf{p}| = 0.430 \times 27.485 = 11.82 \text{ kg m/s}$ A1

END OF SECTION 2 SUGGESTED ANSWERS