

2008 VCAA Specialist Math Exam 2 Solutions

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Section 1

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D	E	B	E	B	C	D	D	B	C	D

12	13	14	15	16	17	18	19	20	21	22
A	E	B	D	C	A	E	C	E	A	B

Q1 $y = \frac{x^4 + a}{x^2} = x^2 + \frac{a}{x^2}, a > 0.$

D

Q2 $x^2 + ax + y^2 + 1 = 0, x^2 + ax + \left(\frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4} - 1,$

$\left(x + \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4} - 1.$ To represent a circle, $\frac{a^2}{4} - 1 > 0,$
 $a^2 > 4. \therefore a < -2$ or $a > 2.$

E

Q3 $f(x) = 3\sin^{-1}(4x-1) + \frac{\pi}{2}$ is an increasing function.

Domain: $-1 \leq 4x-1 \leq 1, 0 \leq 4x \leq 2, 0 \leq x \leq \frac{1}{2}.$

Range: $f(0) = 3\sin^{-1}(-1) + \frac{\pi}{2} = -\pi,$

$f\left(\frac{1}{2}\right) = 3\sin^{-1}(1) + \frac{\pi}{2} = 2\pi. \therefore -\pi \leq y \leq 2\pi$

B

Q4 $m \in (-\infty, -2) \cup (2, \infty),$ i.e. $m \in \mathbb{R} \setminus [-2, 2].$

E

Q5 $\arg(z^7) = 7\text{Arg}(z) = \frac{7\pi}{5}, \therefore \text{Arg}(z^7) = -\frac{3\pi}{5}.$

B

Q6 $z = \frac{3+4i}{1+2i} = \frac{(3+4i)(1-2i)}{(1+2i)(1-2i)} = \frac{11}{5} + \left(-\frac{2}{5}\right)i, \text{Im}(z) = -\frac{2}{5}.$

C

Q7 $(z+2)(\bar{z}+2) = 4, z\bar{z} + 2(z+\bar{z}) = 0.$ Let $z = x+iy,$
 $x^2 + y^2 + 4x = 0, (x+2)^2 + y^2 = 2^2.$

Radius is 2, centre is $(-2, 0).$

D

Q8 $z = -1+i, z$ is in the second quadrant.

$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}, \theta = \tan^{-1}\left(\frac{1}{-1}\right) = \frac{3\pi}{4}.$

$\therefore z = \sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right).$

D

Q9 Choose the particular solution through O, $y = 0.5\sin(2x).$

$\frac{dy}{dx} = \cos(2x).$

B

Q10 $V = 4h, \frac{dV}{dh} = 4.$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}, 0.2 - 0.01\sqrt{h} = 4 \frac{dh}{dt}.$

$\therefore \frac{dh}{dt} = \frac{0.2 - 0.01\sqrt{h}}{4} = \frac{20 - \sqrt{h}}{400}.$

C

Q11 $y' = \frac{dy}{dx} = 2\tan^{-1}(x+1)$

$x_0 = 0 \quad y_0 = 1 \quad y'(0) = 2\tan^{-1}(1) = \frac{\pi}{2}$

$x_1 = 0.2 \quad y_1 = 1 + 0.2 \times \frac{\pi}{2} = 1 + 0.1\pi \quad y'(1) = 2\tan^{-1}(1.2)$

$x_2 = 0.4 \quad y_2 = 1 + 0.1\pi + 0.2 \times 2\tan^{-1}(1.2)$
 $= 1 + 0.1\pi + 0.4\tan^{-1}(1.2)$

D

Q12 The parabola is $y = f(x) = (x+3)(x-1) = x^2 + 2x - 3.$

$\int_{-3}^0 f(x)dx = \left[\frac{x^3}{3} + x^2 - 3x\right]_{-3}^0 = -(-9+9+9) = -9$

A

Q13 $\tilde{r}(t) = 15t\tilde{i} + (20t - 5t^2)\tilde{j}, t \geq 0.$

$\tilde{v}(t) = \frac{d\tilde{r}}{dt} = 15\tilde{i} + (20 - 10t)\tilde{j}.$ At maximum height, $v_y = 0.$

$\therefore 20 - 10t = 0, t = 2, \text{ and } \tilde{r}(2) = 30\tilde{i} + 20\tilde{j}.$

E

Q14 $\tilde{a}$ and $\tilde{b}$ are perpendicular, $\therefore \tilde{a} \cdot \tilde{b} = 0,$

$m^2 + 4m - 12 = 0, (m+6)(m-2) = 0, m = -6 \text{ or } 2.$

B

Q15 $\tilde{P} = \tilde{i}, \tilde{Q} = a(\tilde{i} + \sqrt{3}\tilde{j}), |\tilde{Q}| = 4, \therefore a\sqrt{1^2 + (\sqrt{3})^2} = 4,$

$\therefore a = 2 \text{ and } \tilde{Q} = 2(\tilde{i} + \sqrt{3}\tilde{j}).$

$\tilde{P} + \tilde{Q} = \tilde{i} + 2(\tilde{i} + \sqrt{3}\tilde{j}) = 3\tilde{i} + 2\sqrt{3}\tilde{j},$

$\therefore |\tilde{P} + \tilde{Q}| = \sqrt{3^2 + (2\sqrt{3})^2} = \sqrt{21}.$

D

Q16 Let $u = \tan^{-1}(x), \frac{du}{dx} = \frac{1}{1+x^2}.$

When $x = 0, u = 0;$ when $x = \sqrt{3}, u = \frac{\pi}{3}.$

$\int_0^{\sqrt{3}} \frac{\log_e(\tan^{-1}(x))}{1+x^2} dx = \int_0^{\sqrt{3}} \log_e(u) \frac{du}{dx} dx = \int_0^{\frac{\pi}{3}} \log_e(u) du.$

C

Q17 $\left|\overrightarrow{QR}\right| = \frac{1}{2}\left|\overrightarrow{PQ}\right|, \therefore Q$ divides PR into a ratio of 2 : 1.

$\therefore \tilde{q} = \frac{\tilde{p} + 2\tilde{r}}{3}, \therefore \tilde{r} = \frac{3}{2}\tilde{q} - \frac{1}{2}\tilde{p}.$

A

Q18 Magnitude of $\tilde{F} = \tilde{F} \cdot \frac{\tilde{d}}{|\tilde{d}|}$. E

Comment: Wording problem? According to the information, $\tilde{F}$ causes the object to accelerate in the direction of $\tilde{d}$. $\therefore \tilde{F}$ and $\tilde{d}$ are in the same direction. If $\tilde{F}$ is known, then $|\tilde{F}|$ is the magnitude of $\tilde{F}$. Why would one want to find the magnitude of $\tilde{F}$ the long way?

Q19 $u = \frac{30}{5} = 6$, $t = 6$ and $v = \frac{40}{5} = 8$, use $s = \frac{1}{2}(u + v)t$ to find the displacement $s = \frac{1}{2}(6 + 8)6 = 42$ m.

Distance = 42 m.

Q20 $v = \sin^{-1}(x)$, $a = v \frac{dv}{dx} = \sin^{-1}(x) \times \frac{1}{\sqrt{1-x^2}} = \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}$ E

Q21 $F_{\text{friction}} = 0.1 \times 10g = g$ newtons.

Resultant force driving the system $R = 4g - g = 3g$ newtons.

Acceleration $a = \frac{R}{m} = \frac{3g}{10+4} = \frac{3g}{14}$.

Q22 $a = f(v)$, $\frac{dv}{dt} = f(v)$, $\frac{dt}{dv} = \frac{1}{f(v)}$, $t = \int \frac{1}{f(v)} dv$,

$t_1 = \int_{v_0}^{v_1} \frac{1}{f(v)} dv + t_0$.

Section 2

Q1a $f(x) = \frac{6x\sqrt{x}}{3x^2+1}$, $x \in [0, \infty)$. Let $f'(x) = 0$ to locate the

turning point(s). $\therefore 9\sqrt{x}(1-x^2) = 9\sqrt{x}(1-x)(1+x) = 0$,
 $x = 0$ or 1 .

$f''(1) = -1.125$ is a negative value.

$\therefore$ the maximum turning point is at $x = 1$ and $y = f(1) = \frac{3}{2}$,

i.e. $\left(1, \frac{3}{2}\right)$.

Q1bi Let $f''(x) = 0$ to locate the inflection points.

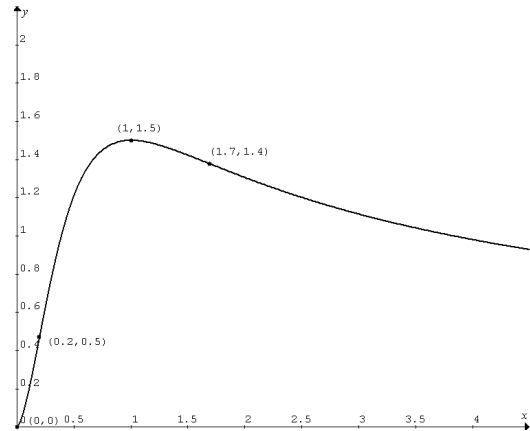
$\therefore 9x^4 - 26x^2 + 1 = 0$.

Q1bii Use graphics calculator to solve for x , and to find y .

$x = 0.19745$, $y = 0.4713$ (0.2, 0.5)

$x = 1.688165$, $y = 1.3781$ (1.7, 1.4).

Q1c



Q1di $y = \frac{6x\sqrt{x}}{3x^2+1}$, $y^2 = \frac{36x^3}{(3x^2+1)^2}$.

$V = \int_0^{\frac{1}{\sqrt{3}}} \pi y^2 dx = 2\pi \int_0^{\frac{1}{\sqrt{3}}} \frac{18x^3}{(3x^2+1)^2} dx$.

Q1dii $u = 3x^2 + 1$, $3x^2 = u - 1$ and $\frac{du}{dx} = 6x$.

$V = 2\pi \int_0^{\frac{1}{\sqrt{3}}} \frac{18x^3}{(3x^2+1)^2} dx = 2\pi \int_0^{\frac{1}{\sqrt{3}}} \frac{3x^2}{(3x^2+1)} \times 6x dx$

$= 2\pi \int_0^{\frac{1}{\sqrt{3}}} \frac{u-1}{u^2} \times \frac{du}{dx} dx = 2\pi \int_1^2 \left(\frac{u-1}{u^2} \right) du = 2\pi \int_1^2 \left(\frac{1}{u} - \frac{1}{u^2} \right) du$

Q1diii $V = 2\pi \left[\log_e u + \frac{1}{u} \right]_1^2 = 2\pi \left(\log_e 2 - \frac{1}{2} \right) = \pi (\log_e 4 - 1)$

cubic units.

Q2a $a = \frac{R}{m} = \frac{390-30}{80} = 4.5 \text{ ms}^{-2}$.

Q2b $u = 0$, $s = 16$, $a = 4.5$. Use $v^2 = u^2 + 2as$ to find v .
 $\therefore v = 12$. The speed is 12 ms^{-1} .

Q2c $a = \frac{R}{m} = \frac{390-30-6v}{80} = \frac{3}{40}(60-v)$, where $v \geq 12$.

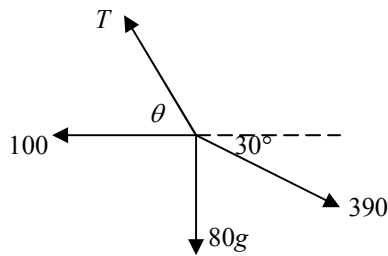
Q2d $\frac{dv}{dt} = \frac{3}{40}(60-v)$, $\frac{dt}{dv} = \frac{40}{3} \times \frac{1}{60-v}$, $t = \frac{40}{3} \int \frac{1}{60-v} dv$.

$\therefore \frac{3}{40}t = -\log_e(60-v) + c$.

When $t = 0$, $v = 12$. $\therefore c = \log_e 48$, and $t = \frac{40}{3} \log_e \left(\frac{48}{60-v} \right)$.

When $v = 18$, $t \approx 1.8$ s.

Q2ei



Q2eii Constant velocity, $\therefore$ zero resultant force.

Horizontal component: $390 \cos 30^\circ - T \cos \theta - 100 = 0$

Vertical component: $T \sin \theta - 390 \sin 30^\circ - 80g = 0$

$$\text{Q2eiii } \cos \theta = \frac{390 \cos 30^\circ - 100}{T}, \sin \theta = \frac{390 \sin 30^\circ + 80g}{T}.$$

$$\therefore \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{390 \sin 30^\circ + 80g}{390 \cos 30^\circ - 100} = 4.118.$$

$$\text{Q2eiv } \theta = \tan^{-1}(4.118) = 76.35^\circ,$$

$$T = \frac{390 \sin 30^\circ + 80g}{\sin 76.35^\circ} \approx 1007 \text{ N}.$$

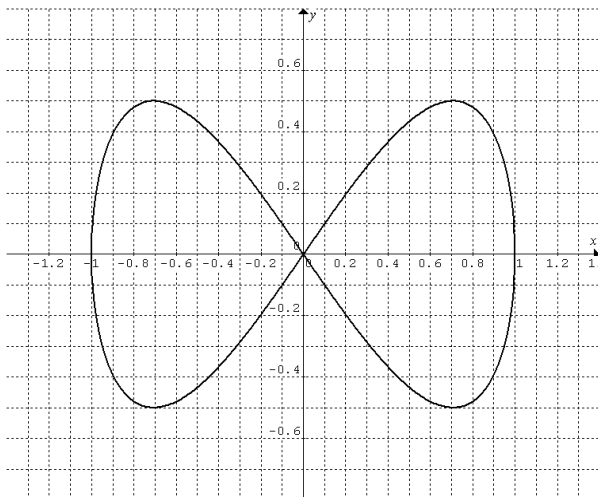
$$\text{Q3ai } \tilde{r}(t) = \sin\left(\frac{t}{3}\right)\tilde{i} + \frac{1}{2}\sin\left(\frac{2t}{3}\right)\tilde{j}, t \geq 0.$$

$$y = \frac{1}{2}\sin\left(\frac{2t}{3}\right) = \sin\left(\frac{t}{3}\right)\cos\left(\frac{t}{3}\right), \therefore y^2 = \sin^2\left(\frac{t}{3}\right)\cos^2\left(\frac{t}{3}\right).$$

$$\text{Q3aii } \therefore y^2 = \sin^2\left(\frac{t}{3}\right)\left[1 - \sin^2\left(\frac{t}{3}\right)\right] = x^2(1 - x^2), \text{ where}$$

$$x = \sin\left(\frac{t}{3}\right).$$

Q3b



$$\text{Q3c } x = \sin\left(\frac{t}{3}\right), \text{ period} = \frac{2\pi}{\frac{1}{3}} = 6\pi.$$

$$\text{Q3d } \tilde{v}(t) = \frac{d\tilde{r}}{dt} = \frac{1}{3}\cos\left(\frac{t}{3}\right)\tilde{i} + \frac{1}{3}\cos\left(\frac{2t}{3}\right)\tilde{j}.$$

$$\text{Speed} = |\tilde{v}(t)| = \frac{1}{3}\sqrt{\cos^2\left(\frac{t}{3}\right) + \cos^2\left(\frac{2t}{3}\right)}$$

The train passes through the origin at $t = 0, 3\pi, 6\pi, \dots$

$$\therefore \text{speed} = \frac{\sqrt{2}}{3} \text{ ms}^{-1}.$$

$$\text{Q3ei Distance} = 4 \int_0^{1.5\pi} |\tilde{v}(t)| dt = \frac{4}{3} \int_0^{1.5\pi} \sqrt{\cos^2\left(\frac{t}{3}\right) + \cos^2\left(\frac{2t}{3}\right)} dt.$$

Q3eii By graphics calculator: Distance ≈ 6.1 m.

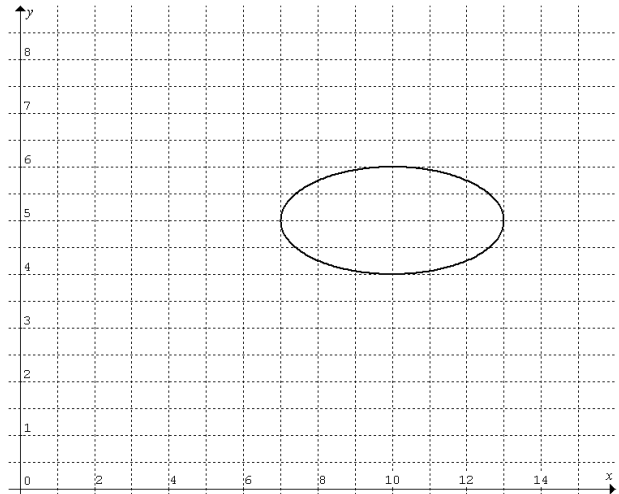
$$\text{Q4a Rabbits: } x = 10 + 3\cos\left(\frac{\pi t}{6}\right), t \geq 0.$$

$$\text{Foxes: } y = 5 + \sin\left(\frac{\pi t}{6}\right), t \geq 0.$$

$$\cos\left(\frac{\pi t}{6}\right) = \frac{x-10}{3}, \sin\left(\frac{\pi t}{6}\right) = y-5.$$

$$\cos^2\left(\frac{\pi t}{6}\right) + \sin^2\left(\frac{\pi t}{6}\right) = 1, \therefore \frac{(x-10)^2}{9} + (y-5)^2 = 1.$$

Q4b



$$\text{Q4ci } x_{\min} = 7 \text{ when } \cos\left(\frac{\pi t}{6}\right) = -1. \frac{\pi t}{6} = \pi, t = 6 \text{ months}.$$

Q4cii When $t = 6$, $y = 5$, i.e. 500 foxes.

$$\text{Q4di } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-0.2y + 0.02xy}{0.5x - 0.1xy} = \frac{xy - 10y}{25x - 5xy}.$$

Q4dii $25 \log_e(y) - 5y - x + 10 \log_e(x) = c$

Implicit differentiation: $\frac{25}{y} \frac{dy}{dx} - 5 \frac{dy}{dx} - 1 + \frac{10}{x} = 0$,

$$\left(\frac{25}{y} - 5\right) \frac{dy}{dx} = 1 - \frac{10}{x},$$

$$\frac{dy}{dx} = \frac{1 - \frac{10}{x}}{\frac{25}{y} - 5} = \frac{1 - \frac{10}{x}}{\frac{25}{y} - 5} \times \frac{xy}{xy} = \frac{xy - 10y}{25x - 5xy}.$$

Q4e As $x \rightarrow x_{\min}$ or $x_{\max}$, $\frac{dy}{dx} \rightarrow \infty$, $\frac{dx}{dy} \rightarrow 0$.

$$\therefore \frac{25x - 5xy}{xy - 10y} \rightarrow 0, \text{ where } x, y > 0.$$

$$\therefore 25x - 5xy \rightarrow 0, y \rightarrow 5.$$

Let $y = 5$, $25 \log_e(5) - 5(5) - x + 10 \log_e(x) = 27.5$,

$$25 \log_e(5) - x + 10 \log_e(x) = 52.5.$$

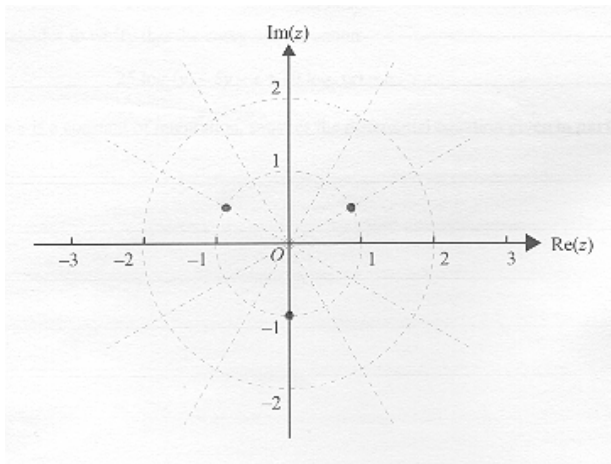
By graphics calculator: $x_{\min} = 6.5871$, $x_{\max} = 14.4269$.

Minimum number of rabbits ≈ 6590 .

Maximum number of rabbits ≈ 14430 .

Q5a $z = \frac{\sqrt{3}}{2} + \frac{1}{2}i = \text{cis}\left(\frac{\pi}{6}\right)$, $z^3 = \text{cis}\left(3 \times \frac{\pi}{6}\right) = \text{cis}\left(\frac{\pi}{2}\right) = i$.

Q5b



Q5c $|z - i| = 1$ is a circle: $x^2 + (y - 1)^2 = 1$.

$\text{Re}(z) = -\frac{1}{\sqrt{3}} \text{Im}(z)$ is a straight line: $y = -\sqrt{3}x$.

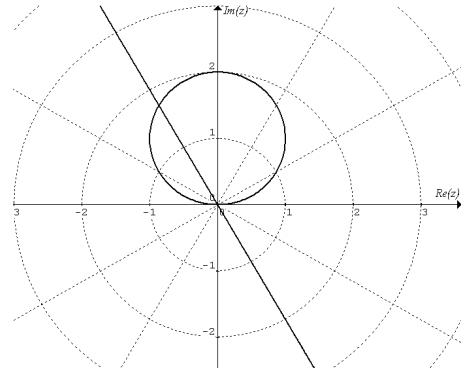
Solve the two equations simultaneously to find the coordinates of the points of intersection.

$$x^2 + (-\sqrt{3}x - 1)^2 = 1, \text{ expand and simplify to } 4x^2 + 2\sqrt{3}x = 0.$$

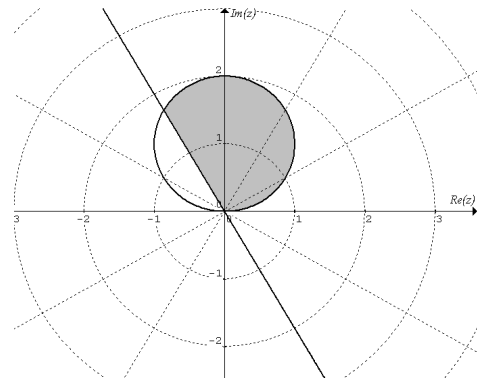
$$\therefore x = 0 \text{ and } y = 0, \text{ or } x = -\frac{\sqrt{3}}{2} \text{ and } y = \frac{3}{2}.$$

The two points are $z = 0$, $z = -\frac{\sqrt{3}}{2} + \frac{3}{2}i$.

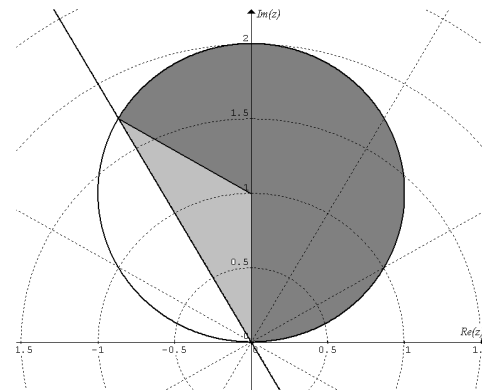
Q5d



Q5e



Q5f



Area of the dark shaded region = $\frac{2}{3}$ of the area of the circle
 $= \frac{2}{3} \pi$.

Area of the light shaded region = $\frac{1}{2} \times 1 \times 1 \times \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{4}$.

Total area = $\frac{2\pi}{3} + \frac{\sqrt{3}}{4} \approx 2.53$ square units.

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors