Insight Publications

INSIGHT

Trial Exam Paper

2008

SPECIALIST MATHEMATICS

Written examination 1

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions.
- mark allocations
- tips on how to approach the questions.

Question 1

Let $u = 10 - 5i$ and $v = 2 - i$.

Find $\frac{iu}{\overline{v}}$ in Cartesian form.

Worked solution

$$\frac{iu}{\overline{v}} = \frac{i(10-5i)}{2+i}$$

$$= \frac{(10i+5)}{(2+i)} \times \frac{(2-i)}{(2-i)}$$ 1A

$$= \frac{20i + 10 + 10 - 5i}{4+1}$$

$$= \frac{15i + 20}{5}$$

$$= 3i + 4$$ 1A

2 marks

Mark allocation

- 1 mark for multiplying by the complex conjugate.
- 1 mark for correct answer.

Question 2

The position of particles A and B at any time t seconds, $t \geq 0$, is given by
$\underset{\sim}{r}_A(t) = (t^2 - 2t)\underset{\sim}{i} + (6t-2)\underset{\sim}{j}$ and $\underset{\sim}{r}_B(t) = (5t-12)\underset{\sim}{i} + (t^2+6)\underset{\sim}{j}$, respectively.

Determine the time when the particles collide.

Worked solution

Particles A and B will collide when they are in the same position at the same time.

$$\underset{\sim}{r}_A(t) = (t^2 - 2t)\underset{\sim}{i} + (6t-2)\underset{\sim}{j}$$

$$\underset{\sim}{r}_B(t) = (5t-12)\underset{\sim}{i} + (t^2+6)\underset{\sim}{j}$$

Equating $\underset{\sim}{i}$ components:

$$t^2 - 2t = 5t - 12$$
$$t^2 - 7t + 12 = 0$$
$$(t-3)(t-4) = 0$$

$t = 3$ and $t = 4$ 1A

Equating $\underset{\sim}{j}$ components:

$$6t - 2 = t^2 + 6$$
$$t^2 - 6t + 8 = 0$$
$$(t-2)(t-4) = 0$$

$t = 2$ and $t = 4$ 1A

The $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components of the position vectors are equal when $t = 4$.

$$\underset{\sim}{r}_A(t) = (4^2 - 2\times 4)\underset{\sim}{i} + (6\times 4 - 2)\underset{\sim}{j} = 8\underset{\sim}{i} + 22\underset{\sim}{j}$$

$$\underset{\sim}{r}_B(t) = (5\times 4 - 12)\underset{\sim}{i} + (4^2 + 6)\underset{\sim}{j} = 8\underset{\sim}{i} + 22\underset{\sim}{j}$$

The particles collide when $t = 4$ seconds. 1A

3 marks

Mark allocation

- 1 mark for equating $\underset{\sim}{i}$ components and finding the associated t values.
- 1 mark for equating $\underset{\sim}{j}$ components and finding the associated t values.
- 1 mark for recognising that the particles collide when $t = 4$.

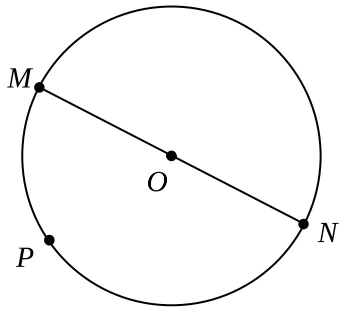
Question 3

MN is the diameter of circle centre O.

P is a point on the circumference of this circle.

Let $\overrightarrow{OP} = \underset{\sim}{p}$ and $\overrightarrow{OM} = \underset{\sim}{m}$.

Use vectors to prove that $\angle MPN$ is a right angle.



Worked solution

$\overrightarrow{OM} = \underset{\sim}{m}$, therefore $\overrightarrow{NO} = \underset{\sim}{m}$ (since O is the midpoint of diameter MN).

$\overrightarrow{OP} = \underset{\sim}{p}$ (given)

Finding vector expressions for $\overrightarrow{MP}$ and $\overrightarrow{NP}$:

$\overrightarrow{MP} = \overrightarrow{MO} + \overrightarrow{OP}$ $\qquad$ $\overrightarrow{NP} = \overrightarrow{NO} + \overrightarrow{OP}$

$\overrightarrow{MP} = -\underset{\sim}{m} + \underset{\sim}{p}$ $\qquad$ $\overrightarrow{NP} = \underset{\sim}{m} + \underset{\sim}{p}$ $\qquad$ 1A

$\overrightarrow{MP} . \overrightarrow{NP} = \left(-\underset{\sim}{m} + \underset{\sim}{p}\right) . \left(\underset{\sim}{m} + \underset{\sim}{p}\right)$

$\overrightarrow{MP} . \overrightarrow{NP} = -\underset{\sim}{m} . \underset{\sim}{m} - \underset{\sim}{m} . \underset{\sim}{p} + \underset{\sim}{p} . \underset{\sim}{m} + \underset{\sim}{p} . \underset{\sim}{p}$

$\overrightarrow{MP} . \overrightarrow{NP} = \underset{\sim}{p} . \underset{\sim}{p} - \underset{\sim}{m} . \underset{\sim}{m}$

$\overrightarrow{MP} . \overrightarrow{NP} = \left|\underset{\sim}{p}\right|^2 - \left|\underset{\sim}{m}\right|^2$ $\qquad$ 1A

Since OP and OM are radii of the circle $\left|\underset{\sim}{p}\right| = \left|\underset{\sim}{m}\right|$,

$\overrightarrow{MP} . \overrightarrow{NP} = 0$ $\qquad$ 1A

Hence, $\overrightarrow{MP}$ and $\overrightarrow{NP}$ are perpendicular.

$\therefore$ $\angle MPN$ is a right angle.

3 marks

Question 3 – continued

Mark allocation

- 1 mark for finding vector expressions for $\overrightarrow{MP}$ and $\overrightarrow{NP}$.
- 1 mark for finding a simplified expression for the dot product.
- 1 mark for showing dot product is zero and deducing $\overrightarrow{MP}$ and $\overrightarrow{NP}$ are perpendicular.

Question 4

a. Show that $\cot(x) - \operatorname{cosec}(2x) = \cot(2x)$

Worked solution

$$\begin{aligned}
\text{LHS} &= \cot(x) - \operatorname{cosec}(2x) \\
&= \frac{\cos(x)}{\sin(x)} - \frac{1}{\sin(2x)} \\
&= \frac{\sin(2x)\cos(x) - \sin(x)}{\sin(x)\sin(2x)} && \text{1M} \\
&= \frac{2\sin(x)\cos(x)\cos(x) - \sin(x)}{\sin(x)\sin(2x)} \\
&= \frac{2\cos^2(x) - 1}{\sin(2x)} \\
&= \frac{\cos(2x)}{\sin(2x)} && \text{1A} \\
&= \cot(2x) \\
&= \text{RHS}
\end{aligned}$$

2 marks

Mark allocation

- 1 mark for simplifying the left hand side of the identity.
- 1 mark for further simplification leading directly to the right hand side of identity.

Question 4 – continued
TURN OVER

b. Hence, solve the equation $\cot(x) - \operatorname{cosec}(2x) = \sqrt{3}$, $x \in [-\pi, \pi]$.

Worked solution

$$\cot(x) - \operatorname{cosec}(2x) = \sqrt{3}$$

$$\cot(2x) = \sqrt{3}$$

$$\tan(2x) = \frac{1}{\sqrt{3}}$$

$2x = \frac{\pi}{6}$ (tan is positive in the first and third quadrants) 1A

$$2x = -2\pi + \frac{\pi}{6}, \quad -\pi + \frac{\pi}{6}, \quad \frac{\pi}{6}, \quad \pi + \frac{\pi}{6}$$

$$2x = -\frac{11\pi}{6}, \quad -\frac{5\pi}{6}, \quad \frac{\pi}{6}, \quad \frac{7\pi}{6}$$

$x = -\frac{11\pi}{12}, \quad -\frac{5\pi}{12}, \quad \frac{\pi}{12}, \quad \frac{7\pi}{12}$ 1A

2 marks

Total 2 + 2 = 4 marks

Mark allocation

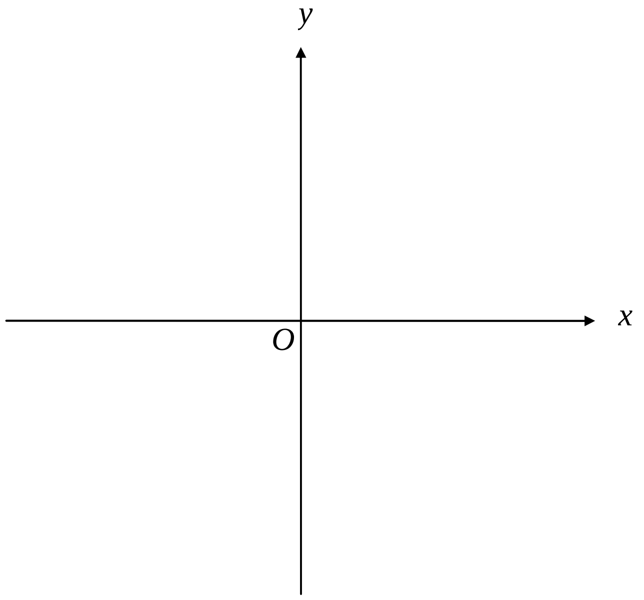
- 1 mark for simplifying to find $2x = \frac{\pi}{6}$.
- 1 mark for finding all solutions.

Tip

- *The word ‘hence’ gives the hint that something from the previous part of the question is used to find the answer. So, use the information given in part **a** to answer part **b**.*

Question 5

a. Sketch the graph of $f(x) = 3\arcsin(x+1) - \frac{\pi}{2}$ on the axes below, showing the intercepts and endpoints in exact form.



Worked solution

Graph of $y = \arcsin(x)$ has been dilated by a factor of 3 from the x-axis, then translated 1 unit left and $\frac{\pi}{2}$ units down.

Domain

$x \in [-2, 0]$

Range

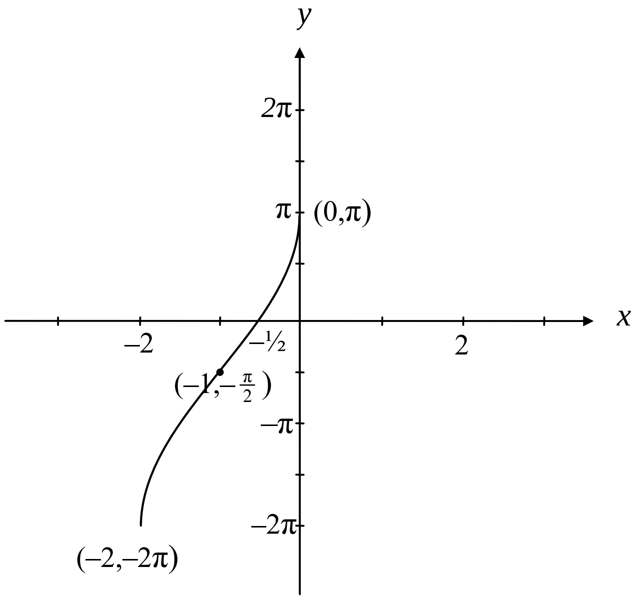
$y \in \left[3\times\left(-\frac{\pi}{2}\right) - \frac{\pi}{2},\ \ 3\times\left(\frac{\pi}{2}\right) - \frac{\pi}{2}\right] = [-2\pi, \pi]$

$\therefore$ Endpoint coordinates are $(-2, -2\pi)$ and $(0, \pi)$.

At x-intercept, $y = 0$:

$$3\arcsin(x+1) - \frac{\pi}{2} = 0$$
$$\arcsin(x+1) = \frac{\pi}{6}$$
$$x+1 = \sin\left(\frac{\pi}{6}\right)$$
$$x+1 = \frac{1}{2}$$
$$x = -\frac{1}{2}$$

Question 5 – continued
TURN OVER



3 marks

Mark allocation

- 1 mark for Endpoints
- 1 mark for Intercepts
- 1 mark for Shape

b. Find $f^{-1}(x)$ stating its domain.

Worked solution

$x = 3\arcsin(y+1) - \frac{\pi}{2}$

Make *y* the subject:

$\arcsin(y+1) = \frac{1}{3}\left(x + \frac{\pi}{2}\right)$

$y + 1 = \sin\left(\frac{1}{3}\left(x + \frac{\pi}{2}\right)\right)$

$f^{-1}(x) = \sin\left(\frac{1}{3}\left(x + \frac{\pi}{2}\right)\right) - 1$ 1A

Domain f^{-1} is $x \in [-2\pi, \pi]$. This is the range of *f*. 1A

2 marks

Total 3 + 2 = 5 marks

Mark allocation

- 1 mark for finding the inverse equation
- 1 mark for finding the range of f^{-1}

Tip

- *An inverse equation is found by swapping x and y.*

Question 6

Find $\int \frac{x+1}{x^2+2}\,dx$.

Worked solution

$\int \frac{x}{x^2+2}\,dx \quad + \quad \int \frac{1}{x^2+2}\,dx$ 1A

Integrating $\int \frac{x}{x^2+2}\,dx$:

$= \frac{1}{2}\int \frac{2x}{x^2+2}\,dx$

$= \frac{1}{2}\int \frac{1}{u}\,du$

$= \frac{1}{2}\log_e\left(x^2+2\right)$ 1A

Integrating $\int \frac{1}{x^2+2}\,dx$:

$= \frac{1}{\sqrt{2}}\int\left(\frac{\sqrt{2}}{\left(\sqrt{2}\right)^2+x^2}\right)dx$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)$

$\therefore \int \frac{x+1}{x^2+2}\,dx = \frac{1}{2}\log_e\left(x^2+2\right)+\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+c$ 1A

3 marks

Mark allocation

- 1 mark for correctly splitting the integral into two parts.
- 1 mark for integrating one part of the integral.
- 1 mark for correct solution.

Tip

- *Solve by splitting the integral into two parts.*

TURN OVER

Question 7

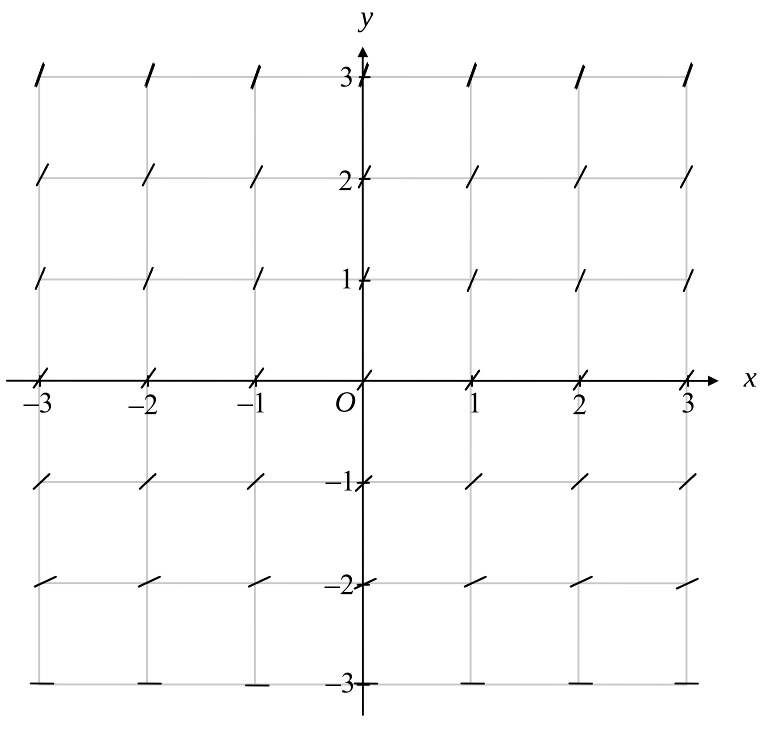
Given the differential equation $\frac{dy}{dx} = \frac{y+3}{2}$

a. Use $y = -3, -2, -1, 0, 1, 2, 3$ to sketch a slope field of the differential equation at each of the values $x = -3, -2, -1, 0, 1, 2, 3$.

Worked solution

y	–3	–2	–1	0	1	2	3
$\frac{dy}{dx}$	0	0.5	1	1.5	2	2.5	3

Correct gradients 1A



1A

2 marks

Mark allocation

- 1 mark for finding the gradients at the given values of x.
- 1 mark for graphing the gradients correctly.

Tip

- *Calculate gradients by substituting the y values into the differential equation. For example, when x = –3,* $\frac{dy}{dx} = \frac{-3+3}{2} = 0$.

b. If $y = -2$ when $x = 1$, solve the differential equation to find y in terms of x.

Worked solution

$$\frac{dx}{dy} = \frac{2}{y+3}, \ y \neq -3$$

$$x = 2\log_e |\ y+3\ | + c, \ y \neq -3 \qquad \text{1A}$$

Given that $y = -2$ when $x = 1$:

$$1 = 2\log_e |-2+3| + c$$

$$1 = 2\log_e (1) + c$$

$$c = 1 \qquad \text{1A}$$

Finding y in terms of x:

$$x = 2\log_e |\ y+3\ | + 1, \ y \neq -3$$

$$\frac{1}{2}(x-1) = \log_e |\ y+3\ |, \ y \neq -3$$

$$y = e^{\frac{1}{2}(x-1)} - 3 \qquad \text{1A}$$

3 marks
Total 2 + 3 = 5 marks

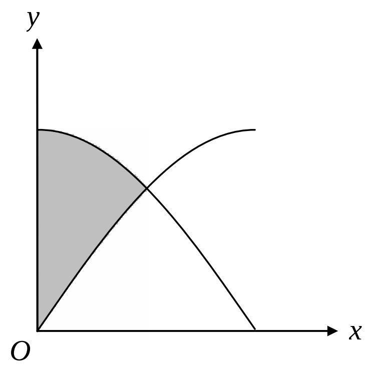
Mark allocation

- 1 mark for integrating to find x in terms of y.
- 1 mark for finding the constant term.
- 1 mark for finding y in terms of x.

Tip

- *Take reciprocals of both sides of the equation.*

Question 8



The area between the graphs of $y = \sin(3x)$ and $y = \cos(3x)$ shaded in the diagram above is rotated around the x-axis to form a solid of revolution.

Find the exact volume of this solid.

Question 8 – continued
TURN OVER

Worked solution

Tip

- *Solve* $\sin(3x) = \cos(3x)$ *to find point of intersection of the curves.*

$$\frac{\sin(3x)}{\cos(3x)} = 1$$

$$\tan(3x) = 1$$

$$3x = \frac{\pi}{4}$$

$$x = \frac{\pi}{12}$$ 1A

The volume of revolution is rotated around the *x*-axis.

Tip

- $\cos(3x) > \sin(3x)$ *over the interval* $0 \le x < \frac{\pi}{12}$.

$$V = \pi\int_0^{\frac{\pi}{12}} \left(\cos^2(3x) - \sin^2(3x)\right) dx$$ 1A

$$V = \pi\int_0^{\frac{\pi}{12}} \cos(6x)\, dx$$

$$V = \pi\left[\frac{1}{6}\sin(6x)\right]_0^{\frac{\pi}{12}}$$ 1A

$$V = \frac{\pi}{6}\left(\sin\left(6\times\frac{\pi}{12}\right) - \sin(0)\right)$$

$$V = \frac{\pi}{6}\left(\sin\left(\frac{\pi}{2}\right) - 0\right)$$

$$V = \frac{\pi}{6}(1-0)$$

$$V = \frac{\pi}{6} \text{ cubic units}$$ 1A

4 marks

Mark allocation

- 1 mark for finding the point of intersection of the curves.
- 1 mark for writing a correct integral to find the volume.
- 1 mark for integrating.
- 1 mark for finding the correct volume.

Question 9

Let $y = x\sqrt{1-x^2} - \cos^{-1}(x)$.

a. Show that $\dfrac{dy}{dx} = 2\sqrt{1-x^2}$.

Worked solution

$$y = x\sqrt{1-x^2} - \cos^{-1}(x)$$

$$\frac{dy}{dx} = \left(1\sqrt{1-x^2} + x \times \frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}} \times (-2x)\right) - \left(\frac{-1}{\sqrt{1-x^2}}\right) \qquad \text{1M}$$

$$\frac{dy}{dx} = \left(\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}\right) + \frac{1}{\sqrt{1-x^2}} \qquad \text{1A}$$

$$\frac{dy}{dx} = \sqrt{1-x^2} + \frac{1-x^2}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \sqrt{1-x^2} + \sqrt{1-x^2}$$

$$\frac{dy}{dx} = 2\sqrt{1-x^2} \qquad \text{1A}$$

3 marks

Mark allocation

- 1 mark for using correct method to differentiate.
- 1 mark for simplifying the expression for the derivative.
- 1 mark for further simplification leading directly to the answer.

Question 9 – continued
TURN OVER

b. Hence, determine the exact value of $\int_{\frac{1}{2}}^{1} \sqrt{1-x^2}\ dx$.

Worked solution

$\int 2\sqrt{1-x^2}\ dx = x\sqrt{1-x^2} - \cos^{-1}(x)$ (From part **a**.)

$= \int_{\frac{1}{2}}^{1} \sqrt{1-x^2}\ dx = \frac{1}{2}\left[x\sqrt{1-x^2} - \cos^{-1}(x)\right]_{\frac{1}{2}}^{1}$

$= \frac{1}{2}\left[\left(1\sqrt{1-1^2} - \cos^{-1}(1)\right) - \left(\frac{1}{2}\sqrt{1-\left(\frac{1}{2}\right)^2} - \cos^{-1}\left(\frac{1}{2}\right)\right)\right]$ 1A

$= \frac{1}{2}\left[(0) - \left(\frac{1}{2}\sqrt{\frac{3}{4}} - \frac{\pi}{3}\right)\right]$

$= \frac{1}{2}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right]$

$= \frac{\pi}{6} - \frac{\sqrt{3}}{8}$ 1A

2 marks
Total 3 + 2 = 5 marks

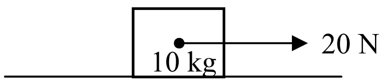
Mark allocation

- 1 mark for using the result of a. to evaluate the integral.
- 1 mark for the answer.

Question 10

A crate of toys of mass 10 kg is sitting on the floor of a room.

a. A child starts to pull the crate with a horizontal force of 20 newtons so that it is on the point of moving. Show that the coefficient of friction between the floor and the crate of toys is $\frac{2}{g}$.



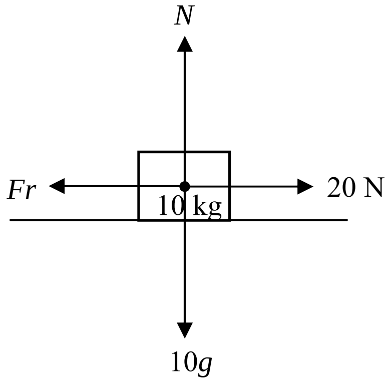
Worked solution

N normal reaction

Fr friction

$10g$ weight force

20 child's pulling force



$Fr = \mu N$

Resolving forces in a vertical direction:

$N = 10g$

The crate is on the point of moving, so the forces are in limiting equilibrium, i.e. $a = 0$.

Resolving forces in a horizontal direction:

$20 - Fr = 10a$ 1A

$20 - \mu N = 10 \times 0$

$20 - \mu \times 10g = 0$ 1A

$\mu = \frac{2}{g}$

2 marks

Mark allocation

- 1 mark for resolving forces.
- 1 mark for simplification leading directly to the answer.

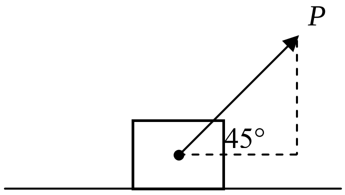
Tip

- *First draw the forces acting on the diagram before doing any calculations.*

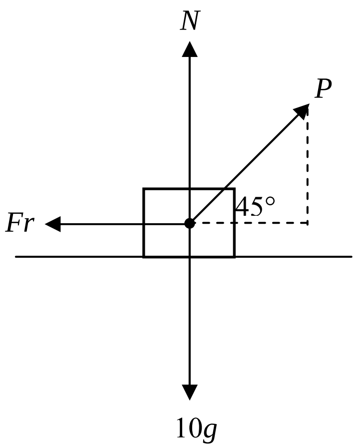
Question 10 – continued
TURN OVER

b. Determine the maximum force, P newtons, that can be applied to the crate at an angle of 45° to the horizontal level without moving it.

Express your answer in the form $P = \frac{ag}{g+b}$, where $a, b \in R$.



Worked solution



$P\sin(45°)$ Vertical component of P.

$P\cos(45°)$ Horizontal component of P.

Resolving forces in a vertical direction:

$$N + P\sin(45°) = 10g$$

$$N + P \times \frac{1}{\sqrt{2}} = 10g$$

$$N = 10g - \frac{P}{\sqrt{2}} \quad \ldots(1)$$ 1A

Resolving forces in a horizontal direction:

The crate is on the point of moving, so $a = 0$.

$$P\cos(45°) - Fr = 10 \times 0$$

$$P\cos(45°) = Fr$$

$$P \times \frac{1}{\sqrt{2}} = \mu N$$

$$\frac{P}{\sqrt{2}} = \frac{2}{g} \times N \quad \ldots(2)$$ 1A

Substituting (1) into (2):

$$\frac{P}{\sqrt{2}} = \frac{2}{g}\left(10g - \frac{P}{\sqrt{2}}\right) \qquad \text{1M}$$

$$P = \frac{2\sqrt{2}}{g}\left(10g - \frac{P}{\sqrt{2}}\right)$$

$$P = 20\sqrt{2} - \frac{2P}{g}$$

$$P + \frac{2P}{g} = 20\sqrt{2}$$

$$g\,P + 2P = 20\sqrt{2}\,g$$

$$P(g+2) = 20\sqrt{2}\,g$$

$$P = \frac{20\sqrt{2}\,g}{g+2} \text{ newtons} \qquad \text{1A}$$

4 marks
Total 2 + 4 = 6 marks

Mark allocation

- 1 mark for finding one equation for N in terms of P.
- 1 mark for finding another equation for N in terms of P.
- 1 mark for attempting to solve these equations to find P.
- 1 mark for the correct answer.

Tip

- *First draw the forces acting on the diagram before doing any calculations.*

END OF SOLUTIONS BOOK