

VCE Specialist Mathematics Units 3 & 4

Written Examination 2

Suggested Solutions

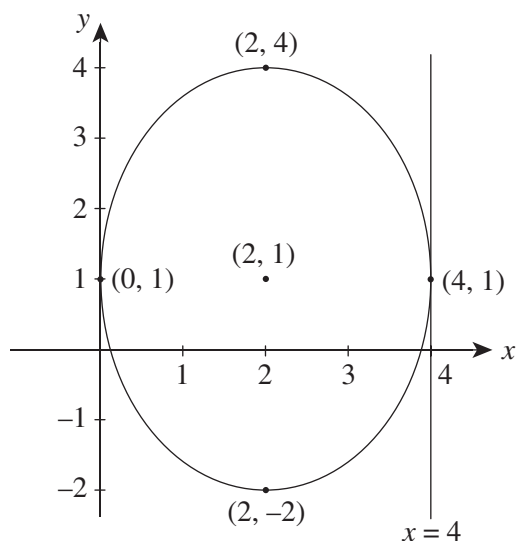
SECTION 1

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E

12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E
21	A	B	C	D	E
22	A	B	C	D	E

SECTION 1**Question 1**

The ellipse can be sketched as follows:



With reference to the general equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, the semi-minor axis length a is 2, the semi-major axis length b is 3 and the centre (h, k) is $(2, 1)$. Hence the equation is $\frac{(x-2)^2}{4} + \frac{(y-1)^2}{9} = 1$.

Answer D

Question 2

For $f(x) = \frac{a + bx^2}{x^2}$, $f(-x) = f(x)$ ('even' function), hence the graph of $y = f(x)$ is symmetrical about the y -axis. By division, $f(x) = \frac{a}{x^2} + b$. Since $f(x)$ is undefined when $x = 0$, the line $x = 0$ is a vertical asymptote for the graph. As the magnitude of x approaches infinity, $\frac{a}{x^2}$ approaches zero and $f(x)$ approaches b . Hence the line $y = b$ is a horizontal asymptote for the graph.

Answer E

Question 3

$$\begin{aligned}
 z &= \frac{4}{\sqrt{3}i - 1} \\
 &= \frac{4}{-1 + \sqrt{3}i} \times \frac{-1 - \sqrt{3}i}{-1 - \sqrt{3}i} \\
 &= \frac{4(-1 - \sqrt{3}i)}{1 + 3} \\
 &= -1 - \sqrt{3}i
 \end{aligned}$$

The complex conjugate of z , $\bar{z} = -1 + \sqrt{3}i$.

The modulus of $\bar{z} = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2 = a$.

The argument of $\bar{z} = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$ in the second quadrant $= \frac{2\pi}{3} = b$.

Answer A**Question 4**

To find the three cube roots of $27i$, let $z^3 = 27i = 0 + 27i$.

Thus $z^3 = 27 \operatorname{cis}\left(\frac{\pi}{2} + 2k\pi\right)$ for k , an integer.

By de Moivre's theorem, $z = 3 \operatorname{cis}\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)$ for k , an integer.

The first cube root ($k = 0$) is $3 \operatorname{cis}\left(\frac{\pi}{6}\right) = 3\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$.

The other values of z may be obtained by progressively adding $\frac{2\pi}{3}$ or letting $k = 1, 2$.

$$3 \operatorname{cis}\left(\frac{5\pi}{6}\right) = 3\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \text{ and } 3 \operatorname{cis}\left(\frac{3\pi}{2}\right) = 3(-i)$$

The truth of each statement can now be determined.

A. On an Argand diagram, z_1, z_2 and z_3 are separated by an angle of $\frac{2\pi}{3}$ radians. True: the three cube roots must be equally spaced on a circle.

B. $z_1 = 3i$. False: one of the roots is $-3i$ but none is $3i$. Also, $(3i)^3 = -27i$. Note that the complex conjugate root theorem applies to polynomial equations with real coefficients.

C. $|z_2| = 3$. True: all roots have modulus 3 (they are spaced on a circle of radius 3).

D. $(z_3)^{-1} = \frac{1}{3} \operatorname{cis}\left(-\frac{5\pi}{6}\right)$. Could be true if $z_3 = 3 \operatorname{cis}\left(\frac{5\pi}{6}\right)$, then by de Moivre's theorem

$$(z_3)^{-1} = \frac{1}{3} \operatorname{cis}\left(-\frac{5\pi}{6}\right).$$

E. $z_1 + z_2 + z_3 = 0$. True: $z_1 + z_2 + z_3 = 3\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) + 3\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) + (-3i) = 0$.

Answer B

Question 5

If $\text{Im}(z^2) \geq 4$, we can write $\text{Im}\{(x + iy)^2\} \geq 4$.

$$\text{Im}\{x^2 + 2xyi + (iy)^2\} \geq 4$$

$$\text{Im}\{x^2 - y^2 + 2xyi\} \geq 4$$

$$2xy \geq 4$$

$$y \geq \frac{2}{x}$$

This region is 'above' each branch of the hyperbola $y = \frac{2}{x}$.

The region defined by $-\frac{3\pi}{4} \leq \text{Arg}(z) \leq \frac{\pi}{4}$ is one half of the complex plane (with the origin excluded)

bounded by the line $\text{Re}(z) = \text{Im}(z)$.

The intersection of the two regions is illustrated by alternative **B**.

Answer B

Question 6

The alternatives show a dilation from the y -axis, a translation parallel to the x -axis and possibly a reflection in the x -axis. Alternative **E**. can be eliminated first because its period of $\frac{2\pi}{b}$ does not match the graph.

Consider the stationary points and vertical asymptotes of $y = \pm \sec(a(x - b))$ and $y = \pm \text{cosec}(a(x - b))$.

Curve equation	Stationary points	Vertical asymptotes (x values)
$y = \sec(a(x - b))$	$(b, 1), \left(\frac{\pi}{a} + b, -1\right), \left(\frac{2\pi}{a} + b, 1\right), \dots$	$\frac{\pi}{2a} + b, \frac{3\pi}{2a} + b, \dots$
$y = -\sec(a(x - b))$	$(b, -1), \left(\frac{\pi}{a} + b, 1\right), \left(\frac{2\pi}{a} + b, -1\right), \dots$	$\frac{\pi}{2a} + b, \frac{3\pi}{2a} + b, \dots$
$y = \text{cosec}(a(x - b))$	$\left(\frac{\pi}{2a} + b, 1\right), \left(\frac{3\pi}{2a} + b, -1\right), \left(\frac{5\pi}{2a} + b, 1\right), \dots$	$b, \frac{\pi}{a} + b, \frac{2\pi}{a} + b, \dots$
$y = -\text{cosec}(a(x - b))$	$\left(\frac{\pi}{2a} + b, -1\right), \left(\frac{3\pi}{2a} + b, 1\right), \left(\frac{5\pi}{2a} + b, -1\right), \dots$	$b, \frac{\pi}{a} + b, \frac{2\pi}{a} + b, \dots$

Comparing these stationary points and asymptotes with those of the graph shows that the appropriate equation is $y = -\text{cosec}(a(x - b)) = \text{cosec}(a(b - x))$ since $\sin(-A) = -\sin(A)$.

Answer D

Question 7

The usual domain of $y = \sin(x)$ so that it is a one-to-one function is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The range of $\sin(x)$ is $[-1, 1]$.

With $f(x) = \sin(2x)$ the period is halved, so the domain of $f(x)$ could be $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$, with the same range as for

$y = \sin(x)$. The domain and range of $f^{-1}(x)$ would then be $[-1, 1]$ and $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$. Although no alternative has

$\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ as its range, the range of alternative C, $\left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$, is suitable because it is one period advanced from $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$. Hence the original domain restriction of $f(x)$ corresponding to this will result in a one-to-one

function, as required.

Answer C

Question 8

The vector resolute of $2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$ in the direction of $3\mathbf{i} - 2\mathbf{k}$ is given by

$$\begin{aligned} & \frac{2 \times 3 - 3 \times 0 - 1 \times (-2)}{3^2 + (-2)^2} \\ &= \frac{8}{13}(3\mathbf{i} - 2\mathbf{k}) \end{aligned}$$

Answer C

Question 9

The gradient of the tangent to the curve $y = \cos^{-1}\left(\frac{x}{2}\right)$ is given by $\frac{dy}{dx}$.

$$\begin{aligned} \frac{dy}{dx} &= -\frac{1}{\sqrt{1 - \left(\frac{x}{2}\right)^2}} \times \frac{1}{2} \text{ applying the chain rule} \\ &= -\frac{1}{\sqrt{4 - x^2}} \end{aligned}$$

When $x = 1$, the gradient is $-\frac{1}{\sqrt{3}}$ and $y = \cos^{-1}\left(\frac{x}{2}\right) = \frac{\pi}{3}$.

The tangent equation is $y - \frac{\pi}{3} = -\frac{1}{\sqrt{3}}(x - 1)$

Answer C

Question 10

In $\int_0^{\frac{\pi}{4}} \sqrt{1 - \sin(2x)}(1 - 2\cos^2(x)) dx$, replace $1 - \sin(2x)$ with u .

$$\begin{aligned}\frac{du}{dx} &= -2\cos(2x) \\ &= -2(2\cos^2(x) - 1) \\ &= 2(1 - 2\cos^2(x))\end{aligned}$$

When $x = \frac{\pi}{4}$, $u = 1 - 1 = 0$.

When $x = 0$, $u = 1 - 0 = 1$.

The integral becomes $\frac{1}{2} \int_1^0 u^{\frac{1}{2}} du$.

Answer E

Question 11

$$\int \sin^3(6x) dx = \int \sin^2(6x) \sin(6x) dx = \int (1 - \cos^2(6x)) \sin(6x) dx$$

Using the substitution $u = \cos(6x)$ so that $\frac{du}{dx} = -6\sin(6x)$, $x = \frac{\pi}{6}$ gives $u = -1$ and $x = 0$ gives $u = 1$.

$$\begin{aligned}\int_0^{\frac{\pi}{6}} (1 - \cos^2(6x)) \sin(6x) dx \\ &= -\frac{1}{6} \int_0^{\frac{\pi}{6}} (1 - \cos^2(6x)) (-6\sin(6x)) dx \\ &= -\frac{1}{6} \int_1^{-1} (1 - u^2) du \\ &= \frac{1}{6} \int_1^{-1} (u^2 - 1) du\end{aligned}$$

Answer D

Question 12

The initial point is $(0, 1)$, i.e. $a = 0$ and $b = 1$.

Euler's method using a step size of 0.1 gives

$$a = 0 \quad f(a) = f(0) = 1$$

$$x_1 = 0.1 \quad f(x_1) = f(0.1) = \cos^3(0.1)$$

Using $y_{n+1} = y_n + hf(x_n)$

$$y_1 = b + hf(a)$$

$$= 1 + 0.1 \cos^3(0)$$

$$= 1.1$$

$$y_2 = y_1 + hf(x_1)$$

$$= 1.1 + 0.1 \cos^3(0.1)$$

Answer C

Question 13

$$10a = -10g - \frac{v^2}{10}$$

$$\frac{dv}{dt} = -g - \frac{v^2}{100}$$

$$\frac{dv}{dt} = -\frac{100g + v^2}{100}$$

Answer D

Question 14

We require $\overrightarrow{AB} = \overrightarrow{DC}$ so that one pair of opposite sides are equal and parallel.

Also, we require that $|\overrightarrow{AB}| = |\overrightarrow{AD}|$. Hence we have adjacent sides of equal length.

Answer A

Question 15

$$\underline{\underline{r}}(t) = \int (e^{-t}\underline{\underline{i}} - 3e^{-3t}\underline{\underline{j}})dt$$

$$= -e^{-t}\underline{\underline{i}} + e^{-3t}\underline{\underline{j}} + \underline{\underline{d}}$$

When $t = 0$, $\underline{\underline{r}} = \underline{\underline{0}}$.

$$\underline{\underline{d}} = \underline{\underline{i}} - \underline{\underline{j}}$$

$$\underline{\underline{r}}(t) = (1 - e^{-t})\underline{\underline{i}} + (e^{-3t} - 1)\underline{\underline{j}}$$

Answer A

Question 16

$$\begin{aligned}
 \vec{a} \cdot \vec{b} &= |\vec{a}| |\vec{b}| \cos(\theta) \\
 \cos(\theta) &= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \\
 &= \frac{(\vec{i} - 2\vec{k}) \cdot (2\vec{i} - \vec{j} + 2\vec{k})}{\sqrt{5} \times 3} \\
 &= \frac{2 - 4}{3\sqrt{5}} \\
 &= \frac{-2}{3\sqrt{5}}
 \end{aligned}$$

Answer C**Question 17**

Distance travelled is the area under the graph.

$$600 = \frac{1}{2}(36 + t) \times 24$$

$$50 = 36 + t$$

$$t = 14$$

The particle travels at 24 m/s for 14 seconds. Hence the particle travels at a constant velocity for $24 \times 14 = 336$ metres.

Answer B**Question 18**

$$\begin{aligned}
 \text{initial momentum} &= 8 \times 7 \\
 &= 56 \text{ (kg m/s)}
 \end{aligned}$$

$$\begin{aligned}
 \text{final momentum} &= 8 \times 1 \\
 &= 8 \text{ (kg m/s)}
 \end{aligned}$$

$$\begin{aligned}
 \text{change in momentum} &= \text{final momentum} - \text{initial momentum} \\
 &= 8 - 56 \\
 &= -48 \text{ (kg m/s)}
 \end{aligned}$$

Answer B**Question 19**

$$\sin(\theta) = \frac{5}{13} \text{ and so } \cos(\theta) = \frac{12}{13}$$

$$mg \sin(\theta) - \mu N = ma \text{ (parallel to the plane)}$$

$$N = mg \cos(\theta) \text{ (perpendicular to the plane)}$$

$$\frac{5mg}{13} - \frac{12\mu mg}{13} = ma$$

$$a = \frac{g}{13}(5 - 12\mu)$$

Answer E

Question 20

$$\vec{r} = -\sqrt{3}\cos(t)\vec{i} + \sin(t)\vec{j}$$

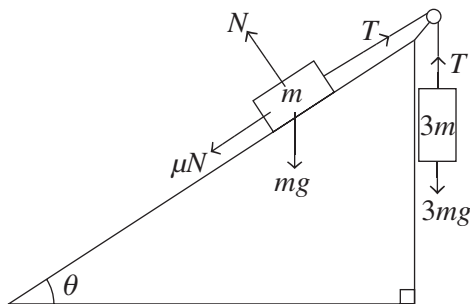
$$\dot{\vec{r}} = \sqrt{3}\sin(t)\vec{i} + \cos(t)\vec{j}$$

$$\ddot{\vec{r}} = \sqrt{3}\cos(t)\vec{i} - \sin(t)\vec{j}$$

From Newton's 2nd law of motion, $\vec{F} = m\ddot{\vec{r}}$.

$$\vec{F} = 2\sqrt{3}\cos(t)\vec{i} - 2\sin(t)\vec{j}$$

$$|\vec{F}| = \sqrt{12\cos^2(t) + 4\sin^2(t)}$$

Answer A**Question 21**

$$3m \text{ particle:} \quad T = 3mg \quad (1)$$

$$m \text{ particle:} \quad T - \mu N - mg \sin(\theta) = 0 \quad (2)$$

$$N = mg \cos(\theta) \quad (3)$$

Substituting (1) and (3) into (2) gives

$$3mg - \mu mg \cos(\theta) - mg \sin(\theta) = 0$$

$$3 - \mu \cos(\theta) - \sin(\theta) = 0$$

$$\mu \cos(\theta) = 3 - \sin(\theta)$$

$$\mu = \frac{3 - \sin(\theta)}{\cos(\theta)}$$

Answer B**Question 22**

Given $v^2 = 12 - 4x^2$

$$\frac{1}{2}v^2 = 6 - 2x^2$$

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Hence $a = -4x$.

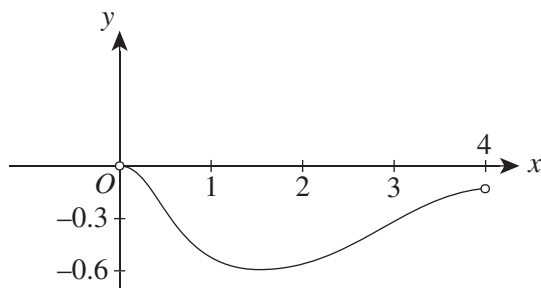
We are looking for a linear graph that passes through the origin with a negative gradient. Graph **D** is correct as the motion could also be along the negative x -axis.

Answer D

SECTION 2

Question 1

a.



Suitable y-axis scale.

A1

Correct graph shape and location of endpoints.

A1

b. The maximum rate of decrease is at the local minimum of the graph of $y = f'(x)$.

A1

The coordinates of the local minimum are (1.47, -0.67).

A1

Hence the corresponding point on the graph of f has coordinates (1.47, 1.49).

c. The maximum rate of decrease is 0.67.

A1

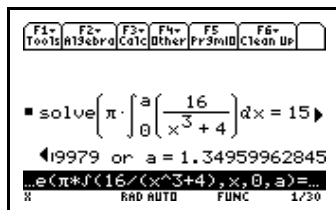
d.
$$\pi \int_0^a \left(\frac{4}{\sqrt{x^3 + 4}} \right)^2 dx = 15$$

A1

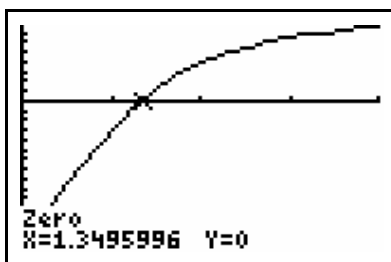
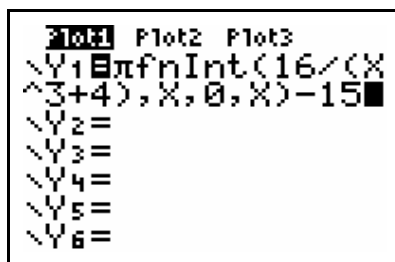
Solving this equation for a gives $a = 1.35$.

A1

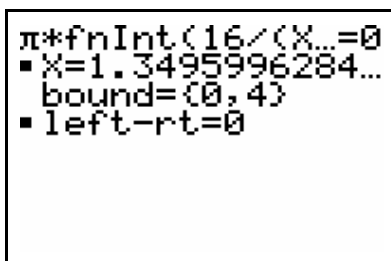
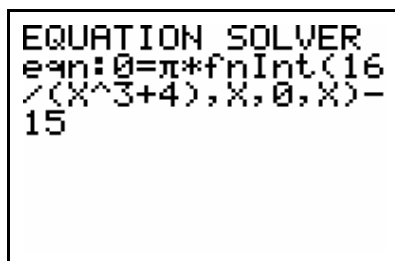
Using CAS:



Using a graphics calculator:



OR



Question 2

a. $\hat{a} = \frac{\underline{a}}{|\underline{a}|} = \frac{1}{\sqrt{3}}(\underline{i} - \underline{j} - \underline{k})$ A1

b. i. $\underline{b} \cdot \underline{c} = (2\underline{i} + 3\underline{j} - \underline{k}) \cdot (4\underline{i} - \underline{j} + 5\underline{k})$
 $= 8 - 3 - 5$ A1
 $= 0$

As $\underline{b} \cdot \underline{c} = 0$ and $|\underline{b}|, |\underline{c}| \neq 0$, then $\underline{b}$ is perpendicular to $\underline{c}$.

ii. Given $\hat{n} = x\underline{i} + y\underline{j} + z\underline{k}$.
 $\underline{b} \cdot \hat{n} = 0, 2x + 3y - z = 0$ (1)
 $\underline{c} \cdot \hat{n} = 0, 4x - y + 5z = 0$ (2)
 $|\hat{n}| = 1, x^2 + y^2 + z^2 = 1$ (3) A1 for (1), (2) and (3)

For example, $2 \times (1) - (2)$ gives $y = z$.

Substituting $y = z$ into (1) for example gives $x = -y$. M1

Substituting $y = z$ and $x = -y$ into (3) gives $x = -\frac{1}{\sqrt{3}}, y = \frac{1}{\sqrt{3}}$ and $z = \frac{1}{\sqrt{3}}$, since $x < 0$.

Hence $\hat{n} = -\frac{1}{\sqrt{3}}(\underline{i} - \underline{j} - \underline{k})$. A1

$\underline{a} = -\sqrt{3}\hat{n}$, i.e. $\underline{a}$ is perpendicular to both $\underline{b}$ and $\underline{c}$, so $\underline{a}, \underline{b}$ and $\underline{c}$ are mutually perpendicular. A1

iii. $V = |\overrightarrow{OA}| \times |\overrightarrow{OB}| \times |\overrightarrow{OC}|$
 $= \sqrt{3} \times \sqrt{14} \times \sqrt{42}$ M1 A1
 $= 42$

c. $\underline{c} = m\underline{a} + n\underline{b}$ where $m, n \neq 0$
 $4\underline{i} + p\underline{j} + 5\underline{k} = m(\underline{i} - \underline{j} - \underline{k}) + n(2\underline{i} + 3\underline{j} - \underline{k})$
 By equating components we obtain:
 $4 = m + 2n$ (1)
 $p = -m + 3n$ (2)
 $5 = -m - n$ (3) A1 for (1), (2) and (3)
 $(1) + (3)$ gives $n = 9$ A1
 Substituting $n = 9$ into (1) or (3) gives $m = -14$. A1
 Hence $p = 41$. A1

Question 3

a. Resolving perpendicular to the slope, $N = 60g \cos(\theta)$.
 Resolving parallel to the slope, $60g \sin(\theta) = 60a$. A1
 Hence $a = g \sin(\theta)$
 $= 9.8 \times \frac{1}{20}$ A1
 $= 0.49 \text{ m/s}^2$ A1

- b.** Michael travels down the slope with constant acceleration. If v is his final speed, then v can be determined from $v^2 = u^2 + 2as$.

$$v^2 = 0^2 + 2 \times 0.49 \times 5 \quad \text{A1}$$

$$v^2 = 4.9$$

$$v = 2.21 \text{ m/s} \quad \text{A1}$$

- c.** The time taken to reach the bottom can be found from $v = u + at$.

$$2.21 = 0 + 0.49t \quad \text{A1}$$

$$t = \frac{2.21}{0.49}$$

To the nearest second, this is 5 seconds. A1

- d.** On the level, the equation of Michael's motion is $a = -\frac{k}{v^2}$, where k is a constant.

- i.** This question concerns distance and speed. Use $a = v \frac{dv}{dx} = -\frac{k}{v^2}$. A1

$$\frac{dv}{dx} = -\frac{k}{v^3}$$

$$\frac{dx}{dv} = -\frac{v^3}{k}$$

$$x = \int -\frac{v^3}{k} dv$$

*From here, technology can be used to find the solution – see alternative solution.

$$x = -\frac{v^4}{4k} + c \quad \text{A1}$$

When $x = 0$, $v = 2.21$

$$0 = \frac{-(2.21)^4}{4k} + c$$

$$c = \frac{(2.21)^4}{4k} \quad \text{M1}$$

When $x = 5$, $v = 1.5$

$$5 = \frac{-(1.5)^4}{4k} + \frac{(2.21)^4}{4k}$$

$$4k = \frac{23.8544 - 5.0625}{5}$$

$$4k = 3.75839$$

$$c = \frac{23.8544}{3.75839}$$

$$x = \frac{23.8544 - v^4}{3.75839} \quad \text{A1}$$

$$\text{When } v = 1, x = \frac{22.8544}{3.75839} = 6.0809.$$

After approximately 6 m of travel on the level, Michael reaches a speed of 1 m/s. A1

***Alternative Solution (using technology):**

$$\text{To find } k, 5 = \int_{2.21}^{1.5} -\frac{v^3}{k} dv$$

$$k = \frac{1}{5} \int_{2.21}^{1.5} -v^3 dv$$

$$\cong 0.9396$$

M1

M1

Using a graphics calculator:

Calculator screen showing the integral calculation: $\frac{1}{5} \text{fnInt}(-V^3, V, 2.21, 1.5)$ resulting in $.9395966405$.

Using CAS:

CAS calculator screen showing the integral calculation: $\frac{1}{5} \cdot \int_{2.21}^{1.5} -v^3 dv$ resulting in $.9395966405$. The screen also shows the input $\frac{1}{5} \int(-v^3, v, 2.21, 1.5)$ and the output $.9395966405$.

$$\text{Then } x = \int_{2.21}^1 \left(-\frac{v^3}{k}\right) dv$$

$$= \frac{1}{0.9396} \int_{2.21}^1 (-v^3) dv$$

$$= 6.0809$$

M1

After approximately 6 m of travel on the level, Michael reaches a speed of 1 m/s.

A1

Using a graphics calculator:

Calculator screen showing the integral calculation: $\frac{1}{.9396} \text{fnInt}(-V^3, V, 2.21, 1)$ resulting in 6.080894213 .

Using CAS:

CAS calculator screen showing the integral calculation: $\frac{1}{.9396} \cdot \int_{2.21}^1 -v^3 dv$ resulting in 6.08089421296 . The screen also shows the input $\frac{1}{.9396} \int(-v^3, v, 2.21, 1)$ and the output 6.08089421296 .

- ii. Since this part concerns time and speed, use $a = \frac{dv}{dt} = -\frac{3.75839}{4v^2}$.

$$\frac{dt}{dv} = -\frac{4v^2}{3.75839}$$

$$t = \int -\frac{4v^2}{3.75839} dv$$

$$t = -\frac{4v^3}{(3.75839 \times 3)} + c$$

A1

When $t = 0$, $v = 2.21$.

$$0 = -\frac{4(2.21)^3}{(3.75839 \times 3)} + c$$

$$c = \frac{4(2.21)^3}{(3.75839 \times 3)}$$

$$t = \frac{4((2.21)^3 - v^3)}{3.75839 \times 3}$$

A1

When $v = 1$, t is given by

$$t = \frac{4((2.21)^3 - 1^3)}{3.75839 \times 3}$$

$$t = 3.474$$

Michael reaches a speed of 1 m/s on the level after 3 seconds (correct to the nearest second).

A1

Alternative Solution (using technology):

$$\frac{dv}{dt} = -\frac{0.9396}{v^2}$$

$$\frac{dt}{dv} = -\frac{v^2}{0.9396}$$

M1

$$t = \int_{2.21}^1 \left(-\frac{v^2}{0.9396} \right) dv$$

M1

$$= 3.474$$

Michael reaches a speed of 1 m/s on the level after 3 seconds (correct to the nearest second).

A1

Using a graphics calculator:

fnInt(-v²/0.9396, v, 2.21, 1)
3.474478856

Using CAS:

$\int_{2.21}^1 \left(-\frac{v^2}{0.9396} \right) dv$
3.47447885625
f(-v²/0.9396, v, 2.21, 1)
RAD EXACT FUNC 1/30

Question 4

a. $|\dot{\mathbf{r}}(t)| = \sqrt{(9t - 3t^2)^2 + (\log_e(1 + (t - 3)^4))^2}$
 $|\dot{\mathbf{r}}(1)| = \sqrt{(9 - 3)^2 + (\log_e(1 + (1 - 3)^4))^2}$ M1

Hence $|\dot{\mathbf{r}}(1)| = 6.64$, i.e. the particle's speed is 6.64 m/s (correct to two decimal places). A1

b. Attempting to solve $9t - 3t^2 = 0$ and $\log_e(1 + (t - 3)^4) = 0$ for t : M1

From $9t - 3t^2 = 0$ we obtain $t = 0, 3$ and from $\log_e(1 + (t - 3)^4) = 0$ we obtain $t = 3$. A1

Hence the particle is at rest at $t = 3$. A1

c. The gradient of the curve is given by $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$. M1

At $t = 1$, $\frac{dy}{dx} = \frac{\log_e(17)}{6} = 0.47$ (correct to two decimal places). A1

d. $y(t) = y(0) + \int_0^t \log_e(1 + (t - 3)^4) dt$ M1

$y(1) = 2 + \int_0^1 \log_e(1 + (t - 3)^4) dt$ A1

Hence the y-coordinate at P is 5.67 (correct to two decimal places). A1

Question 5

a. Let $z = \text{cis}\left(\frac{\pi}{6}\right)$

i. $z^2 = \left(\text{cis}\left(\frac{\pi}{6}\right)\right)^2 = \text{cis}\left(\frac{2\pi}{6}\right)$ using de Moivre's theorem A1

$$= \cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$$
 A1

ii. $z^4 = \left(\text{cis}\left(\frac{\pi}{6}\right)\right)^4 = \text{cis}\left(\frac{4\pi}{6}\right)$ using de Moivre's theorem

$$= \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)$$

$$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$
 A1

$$\text{b. } z^4 - z^2 + 1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + 1$$

$$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i - \frac{1}{2} - \frac{\sqrt{3}}{2}i + 1$$

A1

$$= 0, \text{ as required}$$

A1

- c. i. Using the factor theorem, one linear factor is $z - \text{cis}\left(\frac{\pi}{6}\right)$. The other will be $z - \text{cis}\left(-\frac{\pi}{6}\right)$ (conjugate root theorem).

M1

$$\text{cis}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{i}{2} \text{ and } \text{cis}\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{i}{2}$$

$$\text{In Cartesian form, these factors are } \left(z - \frac{\sqrt{3}}{2} - \frac{i}{2}\right) \text{ and } \left(z - \frac{\sqrt{3}}{2} + \frac{i}{2}\right).$$

A1

- ii. The product of these factors is $\left(z - \frac{\sqrt{3}}{2} - \frac{i}{2}\right)\left(z - \frac{\sqrt{3}}{2} + \frac{i}{2}\right)$.

$$= \left(z - \frac{\sqrt{3}}{2}\right)^2 - \left(\frac{i}{2}\right)^2$$

$$= z^2 - \sqrt{3}z + \frac{3}{4} + \frac{1}{4}$$

$$= z^2 - \sqrt{3}z + 1$$

A1

$$\text{d. } z^4 - z^2 + 1 = z^4 + 2z^2 + 1 - 3z^2$$

$$= (z^2 + 1)^2 - (\sqrt{3}z)^2$$

$$= (z^2 + 1 - \sqrt{3}z)(z^2 + 1 + \sqrt{3}z)$$

A1

Comparing the quadratic factors with the result of c.ii. enables us to obtain the other set of linear

$$\text{factors as } \left(z + \frac{\sqrt{3}}{2} - \frac{i}{2}\right) \text{ and } \left(z + \frac{\sqrt{3}}{2} + \frac{i}{2}\right).$$

A2

Alternative solution:

A more elegant method is to recognise $z^4 - z^2 + 1$ as an even function of z .

Hence if $f(a) = 0$, then $f(-a) = 0$ also.

M1

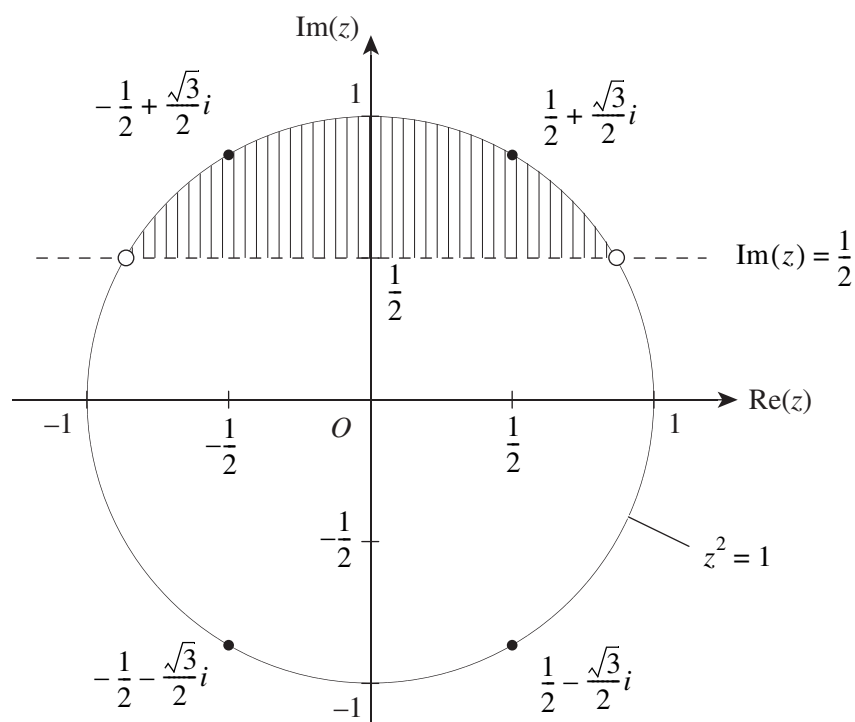
Thus if $(z - a)$ is a factor, so is $(z + a)$,

A1

leading to the same result for the factors.

A1

e.



- i. All points correct and labelled. A1
- ii. Circle $|z| = 1$ and line $\text{Im}(z) = \frac{1}{2}$ shown correctly and labelled. A1
- Region of intersection, with boundary markings, correctly shown. A1