

Working space

Instructions

Answer **all** questions in the spaces provided.

A decimal approximation will not be accepted if an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Take the acceleration due to gravity to have magnitude $g \text{ m/s}^2$, where $g = 9.8$.

Question 1

Given that $P(z) = z^4 - 4z^3 + 6z^2 - 4z + 5 = 0$ and $P(2 + i) = 0$, find all the roots of $P(z) = 0$.

3 marks

TURN OVER

Question 2

$\underline{u}$ and $\underline{v}$ are vectors defined by $\underline{u} = \cos(\theta)\underline{i} + \sin(\theta)\underline{j}$, $\underline{v} = \sin(\theta)\underline{i} + \cos(\theta)\underline{j}$ and $0 < \theta < \frac{\pi}{2}$.

- a. Show that $\underline{u}$ and $\underline{v}$ are unit vectors.

1 mark

- b. Let α be the angle between the vectors $\underline{u}$ and $\underline{v}$. Express α in terms of θ .

1 mark

- c. Find α when $\theta = \frac{\pi}{6}$.

1 mark

- d. If $\theta = \frac{\pi}{3}$, find the vector resolute of $\underline{v}$ in the direction of $\underline{u}$.

2 marks

Question 3

- a. Express $\frac{x+2}{x^2+x}$ in partial fractions with integer numerators.

2 marks

- b. Hence show that $\int_{-4}^{-3} \frac{x+2}{x^2+x} dx = \log_e \left(\frac{a}{b} \right)$ where a and b are positive integers.

Find the values of a and b .

3 marks

Question 4

- a.** Show that, for $0 < x < \frac{1}{3}$, $\frac{d}{dx}(\cos^{-1}(\sqrt{3x})) = \frac{-3}{2\sqrt{3x(1-3x)}}$

[illegible]

2 marks

- b. Hence, find the exact value of** $\int_{\frac{1}{12}}^{\frac{1}{6}} \frac{1}{\sqrt{3x(1-3x)}} dx$

3 marks

Question 5

An object of mass 2 kg falls from rest from a height of 50 metres. Its fall is opposed by an air resistance of magnitude of $0.05v^2$ newton, where v is its velocity.

- a.** Write an equation of motion for the falling object.

1 mark

- b.** Show that $\frac{dx}{dv} = \frac{40v}{40g - v^2}$

2 marks

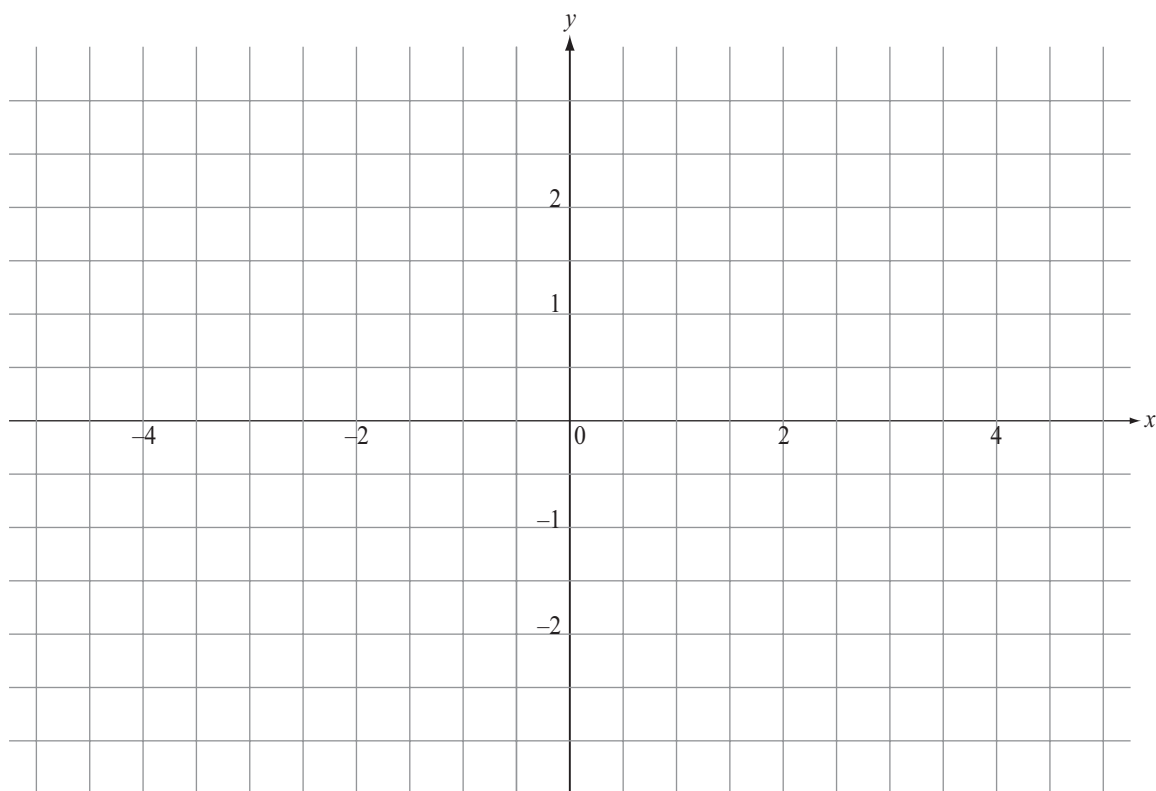
- c.** Hence, find the exact distance travelled for the object to reach a speed of 10 m/s.

3 marks

Question 6

Let $f(x) = \arctan(x) + \frac{\pi}{4}$, $x \in \mathbb{R}$.

- a. On the axes below, sketch the graph of $f(x)$. On the sketch, clearly indicate the asymptotes and axes intercepts.



3 marks

- b. Solve $f(x) = \frac{5\pi}{12}$

1 mark

Question 7

At time t seconds, a particle has position vector

$$\underline{r} = (3 \cos(t) - \sin(2t))\underline{i} + (3 \sin(t) + \cos(2t))\underline{j}, \text{ where } t \geq 0.$$

- a.** Find its velocity vector $\underline{v}$.

2 marks

- b.** Find its maximum speed.

3 marks

- c.** Show that the particle never stops.

1 mark

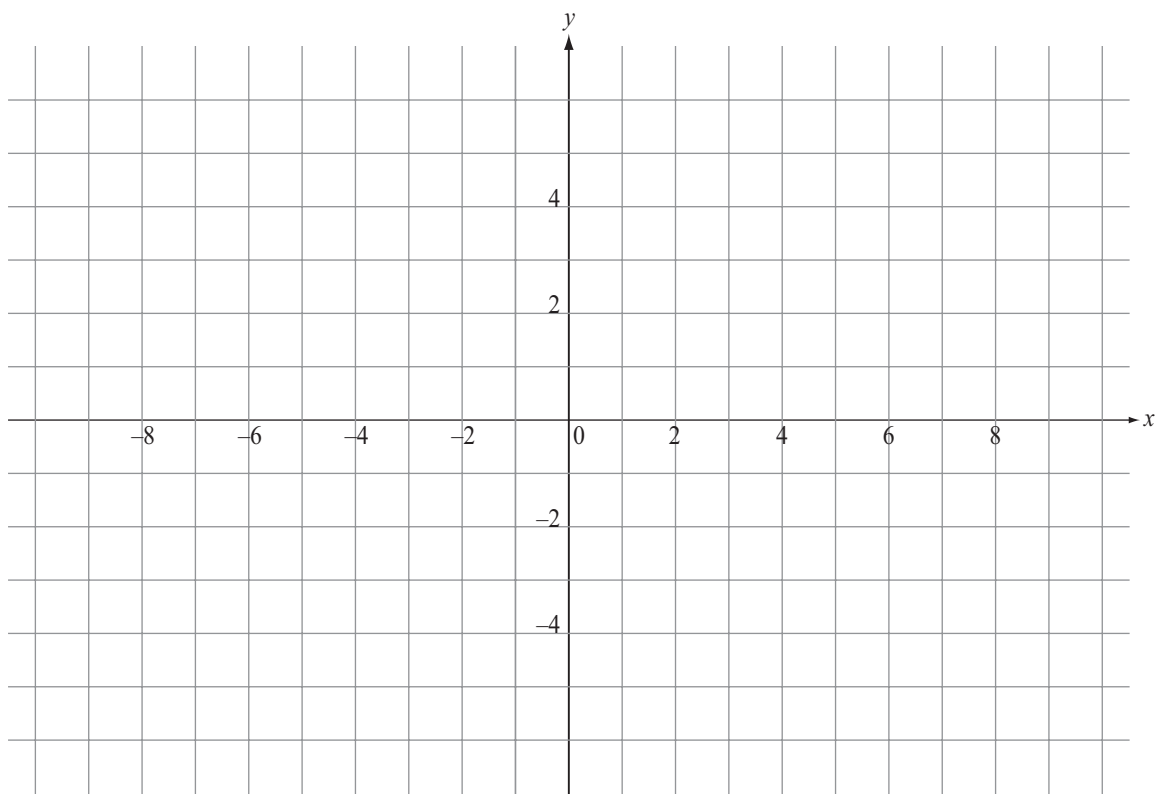
Question 8

The position vector of a particle is given by $\underline{r}(t) = 2\tan(t)\underline{i} + \sec(t)\underline{j}$ where $t \geq 0$.

- a. Find the Cartesian equation of the path of the particle.

2 marks

- b. Sketch the curve on the grid below, showing all important features.



2 marks

- c. Find the exact volume of revolution formed by rotating this curve between $y = 1$ and $y = 2$ about the y -axis.

2 marks

Total 40 marks

Specialist Mathematics Exam 1 2007 Solutions

Question 1

The equation has real coefficients therefore the conjugate root theorem applies.
So $2 - i$ is another root. A1

The two factors can be expressed as a quadratic as follows:

$$(z - 2 - i)(z - 2 + i) = z^2 - 4z + 5$$
A1

Divide $z^2 - 4z + 5$ into $z^4 - 4z^3 + 6z^2 - 4z + 5$ to obtain $z^2 + 1$ M1

$$\begin{array}{r} z^2 + 1 \\ z^2 - 4z + 5 \overline{) z^4 - 4z^3 + 6z^2 - 4z + 5} \\ \underline{z^4 - 4z^3 + 5z^2} \\ z^2 - 4z + 5 \\ \underline{z^2 - 4z + 5} \\ 0 \end{array}$$

$$\therefore (z^2 - 4z + 5)(z^2 + 1) = 0$$

$$(z - 2 - i)(z - 2 + i)(z - i)(z + i) = 0$$

$$\therefore z = 2 + i, 2 - i, i, -i$$

Solutions are: $z = 2 \pm i$ and $z = \pm i$ A1

Question 2

a. $\underline{u} = \cos(\theta)\underline{i} + \sin(\theta)\underline{j}$ and $\underline{v} = \sin(\theta)\underline{i} + \cos(\theta)\underline{j}$

$$\begin{aligned} |\underline{u}| &= \sqrt{\cos^2(\theta) + \sin^2(\theta)} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$
A1

$$\begin{aligned} |\underline{v}| &= \sqrt{\sin^2(\theta) + \cos^2(\theta)} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

Hence, both $\underline{u}$ and $\underline{v}$ are unit vectors.

b. $\cos(\alpha) = \frac{\cos(\theta)\sin(\theta) + \sin(\theta)\cos(\theta)}{\sqrt{1} \times \sqrt{1}}$ M1

$$\begin{aligned} &= 2\sin(\theta)\cos(\theta) \\ &= \sin(2\theta) \end{aligned}$$

$$\alpha = \cos^{-1}(\sin(2\theta)) \text{ or } \alpha = \frac{\pi}{2} - 2\theta$$
A1

c. $\alpha = \cos^{-1}\left(\sin\left(\frac{2 \times \pi}{6}\right)\right)$ A1

$$\begin{aligned} &= \cos^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right) \\ &= \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{\pi}{6} \end{aligned}$$

$$\begin{aligned}
 \text{d. } (\vec{v} \cdot \vec{u})\vec{u} &= \left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\right)\left(\frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j}\right) & \text{M1} \\
 &= \frac{\sqrt{3}}{4}\vec{i} + \frac{3}{4}\vec{j} \text{ or} \\
 &= \frac{1}{4}(\sqrt{3}\vec{i} + 3\vec{j}) & \text{A1}
 \end{aligned}$$

Question 3

$$\text{a. } \frac{x+2}{x^2+x} = \frac{A}{x} + \frac{B}{x+1} \text{ where } A \text{ and } B \text{ are constants.} \quad \text{A1}$$

$$\therefore x+2 = A(x+1) + B(x)$$

$$\text{Let } x = 0 \text{ so } A = 2$$

$$\text{Let } x = -1 \text{ so } B = -1$$

A1 (both A and B correct)

$$\therefore \frac{x+2}{x^2+x} = \frac{2}{x} - \frac{1}{x+1}$$

$$\begin{aligned}
 \text{b. } \int_{-4}^{-3} \left(\frac{x+2}{x^2+x}\right) dx &= \int_{-4}^{-3} \left(\frac{2}{x} - \frac{1}{x+1}\right) dx \\
 &= [2\log_e|x| - \log_e|x+1|]_{-4}^{-3} & \text{A2 for anti-derivatives} \\
 &= (2\log_e 3 - \log_e 2) - (2\log_e 4 - \log_e 3) & \text{Modulus sign missing} = -1 \\
 &= \log_e(27/32)
 \end{aligned}$$

$$\text{Answer: } a = 27, b = 12$$

Note: cannot get this mark from logs of negative numbers. Equivalent multiples of a and b in non-simplified fraction is correct.

Question 4

$$\text{a. Let } u = \sqrt{3x} \text{ and } w = 3x$$

$$u = \sqrt{w} \text{ and so } \frac{du}{dw} = \frac{1}{2\sqrt{w}} \text{ and } \frac{dw}{dx} = 3$$

$$\begin{aligned}
 \frac{du}{dx} &= \frac{du}{dw} \times \frac{dw}{dx} \\
 &= \frac{3}{2\sqrt{3x}} & \text{A1}
 \end{aligned}$$

$$y = \cos^{-1}(u) \text{ and so } \frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}}$$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\
 &= \frac{-1}{\sqrt{1-u^2}} \times \frac{3}{2\sqrt{3x}} & \text{M1} \\
 &= \frac{-1}{\sqrt{1-3x}} \times \frac{3}{2\sqrt{3x}} \\
 &= \frac{-3}{2\sqrt{3x(1-3x)}}
 \end{aligned}$$

Hence shown.

b. $\int_{\frac{1}{12}}^{\frac{1}{6}} \frac{1}{\sqrt{3x(1-3x)}} dx$ A1 for $-\frac{2}{3}$ in front

$$= -\frac{2}{3} \int_{\frac{1}{12}}^{\frac{1}{6}} \frac{-3}{2\sqrt{3x(1-3x)}} dx$$

M1 for recognition

$$= -\frac{2}{3} [\cos^{-1}(\sqrt{3x})]_{\frac{1}{12}}^{\frac{1}{6}}$$

$$= -\frac{2}{3} \left(\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) - \cos^{-1}\left(\frac{1}{2}\right) \right)$$

$$= -\frac{2}{3} \left(\frac{\pi}{4} - \frac{\pi}{3} \right)$$

$$= \frac{\pi}{18}$$

A1

Question 5

a. $2a = 2g - 0.05v^2 \therefore a = g - \frac{v^2}{40}$ A1

b. Using $a = v \frac{dv}{dx}$ in the equation of motion gives:

$$v \frac{dv}{dx} = \frac{2g - 0.05v^2}{2}$$

M1

$$\frac{dv}{dx} = \frac{2g - 0.05v^2}{2v}$$

$$\frac{dx}{dv} = \frac{2v}{2g - 0.05v^2}$$

A1

Multiplying numerator and denominator by 20 gives

$$\frac{dx}{dv} = \frac{40v}{40g - v^2} \text{ as required.}$$

c. The required distance is given by the integral: $\int_0^{10} \frac{40v}{40g - v^2} dv$ A1

Note: The integral must have correct limits and dv . Does not need to have a modulus of

$\frac{40v}{40g - v^2}$, since we are after distance and the graph was not asked for.

$$x = -20 \int_0^{10} \frac{-2v}{-v^2 + 40g} dv$$

M1

$$= [-20 \log_e(40g - v^2)]_0^{10}$$

M1

$$= -20 \log_e(40g - 100) + 20 \log_e(40g)$$

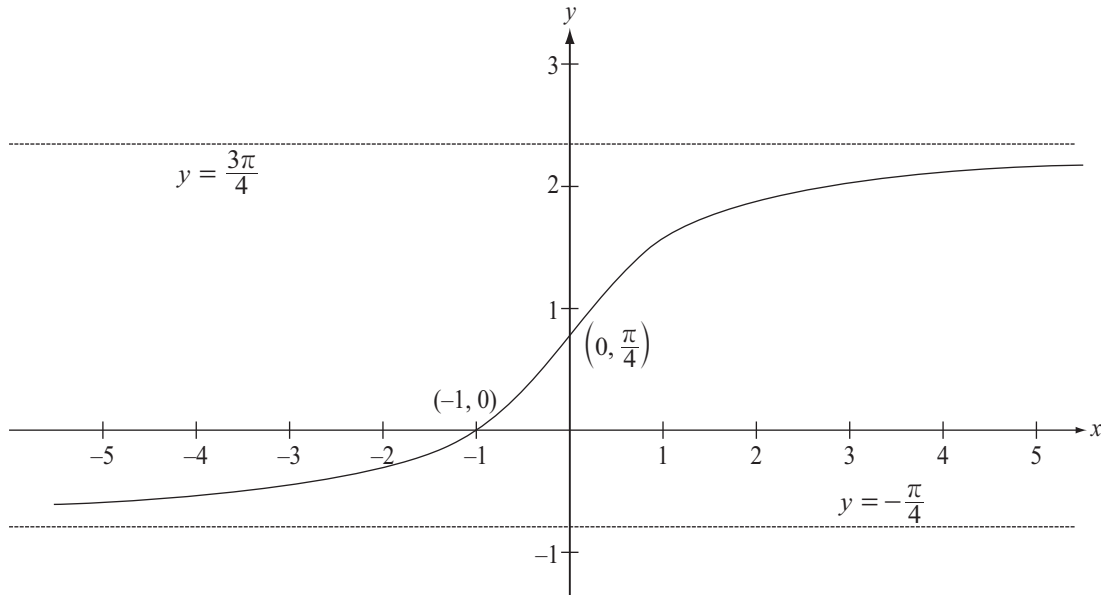
$$= 20 \log_e \left(\frac{40g}{40g - 100} \right)$$

$$= 20 \log_e \left(\frac{2g}{2g - 5} \right)$$

Note: $20 \log_e \left(\frac{40g}{40g - 100} \right)$ can get the last A1 mark. A1

Question 6

a.



x -intercept $(-1, 0)$

A1

y -intercept $(0, \frac{\pi}{4})$

A1

Asymptotes $y = -\frac{\pi}{4}$ and $y = \frac{3\pi}{4}$ and shape.

A1

b. $\arctan(x) + \frac{\pi}{4} = \frac{5\pi}{12}$

$$\arctan(x) = \frac{\pi}{6}$$

$$x = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

A1

Question 7

a. Differentiating $\underline{r}$ with respect to t :

$$\underline{r} = (-3 \sin(t) - 2 \cos(2t))\underline{i} + (3 \cos(t) - 2 \sin(2t))\underline{j}$$

A2 (1 each $\underline{i}$, $\underline{j}$ term)

b. Speed = $|\underline{v}|$

$$= \sqrt{(3 \sin(t) + 2 \cos(2t))^2 + (3 \cos(t) - 2 \sin(2t))^2}$$

M1

$$= \sqrt{9 \sin^2(t) + 12 \sin(t) \cos(2t) + 4 \cos^2(2t) + 9 \cos^2(t) - 12 \cos(t) \sin(2t) + 4 \sin^2(2t)}$$

$$= \sqrt{(9 \sin^2(t) + 9 \cos^2(t)) + 12 (\sin(t) \cos(2t) - \cos(t) \sin(2t)) + (4 \cos^2(2t) + 4 \sin^2(2t))}$$

$$= \sqrt{9 + 4 + 12 \sin(t - 2t)}$$

M1 for using the compound angle formula

$$= \sqrt{13 - 12 \sin(t)}$$

$\therefore$ Maximum speed is $\sqrt{13 + 12}$ when $\sin(t) = -1$

$\therefore$ Maximum speed is 5.

A1

c. $\sqrt{13 - 12 \sin(t)}$

$$-1 \leq \sin(t) \leq 1$$

$$\therefore -12 \leq 12 \sin(t) \leq 12$$

$$\therefore \sqrt{13 - 12} = 1, \sqrt{13 + 12} = 5$$

$\therefore$ speed will always be between 1 and 5

A1

$\therefore$ it never stops

Question 8

a. $\frac{x}{2} = \tan(t)$ and $y = \sec(t)$

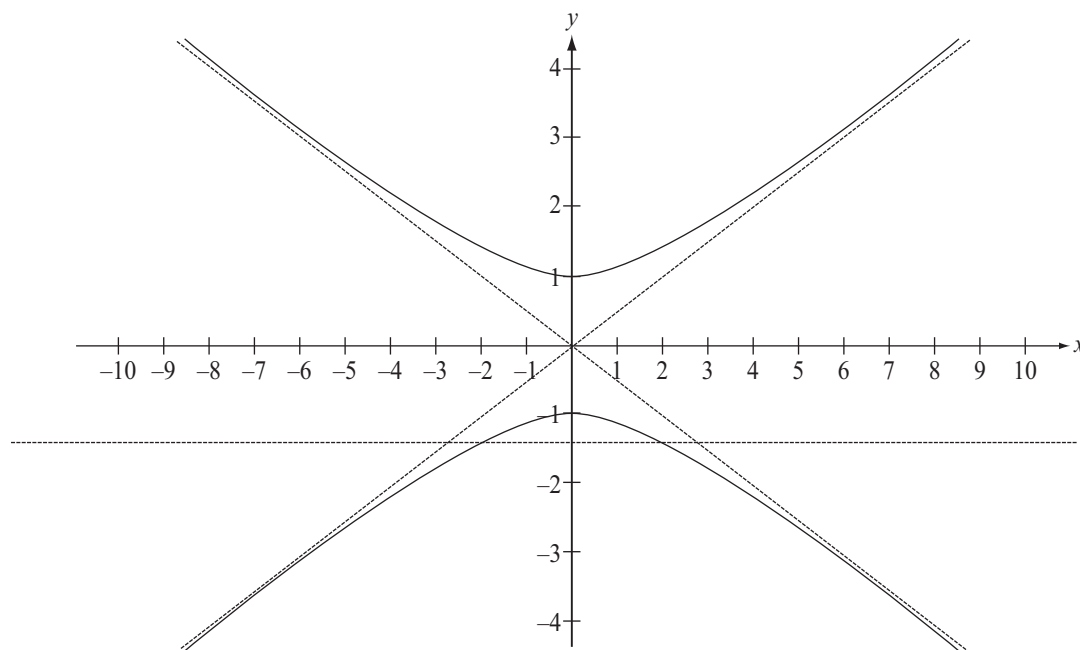
$$1 + \tan^2(t) = \sec^2(t)$$

M1

$$1 + \frac{x^2}{4} = y^2$$

$$1 = \frac{y^2}{1} - \frac{x^2}{4}$$

b.



2 marks: A1 shape and asymptotes $y = \pm \frac{x}{2}$; **A1** y-intercepts $(0, \pm 1)$

c. $\int_1^2 \pi x^2 dy = \int_1^2 4\pi (y^2 - 1) dy$

$$= \left[4\pi \left(\frac{y^3}{3} - y \right) \right]_1^2$$

M1

$$= 4\pi \left[\left(\frac{8}{3} - 2 \right) - \left(\frac{1}{3} - 1 \right) \right]$$

$$= \frac{16\pi}{3} \text{ cubic units}$$

A1