

Year 2007
VCE
Specialist Mathematics
Solutions
Trial Examination 1



KILBAHA MULTIMEDIA PUBLISHING
PO BOX 2227
KEW VIC 3101
AUSTRALIA

TEL: (03) 9817 5374
FAX: (03) 9817 4334
chemas@chemas.com
www.chemas.com

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Question 1

$$\frac{x}{e^{2y}} - 5y + 6x^2 + 3 = 0 \quad \text{taking } \frac{d}{dx} \text{ of each term (implicit differentiation)}$$

$$\frac{d}{dx}(xe^{-2y}) - \frac{d}{dx}(5y) + \frac{d}{dx}(6x^2) + \frac{d}{dx}(3) = 0 \quad \text{M1}$$

product rule in the first term

$$x \frac{d}{dy}(e^{-2y}) \frac{dy}{dx} + e^{-2y} \frac{d}{dx}(x) - \frac{d}{dy}(5y) \frac{dy}{dx} + \frac{d}{dx}(6x^2) + \frac{d}{dx}(3) = 0$$

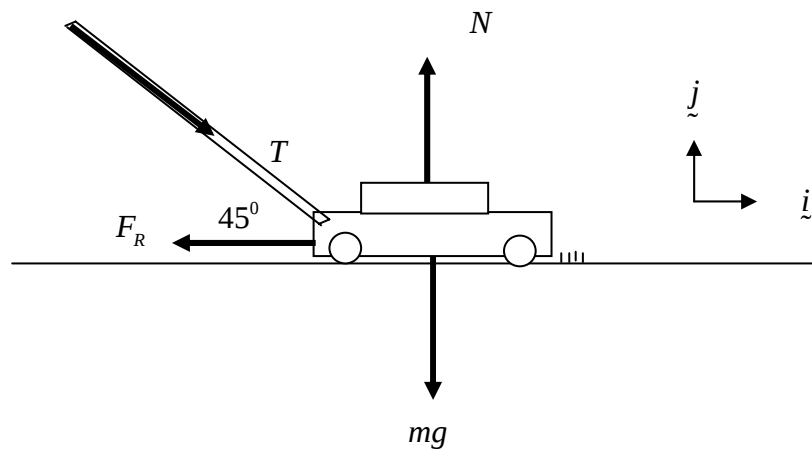
$$-2xe^{-2y} \frac{dy}{dx} + e^{-2y} - 5 \frac{dy}{dx} + 12x = 0 \quad \text{M1}$$

$$\frac{dy}{dx}(2xe^{-2y} + 5) = e^{-2y} + 12x$$

$$\frac{dy}{dx} = \frac{e^{-2y} + 12x}{2xe^{-2y} + 5} \quad \text{A1}$$

Question 2

a.



for all the forces

A1

b. now $T = 12\sqrt{2}$ $m = 10 \text{ kg}$ $g = 9.8$ $\mu = \frac{1}{7}$ $\theta = 45^\circ$

resolving horizontally to the lawn in the $\underline{i}$ direction.

$$(1) \quad T \cos(\theta) - F_R = 0$$

$$\text{from (1)} \quad F_R = T \cos(\theta) = 12\sqrt{2} \cos(45^\circ) = 12 \quad \text{A1}$$

resolving perpendicular to the lawn in the $\underline{j}$ direction.

$$(2) \quad N - T \sin(\theta) - mg = 0$$

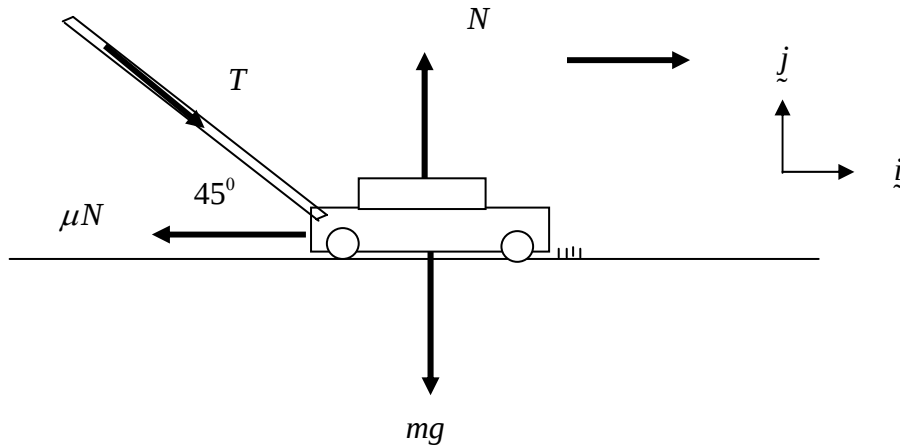
$$\text{from (2)} \quad N = mg + T \sin(\theta) = 10 \times 9.8 + 12\sqrt{2} \sin(45^\circ) = 98 + 12 = 110$$

$$\text{and } \mu N = \frac{110}{7} = 15\frac{5}{7}$$

since $\mu N > F_R$ it is **not** on the point of moving

A1

c.



now $T = \frac{49\sqrt{2}}{2}$ $m = 10 \text{ kg}$ $g = 9.8$ $\mu = \frac{1}{7}$ $\theta = 45^\circ$ $a = ?$

resolving horizontally to the lawn in the $\hat{i}$ direction.

$$(1) \quad T \cos(\theta) - \mu N = ma \quad \text{M1}$$

resolving perpendicular to the lawn in the $\hat{j}$ direction.

$$(2) \quad N - T \sin(\theta) - mg = 0 \quad \text{from (2) } N = T \sin(\theta) + mg \quad \text{into (1)}$$

$$ma = T \cos(\theta) - \mu(T \sin(\theta) + mg)$$

A1

$$ma = T(\cos(\theta) - \mu \sin(\theta)) - \mu mg$$

$$a = \frac{T}{m}(\cos(\theta) - \mu \sin(\theta)) - \mu g$$

$$a = \frac{49\sqrt{2}}{20} \left(\cos(45^\circ) - \frac{1}{7} \sin(45^\circ) \right) - \frac{9.8}{7}$$

$$a = \frac{49\sqrt{2}}{20} \left(\frac{1}{\sqrt{2}} - \frac{1}{7\sqrt{2}} \right) - 1.4$$

$$a = \frac{49\sqrt{2}}{20} \times \frac{6}{7\sqrt{2}} - 1.4$$

$$a = 2.1 - 1.4$$

$$a = 0.7 \text{ m/s}^2$$

A1

Question 3

$$\frac{dy}{dx} = \frac{3x-5}{\sqrt{9-4x^2}} \quad \text{integrating with respect to } x$$

$$y = \int \frac{3x-5}{\sqrt{9-4x^2}} dx \quad \text{separating into two integrals}$$

$$y = \int \frac{3x}{\sqrt{9-4x^2}} dx - \int \frac{5}{\sqrt{9-4x^2}} dx \quad \text{M1}$$

$$\text{in the first integral let } u = 9 - 4x^2, \quad \frac{du}{dx} = -8x$$

$$\text{in the second integral let } v = 2x, \quad \frac{dv}{dx} = 2$$

$$y = -\frac{3}{8} \int u^{-\frac{1}{2}} du - \frac{5}{2} \int \frac{1}{\sqrt{9-v^2}} dv \quad \text{M1}$$

$$y = -\frac{3}{4} u^{\frac{1}{2}} - \frac{5}{2} \sin^{-1}\left(\frac{v}{3}\right) + C$$

$$y = -\frac{3}{4} \sqrt{9-4x^2} - \frac{5}{2} \sin^{-1}\left(\frac{2x}{3}\right) + C \quad \text{A2}$$

to find C use $y(0) = 0$

$$0 = -\frac{9}{4} - 0 + C \quad C = \frac{9}{4}$$

$$y = -\frac{3}{4} \sqrt{9-4x^2} - \frac{5}{2} \sin^{-1}\left(\frac{2x}{3}\right) + \frac{9}{4} \quad \text{A1}$$

Note that a possible correct alternative answer is

$$y = -\frac{3}{4} \sqrt{9-4x^2} + \frac{5}{2} \cos^{-1}\left(\frac{2x}{3}\right) + \frac{9}{4} - \frac{5\pi}{4}$$

Question 4

- a. let $P(z) = z^3 + pz^2 + qz + 15 = 0$ since p and q are real
by the conjugate root theorem $P(1-2i) = P(1+2i) = 0$
so $1+2i$ is also a root

A1

- b. let $\alpha = 1+2i$ $\beta = 1-2i$
now $\alpha + \beta = 2$ and $\alpha\beta = 1-4i^2 = 5$
so the quadratic $z^2 - 2z + 5$ is a factor
 $P(z) = z^3 + pz^2 + qz + 15 = (z^2 - 2z + 5)(z + c) = 0$
now $5c = 15$ so that $c = 3$ and expanding $(z^2 - 2z + 5)(z + 3)$
coefficient of z^2 : $p = 3 - 2 = 1$
coefficient of z : $q = 5 - 6 = -1$
and all the roots are $z = 1 \pm 2i$ and $z = -3$

M1

A2

Question 5

- a. **Method I** using addition theorems $\frac{\pi}{12} = 15^\circ$ $15^\circ = 45^\circ - 30^\circ$

$$\tan\left(\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{6}\right)}{1 + \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{6}\right)}$$

M1

$$\tan\left(\frac{\pi}{12}\right) = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$$

M1

$$\tan\left(\frac{\pi}{12}\right) = \frac{3 - 2\sqrt{3} + 1}{3 - 1} = \frac{4 - 2\sqrt{3}}{2}$$

$$\tan\left(\frac{\pi}{12}\right) = 2 - \sqrt{3}$$

A1

Alternative Method II using double angle formulae let $A = \frac{\pi}{12}$ $2A = \frac{\pi}{6}$

$$\tan(2A) = \frac{2 \tan(A)}{1 - \tan^2(A)}$$

$$\frac{1}{\sqrt{3}} = \frac{2 \tan\left(\frac{\pi}{12}\right)}{1 - \tan^2\left(\frac{\pi}{12}\right)} \quad \text{M1}$$

$$1 - \tan^2\left(\frac{\pi}{12}\right) = 2\sqrt{3} \tan\left(\frac{\pi}{12}\right)$$

$$\tan^2\left(\frac{\pi}{12}\right) + 2\sqrt{3} \tan\left(\frac{\pi}{12}\right) - 1 = 0$$

$$\text{let } u = \tan\left(\frac{\pi}{12}\right) \quad u^2 + 2\sqrt{3}u - 1 = 0$$

$$\Delta = (2\sqrt{3})^2 + 4 = 16$$

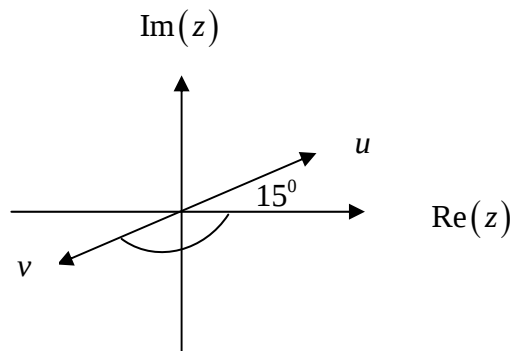
$$u = \tan\left(\frac{\pi}{12}\right) = \frac{-2\sqrt{3} \pm \sqrt{16}}{2} \quad \text{but } \tan\left(\frac{\pi}{12}\right) > 0 \quad \text{take the positive} \quad \text{M1}$$

$$\text{so that } \tan\left(\frac{\pi}{12}\right) = 2 - \sqrt{3} \quad \text{A1}$$

b. let $u = 1 + (2 - \sqrt{3})i$ now $\text{Arg}(u) = \tan^{-1}(2 - \sqrt{3}) = \frac{\pi}{12}$

let $v = -1 + (\sqrt{3} - 2)i = -u = i^2 u$ v is a rotation of 180° from u , so

$$\text{Arg}(v) = \text{Arg}(-1 + (\sqrt{3} - 2)i) = -\pi + \frac{\pi}{12} = -\frac{11\pi}{12} \quad (\text{or } -165^\circ) \quad \text{A1}$$



Question 6

a. Let $y = \tan^{-1}\left(\sqrt{\frac{3}{x}}\right) = \tan^{-1}(u)$ where $u = \sqrt{\frac{3}{x}} = \sqrt{3}x^{-\frac{1}{2}}$

$$\frac{dy}{du} = \frac{1}{1+u^2} \quad \frac{du}{dx} = -\frac{\sqrt{3}}{2}x^{-\frac{3}{2}} = \frac{-\sqrt{3}}{2\sqrt{x^3}} \quad \text{chain rule} \quad \text{M1}$$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{-\sqrt{3}}{2\sqrt{x^3}\left(1+\frac{3}{x}\right)}$$

$$\frac{dy}{dx} = \frac{-\sqrt{3}}{2\sqrt{x^3}\left(\frac{x+3}{x}\right)} \quad \text{since } x > 0$$

$$\frac{dy}{dx} = \frac{-\sqrt{3}}{2\sqrt{x}(x+3)}$$

so shown $\frac{d}{dx}\left(\tan^{-1}\left(\sqrt{\frac{3}{x}}\right)\right) = \frac{-\sqrt{3}}{2\sqrt{x}(x+3)} \quad \text{for } x > 0 \quad \text{A1}$

b. $\int_1^9 \frac{1}{\sqrt{x^3+6x^2+9x}} dx = \int_1^9 \frac{1}{\sqrt{x}(x^2+6x+9)} dx =$

$$\int_1^9 \frac{1}{\sqrt{x}(x+3)^2} dx = \int_1^9 \frac{1}{\sqrt{x}(x+3)} dx \quad \text{since } x > 0$$

$$= -\frac{2}{\sqrt{3}} \left[\tan^{-1}\left(\sqrt{\frac{3}{x}}\right) \right]_1^9 \quad \text{M1}$$

$$= -\frac{2}{\sqrt{3}} \left(\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) - \tan^{-1}(\sqrt{3}) \right)$$

$$= -\frac{2\sqrt{3}}{3} \left(\frac{\pi}{6} - \frac{\pi}{3} \right)$$

$$= \frac{\sqrt{3}\pi}{9} \quad \text{A1}$$

Question 7

a. $\vec{r}(t) = \frac{3}{2}(e^{2t} + e^{-2t})\vec{i} + \frac{5}{2}(e^{2t} - e^{-2t})\vec{j}$ vector equation,

the parametric equations are $x = \frac{3}{2}(e^{2t} + e^{-2t})$ and $y = \frac{5}{2}(e^{2t} - e^{-2t})$

now $x^2 = \frac{9}{4}(e^{4t} + 2 + e^{-4t})$ and $y^2 = \frac{25}{4}(e^{4t} - 2 + e^{-4t})$ A1

so that $\frac{4x^2}{9} = (e^{4t} + 2 + e^{-4t})$ and $\frac{4y^2}{25} = (e^{4t} - 2 + e^{-4t})$

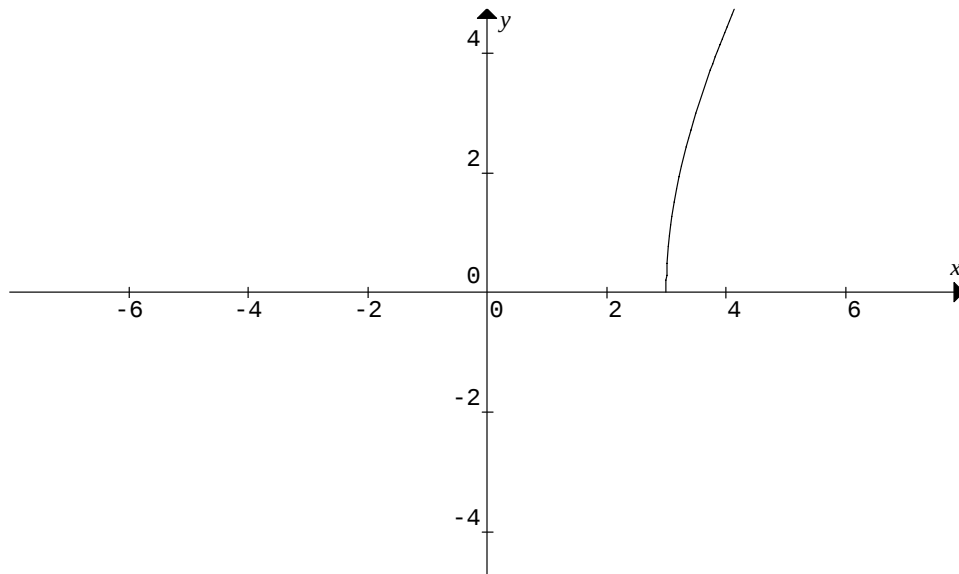
subtracting to eliminate t gives M1

$$\frac{4x^2}{9} - \frac{4y^2}{25} = 4$$

$$\frac{x^2}{9} - \frac{y^2}{25} = 1 \quad a^2 = 9 \quad b^2 = 25$$

since $a > 0$ and $b > 0$ $a = 3$ $b = 5$ A1

- b. since $t \geq 0$ both $x \geq 0$ and $y \geq 0$, the graph is not the whole hyperbola only the upper right branch, touching the x -axis at $(3, 0)$ A1



Question 8

a. $y = 6 \sin\left(\frac{\pi x}{2}\right)$

$$V = \pi \int_0^b 36 \sin^2\left(\frac{\pi x}{2}\right) dx \quad 0 < b < 4$$

$$V = 18\pi \int_0^b (1 - \cos(\pi x)) dx \quad \text{A1}$$

$$V = 18\pi \left[x - \frac{1}{\pi} \sin(\pi x) \right]_0^b \quad \text{M1}$$

$$V = 18\pi \left[b - \frac{1}{\pi} \sin(b\pi) - 0 \right]$$

$$V = 18(b\pi - \sin(b\pi)) \quad \text{A1}$$

b. if $V = 18(b\pi - \sin(b\pi)) = 9(7\pi + 2)$

then $b = \frac{7}{2}$ since $\sin\left(\frac{7\pi}{2}\right) = -1$ A1

Question 9

a. $y = \frac{4}{x^2 - 4x} = \frac{4}{x(x-4)}$

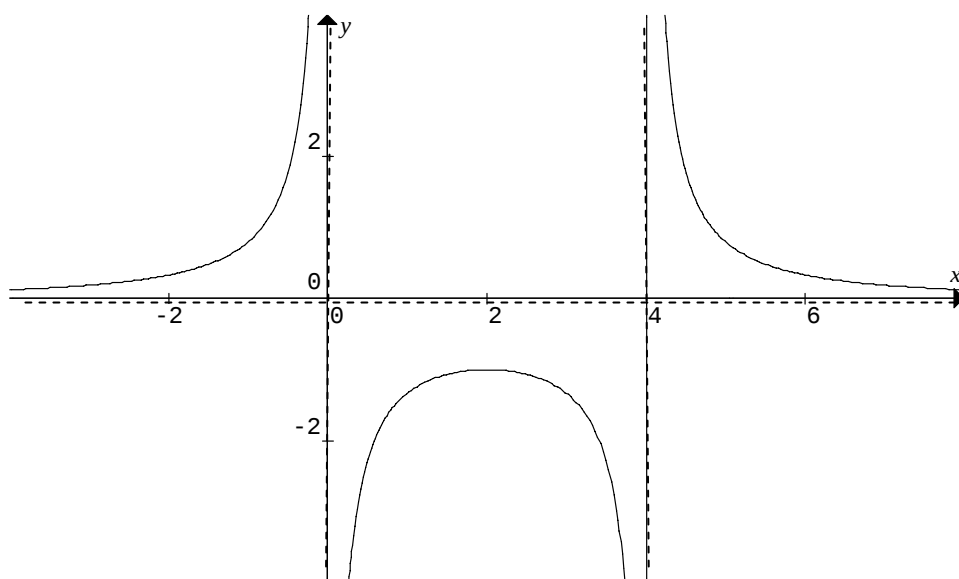
vertical asymptotes at $x = 0$ (the y-axis) and $x = 4$

horizontal asymptotes at $y = 0$ (the x-axis) A1

the turning point is when $2x - 4 = 0$ at $x = 2$ $y(2) = \frac{4}{4-8} = -1$

maximum turning point at $(2, -1)$

correct graph and turning point A1



b. the area $= \int_1^2 \frac{4}{x^2 - 4x} dx = \int_1^2 \frac{4}{x(x-4)} dx$

by partial fractions $\frac{4}{x^2 - 4x} = \frac{4}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4}$ adding the partial fractions

$$= \frac{A(x-4) + Bx}{x(x-4)} = \frac{x(A+B) - 4A}{x^2 - 4x} \quad \text{M1}$$

(1) $A+B=0$ and (2) $-4A=4$ so that $A=-1$ and $B=-A$ $B=1$

the area $A = \int_1^2 \left(\frac{1}{x-4} - \frac{1}{x} \right) dx$ but the area is under the x -axis, so $A < 0$

$$A = \int_1^2 \left(\frac{1}{x} - \frac{1}{x-4} \right) dx \quad \text{now } A > 0 \quad \text{M1}$$

$$= \left[\log_e(|x|) - \log_e(|x-4|) \right]_1^2$$

$$= \left[\log_e \left(\left| \frac{x}{x-4} \right| \right) \right]_1^2 \quad \text{A1}$$

$$= \left(\log_e \left(\left| \frac{2}{-2} \right| \right) - \log_e \left(\left| \frac{1}{-3} \right| \right) \right)$$

$$= \log_e(3) \quad \text{A1}$$

END OF SUGGESTED SOLUTIONS