

2007 VCAA Specialist Maths Exam 2 Solutions
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SECTION 1

1	2	3	4	5	6	7	8	9	10	11
B	E	D	B	E	B	A	C	D	C	D

12	13	14	15	16	17	18	19	20	21	22
A	E	C	A	C	D	E	B	C	A	B

Q1 $\frac{(x-2)^2}{a^2} - \frac{4(y+3)^2}{a^2} = 1, \frac{(x-2)^2}{a^2} - \frac{(y+3)^2}{\left(\frac{a}{2}\right)^2} = 1.$

Asymptotes are $(y+3) = \pm \frac{a}{2}(x-2) = \pm \frac{1}{2}(x-2).$

Product of gradients is $\frac{1}{2} \left(-\frac{1}{2}\right) = -\frac{1}{4}.$

Q2 Ellipse: $\frac{(x-1)^2}{3^2} + \frac{(y-3)^2}{2^2} = 1.$

Let $\frac{x-1}{3} = \cos t, \frac{y-3}{2} = \sin t.$

$x = 1 + 3 \cos t, y = 3 + 2 \sin t$

Q3 For $\tan^{-1} x$, range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$

$b \tan^{-1} x$, range is $\left(-\frac{b\pi}{2}, \frac{b\pi}{2}\right),$

$a + b \tan^{-1} x$, range is $\left(a - \frac{b\pi}{2}, a + \frac{b\pi}{2}\right),$

$a + b \tan^{-1}(x-c)$, range is still the same $\left(a - \frac{b\pi}{2}, a + \frac{b\pi}{2}\right).$

Q4 Given formula: $\cot^2 x + 1 = \operatorname{cosec}^2 x.$

$\therefore \cot^2 x - \operatorname{cosec}^2 x = -1, \therefore \cot^2(4\theta) - \operatorname{cosec}^2(4\theta) = -1.$

Q5 $T = \frac{2\pi}{n} = \frac{5a}{2} - \frac{a}{2} = 2a, \therefore n = \frac{\pi}{a}.$ The given graph is the

translation to the left by $\frac{a}{2}$ of $y = \operatorname{cosec} \frac{\pi}{a} x.$

Q6 The equation has real coefficients, $\therefore$ the complex roots are in conjugate pair(s).

Q7 $\frac{1}{1-z} = \frac{1}{-2+4i} = \frac{-1-2i}{10}.$

Q8 $z = 2 \operatorname{cis} \left(-\frac{\pi}{3}\right),$

$z^4 = 2^4 \operatorname{cis} 4 \left(-\frac{\pi}{3}\right) = 16 \operatorname{cis} \left(-\frac{4\pi}{3}\right) = 16 \operatorname{cis} \left(\frac{2\pi}{3}\right).$

Q9 $\frac{x}{3(x+c)^2} = \frac{\frac{x}{3}}{(x+c)^2} = \frac{A}{x+c} + \frac{B}{(x+c)^2}.$

Q10 $y = \sin^{-1}(2x), x = \frac{1}{2} \sin y, x^2 = \frac{1}{4} \sin^2 y = \frac{1}{8}(1 - \cos 2y).$

When $x = 0, y = 0.$ When $x = \frac{1}{2}, y = \frac{\pi}{2}.$

$V = \int_0^{\frac{\pi}{2}} \pi x^2 dy = \frac{\pi}{8} \int_0^{\frac{\pi}{2}} (1 - \cos 2y) dy.$

Q11 Refer to itute.com Specialist Maths summary sheets.

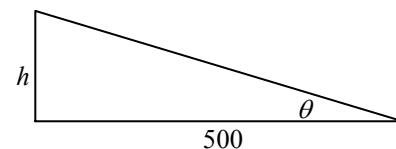
Numerical solution of $\frac{dy}{dx} = f(x)$ when $x = b$, given $y = k$ when

$x = a$, is $y = \int_a^b f(x) dx + k.$

If $\frac{dy}{dx} = \sqrt{\sin x}$ and $y = 1$ when $x = 0,$

then $y = \int_0^{\frac{\pi}{3}} \sqrt{\sin x} dx + 1$ when $x = \frac{\pi}{3}.$

Q12 Let h metres be the altitude of the shuttle at time t seconds.



$h = 500 \tan \theta, v = \frac{dh}{dt} = 500 \sec^2 \theta \frac{d\theta}{dt}.$

When $\theta = \frac{\pi}{6}, \frac{d\theta}{dt} = 0.5, \therefore v = 500 \left(\frac{2}{\sqrt{3}}\right)^2 \times 0.5 \approx 333 \text{ ms}^{-1}.$

Q13 Let $u = \log_e x, \frac{du}{dx} = \frac{1}{x}.$

When $x = 1, u = 0.$ When $x = e^3, u = 3.$

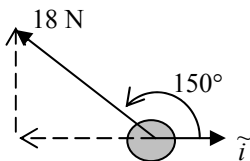
$\therefore \int_1^{e^3} \left(\frac{\log_e x}{x}\right)^3 dx = \int_0^3 u^3 \frac{du}{dx} dx = \int_0^3 u^3 du.$

Q14 $\frac{dN}{dt} \propto N(1000 - N), \therefore \frac{dN}{dt} = kN(1000 - N).$

Q15 Gradient of line = $-\frac{3}{2}$, $\therefore$ gradient of vector $\perp$ to line = $\frac{2}{3}$.

The vector is $3\tilde{i} + 2\tilde{j}$.

Q16



$$\vec{F} = 18\cos 150^\circ \tilde{i} + 18\sin 150^\circ \tilde{j} = -9\sqrt{3}\tilde{i} + 9\tilde{j}.$$

$$\text{Q17 } \cos \theta = \frac{\tilde{a} \cdot \tilde{b}}{|\tilde{a}||\tilde{b}|} = \frac{-4}{9}, \therefore \theta = \cos^{-1}\left(\frac{-4}{9}\right) = \pi - \cos^{-1}\left(\frac{4}{9}\right).$$

$$\text{Q18 } \hat{u} = \frac{1}{3}(2\tilde{i} - \tilde{j} - 2\tilde{k}), v = a\tilde{i} + 2\tilde{j} - \tilde{k} \text{ and } \tilde{v} \cdot \hat{u} = 1.$$

$$\therefore \frac{1}{3}(2a - 2 + 2) = 1, \therefore a = \frac{3}{2}.$$

$$\text{Q19 } m = \frac{18}{1.5} = 12 \text{ kg}, v = u + at = 5 + 1.5 \times 4 = 11 \text{ ms}^{-1}.$$

$$p = mv = 12 \times 11 = 132 \text{ kgms}^{-1}.$$

Q20 The box slides up the plane, $\therefore$ friction = $0.2N$ down the plane.

$$\text{Q21 } v = \sqrt{3x^2 - x^3 + 16}, \therefore \frac{1}{2}v^2 = \frac{1}{2}(3x^2 - x^3 + 16).$$

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2}(6x - 3x^2) = 3x - \frac{3x^2}{2},$$

$$\therefore F = ma = 12\left(3x - \frac{3x^2}{2}\right).$$

$$\text{Q22 } ma = F, \therefore m \frac{dv}{dt} = P - mkv^2.$$

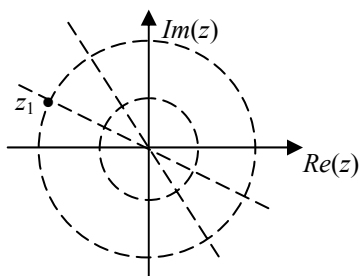
SECTION 2

Q1a $z_1 = -\sqrt{3} + i$ is in the second quadrant.

$$r = \sqrt{(-\sqrt{3})^2 + 1^2} = 2 \text{ and } \theta = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = \frac{5\pi}{6}.$$

$$\therefore z_1 = 2\text{cis}\left(\frac{5\pi}{6}\right).$$

Q1b



$$\text{Q1c } z = \frac{2\sqrt{3} \pm \sqrt{12-16}}{2} = \sqrt{3} - i \text{ or } \sqrt{3} + i.$$

$$\text{Q1d } z_1 = -\sqrt{3} + i$$

$$\therefore \sqrt{3} - i = -z_1 \text{ and } \sqrt{3} + i = -\bar{z}_1 \text{ or } \overline{-z_1}.$$

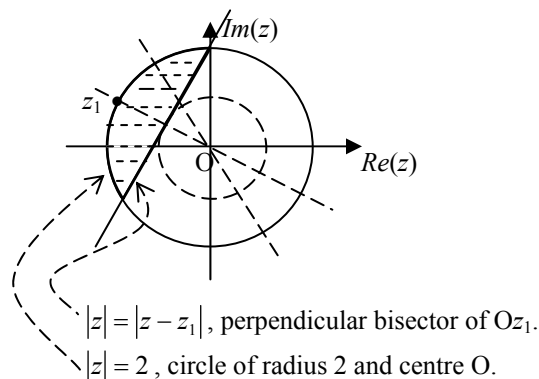
$$\text{Q1e } |z| = |z - z_1|. \text{ Let } z = x + yi.$$

$$\sqrt{x^2 + y^2} = \sqrt{(x + \sqrt{3})^2 + (y - 1)^2}, \text{ simplify to } y = \sqrt{3}x + 2.$$

$$\text{Q1f } \bar{z}_1 = -\sqrt{3} - i, x = -\sqrt{3} \text{ and } y = -1 \text{ satisfy } y = \sqrt{3}x + 2.$$

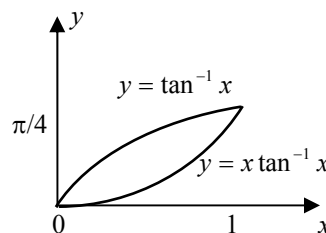
$$\therefore \bar{z}_1 = -\sqrt{3} - i \text{ satisfies } |z| = |z - z_1|.$$

Q1g



$$\text{Q2a } f(x) = x \tan^{-1} x, f'(x) = \tan^{-1} x + \frac{x}{1+x^2}, f'(0) = 0.$$

Q2b



$$\text{Q2ci and ii } \int_0^1 x \tan^{-1} x dx = 0.285 \text{ by graphics calculator.}$$

$$\text{Q2d } f'(x) = \tan^{-1} x + \frac{x}{1+x^2},$$

$$\int f'(x) dx = \int \tan^{-1} x dx + \int \frac{x}{1+x^2} dx,$$

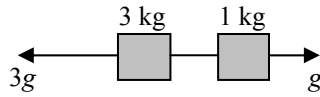
$$f(x) = \int \tan^{-1} x dx + \int \frac{\frac{1}{2} du}{u}, \text{ where } u = 1 + x^2.$$

$$\int \tan^{-1} x dx = x \tan^{-1} x - \frac{1}{2} \log_e(1 + x^2).$$

$$\text{Q2e } \int_0^1 \tan^{-1} x dx = \left[x \tan^{-1} x - \frac{1}{2} \log_e(1 + x^2) \right]_0^1 = \frac{\pi}{4} - \frac{1}{2} \log_e 2.$$

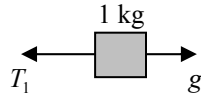
$$\text{Q2f Enclosed area} = \frac{\pi}{4} - \frac{1}{2} \log_e 2 - 0.285 \approx 0.15.$$

Q3a An equivalent diagram:



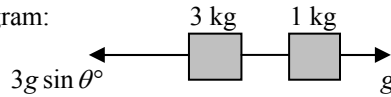
$$a = \frac{R}{m} = \frac{3g - g}{4} = \frac{g}{2}.$$

Q3b Consider the 1-kg mass:



$$R = ma, T_1 - g = 1 \times \frac{g}{2}, T_1 = \frac{3g}{2}.$$

Q3c An equivalent diagram:



$$a = \frac{R}{m}, b = \frac{3g \sin \theta - g}{4} = \frac{(3 \sin \theta - 1)g}{4}.$$

$$\text{When } \theta = 30^\circ, b = \frac{g}{8}.$$

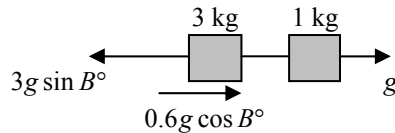
$$\text{Q3d Let } b = \frac{(3 \sin \theta - 1)g}{4} = 0, \therefore \sin \theta = \frac{1}{3}, \theta = 19.5^\circ.$$

Q3e The normal reaction on the 3-kg mass = $3g \cos \theta$.

The force of friction = $\mu \times 3g \cos \theta = 0.6g \cos \theta$.

On the verge of sliding down, the force of friction points up along the plane.

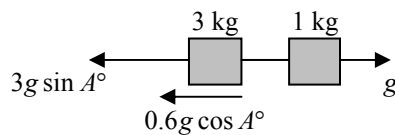
An equivalent diagram:



$$R = 3g \sin B - 0.6g \cos B - g = 0, B = 30.4^\circ \text{ by graphics calculator.}$$

On the verge of sliding up, the force of friction points down along the plane.

An equivalent diagram:



$$R = g - 3g \sin A - 0.6g \cos A = 0, A = 7.8^\circ \text{ by graphics calculator.}$$

$$\text{Q4a } \tilde{v} = 30\tilde{i} - 40\tilde{j} - 4\tilde{k}.$$

$$\tilde{r}(t) = \int \tilde{v} dt = 30t\tilde{i} - 40t\tilde{j} - 4t\tilde{k} + \tilde{c}.$$

Use $\tilde{r}(10) = -500\tilde{i} + 2500\tilde{j} + 200\tilde{k}$ to find

$$\tilde{c} = -800\tilde{i} + 2900\tilde{j} + 240\tilde{k}.$$

$$\therefore \tilde{r}(t) = (30t - 800)\tilde{i} + (2900 - 40t)\tilde{j} + (240 - 4t)\tilde{k}.$$

Q4b Landing when $240 - 4t = 0, t = 60$.

$$\tilde{r}(60) = 1000\tilde{i} + 500\tilde{j}, \text{ distance} = \sqrt{1000^2 + 500^2} = 1118 \text{ m.}$$

Q4c Consider the velocity vector $\tilde{v} = 30\tilde{i} - 40\tilde{j} - 4\tilde{k}$.

$$\tan \theta = \frac{4}{\sqrt{30^2 + 40^2}} = \frac{4}{50}, \theta \approx 4.6^\circ.$$

Q4d Closest when $\tilde{v} \perp \tilde{r}(t)$, i.e. $\tilde{v} \cdot \tilde{r}(t) = 0$.

$$\therefore 30(30t - 800) - 40(2900 - 40t) - 4(240 - 4t) = 0, \therefore t \approx 56.$$

$$\text{Q4e Speed} = |\tilde{v}| = \sqrt{30^2 + (-40)^2 + (-4)^2} = 50.1597 \text{ ms}^{-1}.$$

$$\text{Distance} = \text{speed} \times \text{time} = 50.1597 \times 60 \approx 3010.$$

$$\text{Q5a } v(t) = 20 - 2 \tan^{-1} t, 17 = 20 - 2 \tan^{-1} t, t = \tan \frac{3}{2} \approx 14.1$$

$$\text{Q5b As } t \rightarrow \infty, \tan^{-1} t \rightarrow \frac{\pi}{2}, v \rightarrow (20 - \pi)^+,$$

$$\therefore v > 20 - \pi > 16.$$

$$\text{Q5c Distance (polluting car)} = \int_0^T (20 - 2 \tan^{-1} t) dt.$$

$$\text{Q5d Distance (police car)} = \int_3^8 13 \cos^{-1} \left(\frac{13 - 2t}{7} \right) dt.$$

Polluting car ahead of police car by

$$\int_0^8 (20 - 2 \tan^{-1} t) dt - \int_3^8 13 \cos^{-1} \left(\frac{13 - 2t}{7} \right) dt \approx 60.7 \text{ by graphics calculator.}$$

$$\text{Q5e When } t \geq 8, \text{ speed of police car} = 13 \cos^{-1} \left(-\frac{3}{7} \right).$$

Between $t = 8$ and $t = T_c$,

$$\text{total distance (police car)} = (T_c - 8) \times 13 \cos^{-1} \left(-\frac{3}{7} \right),$$

$$\text{total distance (polluting car)} = \int_8^{T_c} (20 - 2 \tan^{-1} t) dt.$$

$$\therefore (T_c - 8) \times 13 \cos^{-1} \left(-\frac{3}{7} \right) = 60.7 + \int_8^{T_c} (20 - 2 \tan^{-1} t) dt.$$

Q5f For $t \geq 8, 20 - 2 \tan^{-1} t \approx 17$,

$$\therefore \int_8^{T_c} (20 - 2 \tan^{-1} t) dt \approx 17(T_c - 8).$$

$$\therefore (T_c - 8) \times 13 \cos^{-1} \left(-\frac{3}{7} \right) \approx 60.7 + 17(T_c - 8).$$

$$\therefore T_c \approx 15.$$

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