

INSIGHT

Trial Exam Paper

2007

SPECIALIST

MATHEMATICS

Written examination 1

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions.
- mark allocations
- tips on how to approach the questions.

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Question 1

Let $x = \sqrt{t+4}$ and $y = 1-t$ for $-4 \leq t \leq 4$.

1a. Find the Cartesian equation of the curve.

Worked solution

$$x = \sqrt{t+4} \dots (1) \text{ for } -4 \leq t \leq 4$$

$$y = 1-t \dots (2)$$

$$\begin{aligned} \text{From (1)} \quad x^2 &= t+4 \\ t &= x^2 - 4 \end{aligned}$$

1A

Substitute into (2)

$$y = 1 - (x^2 - 4)$$

$$y = 5 - x^2$$

1A

2 marks

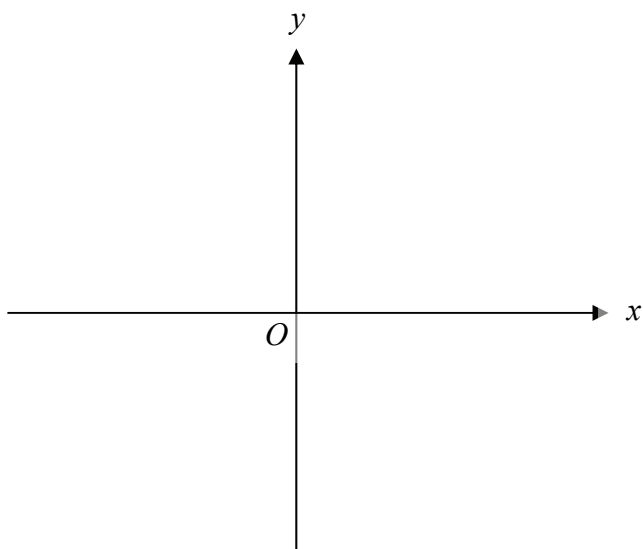
Mark allocation

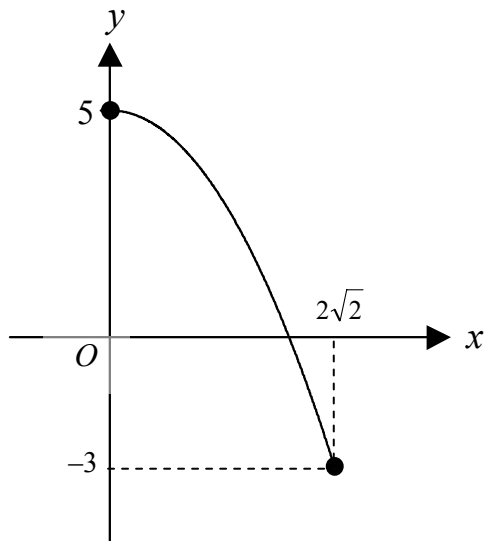
- 1 mark for correctly expressing t in terms of x^2
- 1 mark for the correct answer

Tip

- Use substitution to eliminate t from the parametric equations.

1b. Sketch a graph of the curve showing all features clearly.



Worked solution

2 marks

Mark allocation

- 1 mark for correct shape
- 1 mark for both endpoints correct

Tip

- *Use the restrictions on t to find the endpoints*

$$\begin{aligned} \text{When } t = -4 \quad x &= \sqrt{-4 + 4} = 0 \\ y &= 1 - (-4) = 5 \end{aligned}$$

$$\begin{aligned} \text{When } t = 4 \quad x &= \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2} \\ y &= 1 - 4 = -3 \end{aligned}$$

Question 2

Express $(\sqrt{3} - i)^7$ in the form $x + iy$ where $x, y \in R$

Worked solution

Let $r\text{cis}\theta = (\sqrt{3} - i)$ Express $\sqrt{3} - i$ in polar form

$$r = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$$

$$\tan \theta = \frac{-1}{\sqrt{3}}$$

$$\theta = \frac{11\pi}{6} \quad \text{Fourth quadrant angle}$$

$$\therefore \theta = -\frac{\pi}{6} \quad \text{Calculate equivalent angle } \theta \in (-\pi, \pi]$$

$$(\sqrt{3} - i) = 2\text{cis}\left(-\frac{\pi}{6}\right) \quad 1A$$

$$(\sqrt{3} - i)^7 = 2^7 \text{cis}\left(7 \times -\frac{\pi}{6}\right) \quad \text{Applying De Moivre's theorem} \quad 1A$$

$$= 128\text{cis}\left(-\frac{7\pi}{6}\right)$$

$$= 128\left(\cos\left(-\frac{7\pi}{6}\right) + i\sin\left(-\frac{7\pi}{6}\right)\right)$$

$$= 128\left(\cos\left(\frac{7\pi}{6}\right) - i\sin\left(\frac{7\pi}{6}\right)\right)$$

$$= 128\left(-\cos\left(\frac{\pi}{6}\right) - i \times -\sin\left(\frac{\pi}{6}\right)\right) \quad 1M$$

$$= 128\left(-\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$$

$$= 128\left(-\frac{\sqrt{3}}{2} + i \times \frac{1}{2}\right)$$

$$= -64\sqrt{3} + 64i \quad 1A$$

4 marks

Mark allocation

- 1 mark for expressing $\sqrt{3} - i$ in correct polar form
- 1 mark for correctly applying de Moivre's theorem
- 1 mark for correct method of simplification
- 1 mark for correct answer

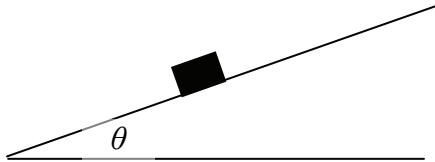
Tip

- Express complex number in polar form and then apply De Moivre's theorem

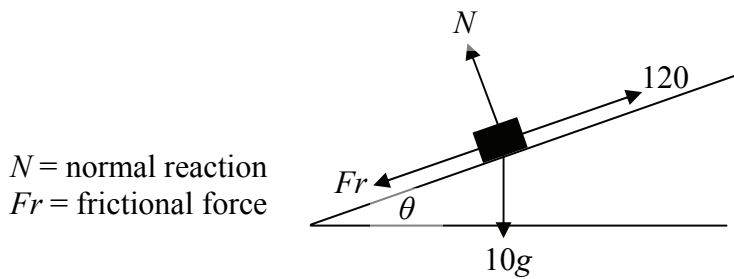
Question 3

A 10 kg mass is pulled up a rough plane inclined at an angle of θ to the horizontal by a force of 120 newtons acting parallel to the plane.

The coefficient of friction between the mass and the plane is $\frac{1}{3}$, $\cos(\theta) = \frac{3}{5}$ and the acceleration due to gravity is $g \text{ m/s}^2$.



3a. Show all forces acting on the mass on the diagram above.

Worked solution

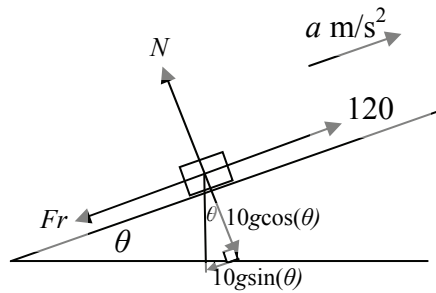
1 mark

Mark allocation

- 1 mark for all forces correctly shown

3b. Find the acceleration of the mass up the plane in terms of g .

Worked solution



Equation of motion up the plane

$$R = ma$$

$$R = \text{resultant force} \quad Fr = \mu N$$

$$120 - Fr - mg \sin(\theta) = ma$$

$$\mu = \frac{1}{3} = \text{coefficient of friction}$$

$$120 - \mu N - 10g \sin(\theta) = 10a \quad \dots(1)$$

1A

Resolving forces perpendicular to the plane

$$N = 10g \cos(\theta) \quad \dots(2)$$

Substitute (2) into (1)

$$120 - \frac{1}{3} \times 10g \cos(\theta) - 10g \sin(\theta) = 10a$$

1M

$$a = 12 - \frac{1}{3} \times g \cos(\theta) - g \sin(\theta)$$

$$\text{Given } \cos(\theta) = \frac{3}{5}$$

$$\sin(\theta) = \sqrt{1 - \cos^2(\theta)} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$$

1A

$$a = 12 - \frac{1}{3} \times g \times \frac{3}{5} - g \times \frac{4}{5}$$

$$a = 12 - \frac{1}{5}g - \frac{4}{5}g$$

$$a = 12 - \frac{5}{5}g$$

$$a = 12 - g \text{ m/s}^2$$

1A

4 marks

Mark allocation

- 1 mark for correctly resolving forces parallel to the plane
- 1 mark for using a correct method to give the acceleration in terms of θ and g .
- 1 mark for correctly finding $\sin \theta = \frac{4}{5}$
- 1 mark for correct answer

Tips

- *Resolve forces parallel and perpendicular to the plane*
- *$\sin \theta$ is not given and should be found from value of $\cos \theta$*

Question 4

4a. Show that $\frac{\sin(x)}{1 - \cos(x)} = \cot\left(\frac{x}{2}\right)$

Worked solution

$$\begin{aligned}
 \text{LHS} &= \frac{\sin(x)}{1 - \cos(x)} \\
 &= \frac{2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right)}{1 - \cos\left(2\left(\frac{x}{2}\right)\right)} && 1\text{M} \\
 &= \frac{2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right)}{1 - \left(1 - 2 \sin^2\left(\frac{x}{2}\right)\right)} \\
 &= \frac{2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right)}{2 \sin^2\left(\frac{x}{2}\right)} \\
 &= \frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} && 1\text{A} \\
 &= \cot\left(\frac{x}{2}\right) \\
 &= \text{RHS}
 \end{aligned}$$

2 marks

Mark allocation

- 1 mark for correct application of the double angle formulae
- 1 mark for correct simplification leading to the correct answer

Tips

- Express $\sin(x)$ and $\cos(x)$ in terms of $\frac{\theta}{2}$ using double angle formulas.

4b. Hence or otherwise solve the equation $\sin(x) = \cos(x) - 1$ over $0 \leq x \leq 2\pi$

Worked solution

$$\frac{\sin(x)}{\cos(x) - 1} = 1$$

$$\frac{\sin(x)}{1 - \cos(x)} = -1 \quad 1A$$

$$\cot\left(\frac{x}{2}\right) = -1$$

$$\tan\left(\frac{x}{2}\right) = -1$$

$$\frac{x}{2} = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$x = \frac{3\pi}{2} \quad 1A$$

2 marks

Mark Allocation

- 1 mark for rearranging the equation to use information given in part a of the question
- 1 mark for the correct answer

Tip

- *The word 'hence' gives the hint that something from the previous part of the question is used to find the answer.*

Question 5

The position of a particle at time t seconds, $t \geq 0$, is given by the vector $\underline{r} = t \underline{i} + (1 - 2t) \underline{j} + (t - 6) \underline{k}$. Find the time when the particle's velocity vector is perpendicular to its position vector.

Worked solution

Position vector: $\underline{r} = t \underline{i} + (1 - 2t) \underline{j} + (t - 6) \underline{k}$

Differentiate to find velocity vector: $\dot{\underline{r}} = \underline{i} - 2 \underline{j} + \underline{k}$ 1A

Vectors are perpendicular when $\underline{r} \cdot \dot{\underline{r}} = 0$

$$\underline{r} \cdot \dot{\underline{r}} = \left(t \underline{i} + (1 - 2t) \underline{j} + (t - 6) \underline{k} \right) \cdot \left(\underline{i} - 2 \underline{j} + \underline{k} \right) = 0 \quad 1A$$

$$\underline{r} \cdot \dot{\underline{r}} = t - 2(1 - 2t) + (t - 6) = 0$$

$$t - 2 + 4t + t - 6 = 0$$

$$6t - 8 = 0$$

$$t = \frac{4}{3} \text{ seconds} \quad 1A$$

3 marks

Mark allocation

- 1 mark for finding the correct velocity vector
- 1 mark for taking the dot product of the position and velocity vectors and setting this to zero.
- 1 mark for the correct answer

Tip

- *The dot product is zero when two vectors are perpendicular.*

Question 6

Consider the relation $xy + \frac{y^2}{x} = 2$.

6a. Find an expression for $\frac{dy}{dx}$ in terms of x and y .

Worked solution

$$xy + \frac{y^2}{x} = 2$$

$$x^2y + y^2 = 2x$$

$$2xy + x^2 \frac{dy}{dx} + 2y \frac{dy}{dx} = 2 \quad 1M$$

$$\frac{dy}{dx}(x^2 + 2y) = 2 - 2xy \quad 1A$$

$$\frac{dy}{dx} = \frac{2(1 - xy)}{(x^2 + 2y)} \quad 1A$$

3 marks

Mark allocation

- 1 mark for correctly applying implicit differentiation techniques
- 1 mark for factorising correctly to collect the terms containing $\frac{dy}{dx}$
- 1 mark for correct answer

Tip

- *The quotient rule can be used to find the answer, however, it is much easier to first simplify as shown and use the product rule.*

6b. Hence find the equations of the tangents to the curve when $x = 1$

Worked solution

Finding y when $x = 1$

$$xy + \frac{y^2}{x} = 2$$

$$1y + \frac{y^2}{1} = 2$$

$$y^2 + y - 2 = 0$$

$$(y + 2)(y - 1) = 0$$

$$y = -2, y = 1$$

1A

Gradient of tangent at $x = 1, y = -2$

$$\frac{dy}{dx} = \frac{2(1 - xy)}{(x^2 + 2y)} = \frac{2(1 - 1 \times -2)}{1^2 + 2 \times -2} = \frac{6}{-3} = -2$$

Equation of tangent at $x = 1, y = -2$

$$y - (-2) = -2(x - 1)$$

$$y = -2x$$

1A

Gradient of tangent at $x = 1, y = 1$

$$\frac{dy}{dx} = \frac{2(1 - xy)}{(x^2 + 2y)} = \frac{2(1 - 1 \times 1)}{1^2 + 2 \times 1} = \frac{0}{3} = 0$$

Equation of tangent at $x = 1, y = 1$

$$y = 1$$

1A

3 marks

Mark Allocation

- 1 mark for finding the y -coordinates of the points where $x = 1$
- 1 mark for finding the correct equation of one tangent
- 1 mark for finding the other tangent equation

Tips

- *Recognise there will be two tangents from the wording of the question.*
- *A line with zero gradient will be parallel to the x -axis*

Question 7

$$f : D \rightarrow R, \quad f(x) = \arccos\left(\frac{1}{\sqrt{x}}\right)$$

7a. Determine the domain D of function f

Worked solution

$$-1 \leq \frac{1}{\sqrt{x}} \leq 1$$

$$\Rightarrow \quad -1 \leq \frac{1}{\sqrt{x}} \quad \text{and} \quad \frac{1}{\sqrt{x}} \leq 1$$

$$-\sqrt{x} \leq 1 \quad \text{and} \quad 1 \leq \sqrt{x}$$

$$\sqrt{x} \geq -1 \quad \text{and} \quad x \geq 1$$

$$x \geq (-1)^2$$

$$\therefore x \geq 1$$

$$\text{Domain of } f, D = [1, \infty)$$

1A

1 mark

Mark allocation

- 1 mark for correct domain.

7b. Find $f'(x)$

Worked solution

$$\text{Let } y = \arccos\left(\frac{1}{\sqrt{x}}\right)$$

$$u = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$$

$$y = \arccos(u)$$

$$\frac{du}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$$

$$\frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}} \times \left(-\frac{1}{2}x^{-\frac{3}{2}}\right)$$

1A

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{1}{\sqrt{x}}\right)^2}} \times \frac{-1}{2\sqrt{x^3}}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\frac{1}{x}}} \times \frac{-1}{2x\sqrt{x}}$$

1M

$$\therefore f'(x) = \frac{dy}{dx} = \frac{1}{2x\sqrt{x-1}}$$

1A

3 marks

Mark allocation

- 1 mark for correctly applying the chain rule
- 1 mark for some correct simplification
- 1 mark for answer fully simplified

Tip

- *The chain rule needs to be used.*

Question 8

Solve the differential equation $\frac{dx}{dt} = \frac{t^2 + 3}{t^2}$ given $x = 1$ when $t = 1$

Worked solution

$$\frac{dx}{dt} = \frac{t^2}{t^2 + 3}$$

$$\frac{dx}{dt} = 1 - \frac{3}{t^2 + 3} \quad 1A$$

$$x = \int 1 - \frac{3}{t^2 + 3} dt$$

$$x = t - \int \frac{3}{t^2 + 3} dt$$

$$x = t - \sqrt{3} \int \frac{\sqrt{3}}{t^2 + 3} dt$$

$$x = t - \sqrt{3} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + c \quad 1A$$

When $t = 1$, $x = 1$

$$\Rightarrow 1 = 1 - \sqrt{3} \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + c$$

$$\Rightarrow c = \sqrt{3} \times \frac{\pi}{6}$$

$$\therefore x = t - \sqrt{3} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + \frac{\sqrt{3}\pi}{6} \quad 1A$$

3 marks

Mark allocation

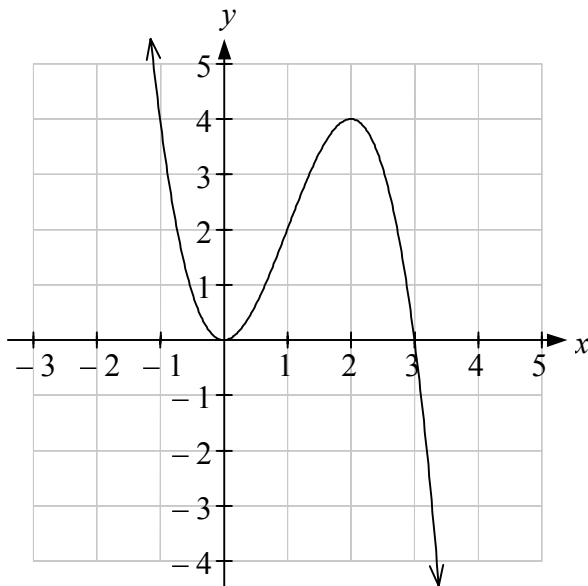
- 1 mark for converting the differential equation to $\frac{dx}{dt}$ and dividing t^2 by $t^2 + 3$.
- 1 mark for correct integration
- 1 mark for finding the correct answer

Tips

- *Recognise to take the reciprocal of the differential equation*
- *The numerator needs to be of lower degree than the denominator*

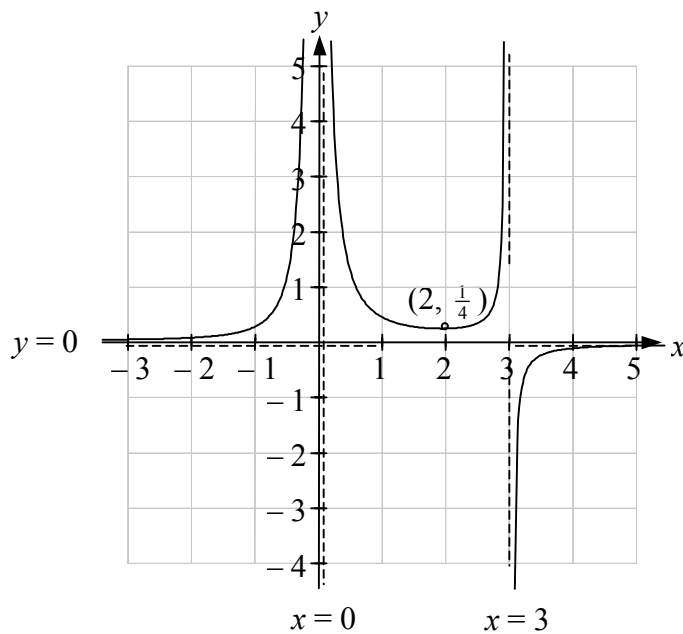
Question 9

The graph of $f(x) = 3x^2 - x^3$ is shown on the axes below.



9a. Draw the graph of $g(x) = \frac{1}{3x^2 - x^3}$ on the axes above showing all features clearly.

Worked solution



2 marks

Mark allocation

- 1 mark for correct shape
- 1 mark for correct asymptotes and turning point

Question 9 – continued

9b. Given $\frac{1}{3x^2 - x^3} = \frac{Ax + B}{x^2} + \frac{C}{3 - x}$. Find the exact values of A , B , and C .

Worked solution

$$\frac{1}{3x^2 - x^3} = \frac{1}{x^2(3 - x)} = \frac{Ax + B}{x^2} + \frac{C}{3 - x}$$

$$\frac{1}{x^2(3 - x)} = \frac{Ax + B}{x^2} + \frac{C}{3 - x}$$

$$\frac{1}{x^2(3 - x)} = \frac{(Ax + B)(3 - x)}{x^2(3 - x)} + \frac{Cx^2}{x^2(3 - x)}$$

$$1 = 3Ax - Ax^2 + 3B - Bx + Cx^2$$

$$1 = (C - A)x^2 + (3A - B)x + 3B \quad 1M$$

Equating coefficients of powers of x

$$0 = C - A$$

$$0 = 3A - B$$

$$1 = 3B$$

$$\therefore B = \frac{1}{3}, \quad A = \frac{1}{9}, \quad C = \frac{1}{9} \quad 1A$$

2 marks

Mark allocation

- 1 mark for correct simplifications leading to simultaneous equations in A , B , C
- 1 mark for correct values of A , B , C

- 9c.** Find the exact area between the graph of $g(x) = \frac{1}{3x^2 - x^3}$, the x -axis and the lines $x = 1$ and $x = 2$

Worked solution

$$\begin{aligned}
 \text{Area} &= \int_1^2 \frac{1}{3x^2 - x^3} dx \\
 &= \int_1^2 \frac{x+3}{9x^2} + \frac{1}{9(3-x)} dx && 1A \\
 &= \frac{1}{9} \int_1^2 \frac{x}{x^2} + \frac{3}{x^2} + \frac{1}{(3-x)} dx \\
 &= \frac{1}{9} \left[\log_e |x| - 3x^{-1} - \log_e |3-x| \right]_1^2 && 1M \\
 &= \frac{1}{9} \left[\log_e \left| \frac{x}{3-x} \right| - \frac{3}{x} \right]_1^2 \\
 &= \frac{1}{9} \left[\left(\log_e(2) - \frac{3}{2} \right) - \left(\log_e\left(\frac{1}{2}\right) - 3 \right) \right] \\
 &= \frac{1}{9} \left[\log_e(2) - \frac{3}{2} - \log_e\left(\frac{1}{2}\right) + 3 \right] \\
 &= \frac{1}{9} \left[\log_e(2) + \log_e(2) + \frac{3}{2} \right] \\
 &= \frac{1}{9} \left[2\log_e(2) + \frac{3}{2} \right] \\
 &= \frac{2}{9} \log_e(2) + \frac{1}{6} && 1A
 \end{aligned}$$

3 marks

Mark allocation

- 1 mark for writing the correct integral and recognising the need to express as partial fractions
- 1 mark for using correct methods to integrate
- 1 mark for the correct exact answer

Tip

- *The quotient can be simplified in the following way to make the integration easier*

$$\begin{aligned}
 \frac{1}{3x^2 - x^3} &= \frac{\frac{1}{9}x + \frac{1}{3}}{x^2} + \frac{\frac{1}{9}}{3-x} \\
 \Rightarrow \frac{1}{3x^2 - x^3} &= \frac{x+3}{9x^2} + \frac{1}{9(3-x)}
 \end{aligned}$$

END OF WORKED SOLUTIONS BOOK