
Section 1 – Multiple-choice answers

1.	D	7.	A	13.	E	19.	A
2.	B	8.	E	14.	B	20.	D
3.	E	9.	B	15.	B	21.	D
4.	B	10.	E	16.	C	22.	B
5.	C	11.	A	17.	E		
6.	C	12.	D	18.	C		

Section 1- Multiple-choice solutions

Question 1

The hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ has asymptotes given by $y - k = \pm \frac{b}{a}(x - h)$.

Now $y = \pm 4(x - 2)$, so (h, k) is $(2, 0)$ so options A and B can be eliminated.

Also, $\frac{b}{a} = 4$. Only option D offers this since $\frac{\sqrt{64}}{\sqrt{4}} = \frac{8}{2} = 4$.

The answer is D.

Question 2

The graph of $y = f(x)$ has one vertical asymptote if the equation $x^2 + px + q = 0$ has one solution.

The quadratic equation $x^2 + px + q = 0$ has one solution if

$$p^2 - 4 \times 1 \times q = 0$$

$$p^2 = 4q$$

The answer is B.

Question 3

For the inverse circular function $y = \arcsin(x)$, $d = [-1, 1]$ and $r = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

For the inverse circular function $f(x) = \frac{\pi}{2} + \arcsin\left(\frac{x}{a}\right)$, $d_f = [-a, a]$

since $1 \leq \frac{x}{a} \leq 1$

$$-a \leq x \leq a$$

$$\text{and } r_f = \left[-\frac{\pi}{2} + \frac{\pi}{2}, \frac{\pi}{2} + \frac{\pi}{2}\right] \\ = [0, \pi]$$

The answer is E.

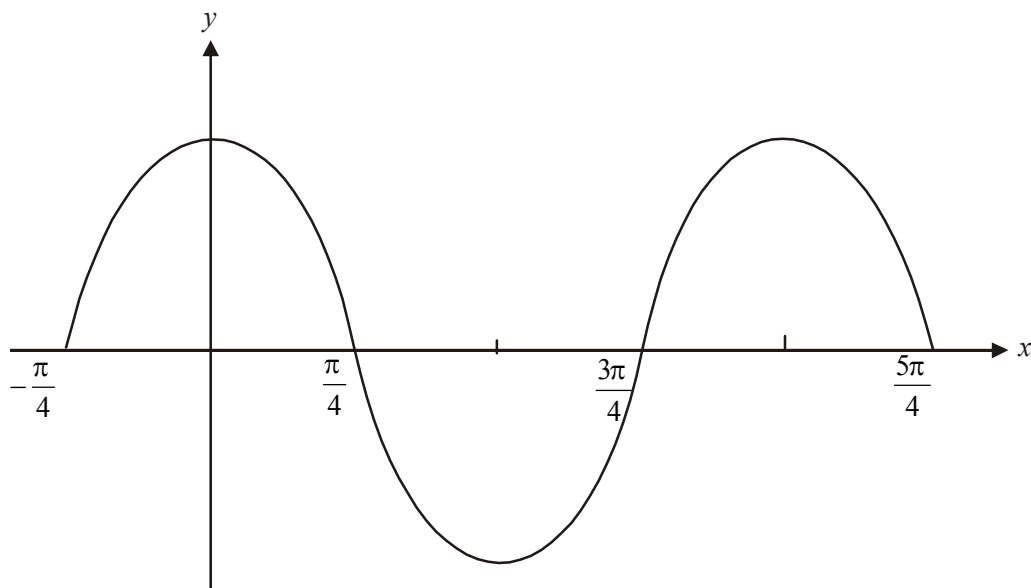
Question 4

$$y = \operatorname{cosec}(a(x-b))$$

$$= \frac{1}{\sin(a(x-b))}$$

Asymptotes occur at $x = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$ and $\frac{5\pi}{4}$.

Consider the graph of $y = \sin(a(x-b))$ at which $y = 0$ at $x = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$ and $\frac{5\pi}{4}$.



period $= \pi = \frac{2\pi}{a}$ so $a = 2$ For this graph above, $y = \sin 2\left(x + \frac{\pi}{4}\right)$.

Also, for the graph of $y = \operatorname{cosec}(a(x-b))$,

$y > 0$ for $x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$ and $y < 0$ for $x \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$. This is true also for the

graph of $y = \sin(a(x-b))$ where $b = -\frac{\pi}{4}$ but not for $b = \frac{\pi}{4}$. So $a = 2$ and $b = -\frac{\pi}{4}$.

The answer is B.

Question 5

$$\begin{aligned}
 v = a + ai \quad |v| &= \sqrt{a^2 + a^2} & \arg(v) &= \tan^{-1}\left(\frac{a}{a}\right) \text{ (first quadrant)} \\
 &= \sqrt{2a^2} & &= \frac{\pi}{4} \\
 &= \sqrt{2}a & \text{since } a \text{ is a positive constant.}
 \end{aligned}$$

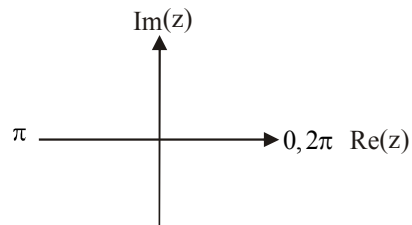
$$\begin{aligned}
 \frac{v}{w} &= \frac{\sqrt{2}a \operatorname{cis}\left(\frac{\pi}{4}\right)}{\operatorname{cis}\left(\frac{\pi}{3}\right)} \\
 &= \sqrt{2}a \operatorname{cis}\left(-\frac{\pi}{12}\right)
 \end{aligned}$$

The answer is C.

Question 6

The solutions to the equation $z^{12} = a$ are equally spaced around a circle, $\frac{2\pi}{12} = \frac{\pi}{6}$ apart.

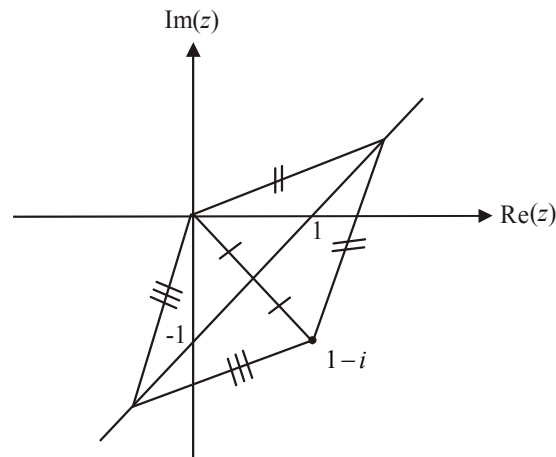
Since one solution is $a^{\frac{1}{12}} \operatorname{cis}\left(\frac{7\pi}{6}\right)$, the others that will have an imaginary part that is less than zero; that is, that have an argument that is greater than π and less than 2π are $a^{\frac{1}{12}} \operatorname{cis}\left(\frac{8\pi}{6}\right)$, $a^{\frac{1}{12}} \operatorname{cis}\left(\frac{9\pi}{6}\right)$, $a^{\frac{1}{12}} \operatorname{cis}\left(\frac{10\pi}{6}\right)$ and $a^{\frac{1}{12}} \operatorname{cis}\left(\frac{11\pi}{6}\right)$.



So there are five in total.

Note that the solutions $a^{\frac{1}{12}} \operatorname{cis}\left(\frac{6\pi}{6}\right)$ and $a^{\frac{1}{12}} \operatorname{cis}(0)$ have an imaginary part equal to zero.

The answer is C.

Question 7

Let z be any point lying on the line.

The distance between any point z on the line and the origin is given by $|z|$. This is the same as the distance between z and the point $1-i$; that is, $|z - (1-i)| = |z - 1 + i|$.

So the required equation of the line is $|z| = |z - 1 + i|$.

The answer is A.

Question 8

Since z lies in the second quadrant, $a < 0$ and $b > 0$.

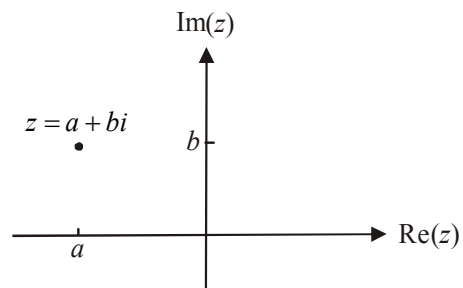
Multiplying z by i rotates z by 90° in an anticlockwise direction which means that zi lies in the third quadrant.

$$\begin{aligned} |z| &= \sqrt{a^2 + b^2} \\ &= \sqrt{2a^2} \text{ if } |a| = b \\ &= \sqrt{2} |a| \end{aligned}$$

$$\begin{aligned} \text{Arg}(z) &= \tan^{-1}(-1) \text{ if } |a| = b \\ &= \frac{3\pi}{4} \text{ since } z \text{ is in the second quadrant} \end{aligned}$$

So $\text{Arg}(z) \neq \frac{\pi}{4}$

The answer is E.



Question 9

$$x^2 - 3y^3 = 1$$

$$\begin{aligned} \text{When } x = 2, \quad 2^2 - 3y^3 &= 1 \\ -3y^3 &= -3 \\ y &= 1 \end{aligned}$$

$$x^2 - 3y^3 = 1$$

$$2x - 3 \times 3y^2 \times \frac{dy}{dx} = 0 \quad (\text{implicit differentiation})$$

$$-9y^2 \frac{dy}{dx} = -2x$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{-2x}{-9y^2} \\ &= \frac{2x}{9y^2} \end{aligned}$$

$$\text{When } x = 2, y = 1 \text{ and } \frac{dy}{dx} = \frac{4}{9}.$$

The answer is B.

Question 10

$$\int_0^{\frac{\pi}{6}} \sin^3(x) \cos^4(x) dx$$

$$= \int_0^{\frac{\pi}{6}} \sin^2(x) \cos^4(x) \sin(x) dx$$

$$= \int_0^{\frac{\pi}{6}} (1 - \cos^2(x)) \cos^4(x) \sin(x) dx$$

$$= \int_1^{\frac{\sqrt{3}}{2}} (1 - u^2) u^4 \times -\frac{du}{dx} dx$$

$$= \int_{\frac{\sqrt{3}}{2}}^1 (u^4 - u^6) du$$

The answer is E.

$$\text{Let } u = \cos(x)$$

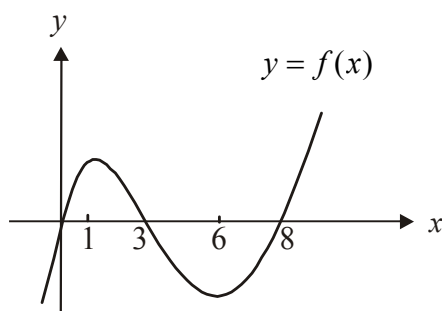
$$\frac{du}{dx} = -\sin(x)$$

$$x = \frac{\pi}{6}, \quad u = \frac{\sqrt{3}}{2}$$

$$x = 0, \quad u = 1$$

Question 11

Do a quick sketch of $y = f(x)$.



The graph of $y = f(x)$ is the gradient function of the graph of $y = F(x)$.

At $x = 0, 3$ and 8 , the gradient of $y = F(x)$ is zero, so there must be a turning point or a stationary point of inflection at $x = 0, x = 3$ and $x = 8$.

Only options A and D show this.

The gradient of the graph of $y = F(x)$ is positive for $0 < x < 3$ and $x > 8$ and negative for $x < 0$ and $3 < x < 8$.

Only option A shows this.

The answer is A.

Question 12

From the slope field when $x = 0$, $\frac{dy}{dx} = 0$. This eliminates options A, B, C and E.

When $x = 0$, $\frac{dy}{dx}$ is defined; that is, values of $\frac{dy}{dx}$ are indicated by the lines. Conversely when

$y = 0$, no such lines are apparent because $\frac{dy}{dx}$ is not defined for $y = 0$. Option D shows this.

Also for a given value of y ; for example $y = 1$, as x increases from zero, the gradient becomes steeper which is indicated by a larger negative number. For another given value of y ; for example $y = -1$, as x increases from zero, the gradient becomes steeper which is indicated by a larger positive number and so on.

The answer is D.

Question 13

The two particles meet iff

$$4 = t + 1 \quad \text{AND} \quad 2t - 6 = 8$$

$$t = 3$$

$$t = 7$$

Since the particles are in the same spot at different times, they never meet.

The answer is E.

Question 14

$$\vec{x}(t) = \frac{1}{t} \vec{i} + \sqrt{t} \vec{j} + e^{2t} \vec{k}$$

$$\dot{\vec{x}}(t) = -1t^{-2} \vec{i} + \frac{1}{2} t^{-\frac{1}{2}} \vec{j} + 2e^{2t} \vec{k}$$

$$\dot{\vec{x}}(1) = -\vec{i} + \frac{1}{2} \vec{j} + 2e^2 \vec{k}$$

The direction of motion at $t=1$ is given by $\dot{\vec{x}}(1)$.

The answer is B.

Question 15

Sketch the graph of $y = x^5 + 2x^3 + x + 1$ and find the enclosed area.

This area occurs between $x = -0.5698403$ and $x = 0$.

The required area is closest to 0.34905 square units.

The answer is B.

Question 16

$$\frac{dy}{dx} = \sqrt{x-9},$$

$$x_0 = 9,$$

$$y_0 = 0, \quad h = 0.1$$

$$x_{n+1} = x_n + h,$$

$$y_{n+1} = y_n + hf(x_n)$$

$$\text{So, } x_1 = 9 + 0.1$$

$$y_1 = 0 + 0.1 \times \sqrt{9-9}$$

$$= 9.1$$

$$= 0$$

$$x_2 = 9.1 + 0.1$$

$$y_2 = 0 + 0.1 \times \sqrt{9.1-9}$$

$$= 9.2$$

$$= 0.1 \times \sqrt{0.1}$$

$$= 0.0316...$$

When $x = 9.2$, the approximation for y is 0.0316...

The answer is C.

Question 17

$\underline{a}$ and $\underline{b}$ are at right angles because the triangle formed by $\underline{a}$, $\underline{b}$ and $\underline{c}$ has its vertices lying on a semicircle with one side forming the straight edge. So option A is correct. Option B is also correct since

$$\begin{aligned}\underline{a} \cdot \underline{a} &= |\underline{a}| |\underline{a}| \cos 0^\circ \\ &= |\underline{a}|^2\end{aligned}$$

Similarly $\underline{b} \cdot \underline{b} = |\underline{b}|^2$ and $\underline{c} \cdot \underline{c} = |\underline{c}|^2$.

So because of Pythagoras theorem options B and C are correct.

$$\begin{aligned}\underline{a} \cdot (\underline{c} + \underline{b}) &= \underline{a} \cdot \underline{a} \\ &= |\underline{a}|^2\end{aligned}$$

So option D is correct.

Option E is incorrect. The angle θ is the angle between $\underline{a}$ and $\underline{c}$ not between $\underline{a}$ and $\underline{b}$.

The answer is E.

Question 18

$$\begin{aligned}& \left| 6\underline{i} + 2\underline{j} + 2\sqrt{6}\underline{k} \right| \\ &= \sqrt{36 + 4 + 24} \\ &= \sqrt{64} \\ &= 8\end{aligned}$$

A vector with a magnitude of 4 that is parallel to the vector $6\underline{i} + 2\underline{j} + 2\sqrt{6}\underline{k}$ is

$$\begin{aligned}& 4 \times \frac{1}{8} (6\underline{i} + 2\underline{j} + 2\sqrt{6}\underline{k}) \\ &= 3\underline{i} + \underline{j} + \sqrt{6}\underline{k}\end{aligned}$$

The answer is C.

Question 19

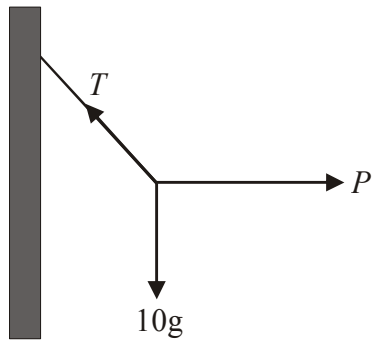
The change in temperature of the cake with respect to time is given by

$$\frac{dT}{dt} = -k(T - 22), \quad T(0) = 180$$

The answer is A.

Question 20

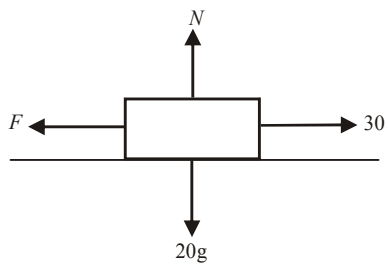
The correct force diagram is given by



The answer is D.

Question 21

Draw a force diagram.



$$\begin{aligned}\text{Now } \mu N &= 0.2 \times 20g \\ &= 4g\end{aligned}$$

$$\text{Since } F = 30$$

$$F < \mu N$$

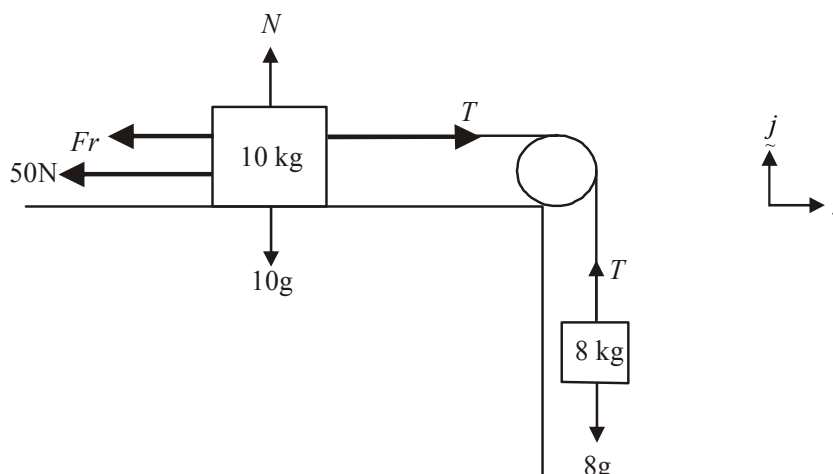
$$\text{i.e. } F < 4g$$

So the box is not at the point of sliding along the surface.

The box remains at rest because $F < 4g$.

The answer is D.

Question 22



Around the 8kg mass:

$$\underline{\underline{R}} = m \underline{\underline{a}}$$

$(T - 8g)\hat{j} = -8a\hat{j}$ assuming that the 8kg mass accelerates downwards

$$T - 8g = -8a$$

$$T = 8g - 8a \quad (\text{A})$$

Around the 10kg mass:

$$\underline{\underline{R}} = m \underline{\underline{a}}$$

$(T - Fr - 50)\hat{i} + (N - 10g)\hat{j} = 10a\hat{i}$ assuming that the 10 kg mass accelerates to the right

$$T - \mu N - 50 = 10a \quad N = 10g$$

$$T - 0.1 \times 10g - 50 = 10a$$

$$T = 10a + g + 50 \quad (\text{B})$$

Substitute (A) into (B):

$$8g - 8a = 10a + g + 50$$

$$-18a = -7g + 50$$

$$a = \frac{7g - 50}{18}$$

Since $a > 0$, the 10kg mass does move to the right and the 8kg mass does move downwards.

The answer is B.

SECTION 2

Question 1

a.

$$\vec{a} = \vec{OA} = m\vec{i} + 2\vec{j}$$

$$\vec{b} = \vec{OB} = \vec{i} + 6\vec{j}$$

$$\vec{c} = \vec{OC} = n\vec{i} + 6\vec{j}$$

$$\vec{d} = \vec{OD} = 3\vec{i} + 2\vec{j}$$

$$\begin{aligned}\text{Now, } \vec{AB} &= \vec{AO} + \vec{OB} \\ &= -m\vec{i} - 2\vec{j} + \vec{i} + 6\vec{j} \\ &= (1-m)\vec{i} + 4\vec{j}\end{aligned}$$

$$\begin{aligned}\vec{AD} &= \vec{AO} + \vec{OD} \\ &= -m\vec{i} - 2\vec{j} + 3\vec{i} + 2\vec{j} \\ &= (3-m)\vec{i}\end{aligned}$$

(1 mark)

- b. If $ABCD$ is a rhombus then $|\vec{AB}| = |\vec{AD}|$ and $\vec{AB} = \vec{DC}$ and $\vec{BC} = \vec{AD}$.

Now, $|\vec{AB}| = |\vec{AD}|$

$$\sqrt{(1-m)^2 + 4^2} = \sqrt{(3-m)^2}$$

$$1 - 2m + m^2 + 16 = 9 - 6m + m^2$$

$$4m = -8$$

$$m = -2$$

(1 mark)

If $\vec{AB} = \vec{DC}$, then $(1-m)\underline{i} + 4\underline{j} = (n-3)\underline{i} + 4\underline{j}$

$$1-m = n-3$$

$$n+m = 4$$

(1 mark)

Substituting $m = -2$ into $n+m = 4$ gives $n-2 = 4$

$$n = 6$$

So $m = -2$ and $n = 6$

(1 mark)

- c. To Show $\vec{AC} \cdot \vec{BD} = 0$

(1 mark)

Now, $\underline{a} = -2\underline{i} + 2\underline{j}$

$$\underline{b} = \underline{i} + 6\underline{j}$$

$$\underline{c} = 6\underline{i} + 6\underline{j}$$

$$\underline{d} = 3\underline{i} + 2\underline{j}$$

$$LS = \left(\vec{AO} + \vec{OC} \right) \cdot \left(\vec{BO} + \vec{OD} \right)$$

$$= \left(2\underline{i} - 2\underline{j} + 6\underline{i} + 6\underline{j} \right) \cdot \left(-\underline{i} - 6\underline{j} + 3\underline{i} + 2\underline{j} \right)$$

$$= \left(8\underline{i} + 4\underline{j} \right) \cdot \left(2\underline{i} - 4\underline{j} \right)$$

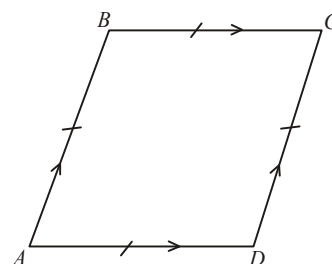
$$= 16 - 16$$

$$= 0$$

$$= RS$$

(1 mark)

Have shown.

Since $\vec{AC} \cdot \vec{BD} = 0$, then $\vec{AC}$ is perpendicular to $\vec{BD}$.

d.

$$\begin{aligned}
 \vec{AE} &= \left(\vec{AB} \cdot \hat{\vec{AD}} \right) \hat{\vec{AD}} \\
 &= \left((3\hat{i} + 4\hat{j}) \cdot \frac{1}{5}(5\hat{i}) \right) \frac{1}{5}(5\hat{i}) \\
 &= 3 \times \frac{1}{5}(5\hat{i}) \\
 &= 3\hat{i}
 \end{aligned}$$

(1 mark)

(1 mark)

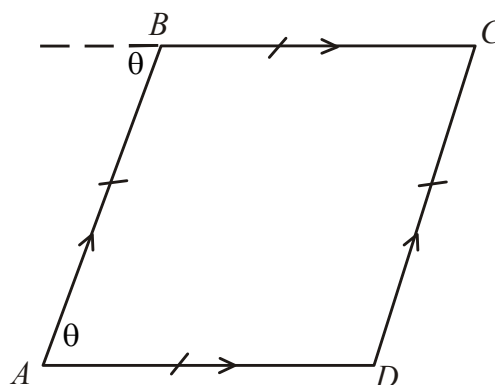
e.

$$\begin{aligned}
 \vec{AB} \cdot \vec{AD} &= |\vec{AB}| |\vec{AD}| \cos \theta \\
 (3\hat{i} + 4\hat{j}) \cdot (5\hat{i}) &= \sqrt{9+16} \sqrt{5^2} \cos(\theta^\circ) \\
 15 &= 25 \cos(\theta^\circ) \\
 \cos(\theta^\circ) &= \frac{3}{5}
 \end{aligned}$$

(1 mark)

(1 mark)

f.

In the rhombus $ABCD$, θ° is the angle between $\vec{AB}$ and $\vec{AD}$ so the angle between $\vec{BA}$ and $\vec{BC}$ will be $(180 - \theta)^\circ$ because of alternate angles in parallel lines. (1 mark)

Now,

$$\cos(180 - \theta)^\circ$$

$$= -\cos(\theta^\circ) \quad (\text{because } \theta < 90^\circ \text{ and } (180^\circ - \theta) \text{ is a second quadrant angle})$$

$$= -\frac{3}{5}$$

(1 mark)

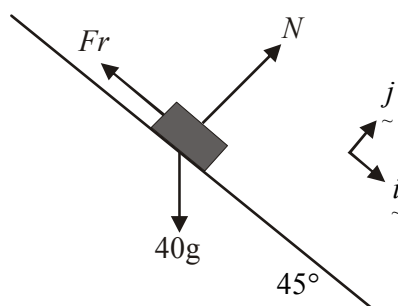
So the angle between $\vec{BA}$ and $\vec{BC}$ is $\cos^{-1}\left(-\frac{3}{5}\right) = 126^\circ 52'$ to the nearest minute.

(1 mark)

Total 13 marks

Question 2

a.



(1 mark)

b.

$$R = m a$$

$$(40g \sin(45^\circ) - Fr)\underline{i} + (N - 40g \cos 45^\circ)\underline{j} = 40a \underline{i} \quad (1 \text{ mark})$$

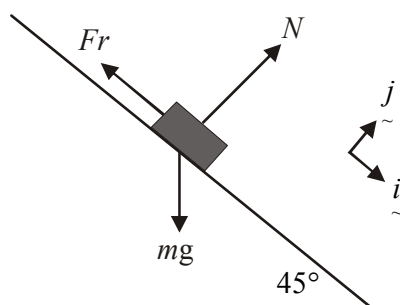
$$\text{So, } \frac{40g}{\sqrt{2}} - Fr = 40a \text{ and } N - \frac{40g}{\sqrt{2}} = 0$$

$$20\sqrt{2}g - 0.1 \times 20\sqrt{2}g = 40a \quad N = 20\sqrt{2}g \quad (1 \text{ mark})$$

$$a = \frac{18\sqrt{2}g}{40}$$

$$= \frac{9\sqrt{2}g}{20} \text{ m/s}^2 \quad (1 \text{ mark})$$

as required.

c. Let the mass of the person on the slide be m kg.

$$R = m a$$

$$(mg \sin(45^\circ) - Fr)\underline{i} + (N - mg \cos 45^\circ)\underline{j} = ma \underline{i}$$

$$\frac{mg}{\sqrt{2}} - 0.1 \times \frac{mg}{\sqrt{2}} = ma \quad N = \frac{mg}{\sqrt{2}} \quad (1 \text{ mark})$$

$$m \left(\frac{g - 0.1g}{\sqrt{2}} \right) = ma$$

$$a = \frac{0.9\sqrt{2}g}{2}$$

$$= \frac{9\sqrt{2}g}{20} \text{ as required} \quad (1 \text{ mark})$$

- d. Since Rhiannon's acceleration down the slide is constant, and we know that $u = 0$,

$a = \frac{9\sqrt{2}g}{20}$ and $s = 15$, we can use the formula

$$s = ut + \frac{1}{2}at^2 \quad (1 \text{ mark})$$

$$15 = 0 + \frac{1}{2} \times \frac{9\sqrt{2}g}{20} t^2$$

$$t = \pm\sqrt{4 \cdot 8102...}$$

$$= 2.1932... \text{ since } t \geq 0$$

So Rhiannon is on the slide for 2.19 seconds (correct to 2 decimal places)

(1 mark)

- e. Now, a is constant i.e. $a = \frac{9\sqrt{2}g}{20}$, $u = 0$ and $t = 2 \cdot 1932...$ (from part d.). Note that the “unrounded” value of 2.1932... from part d. should be carried through in these calculations.

We have,

$$v = u + at$$

$$v = 0 + \frac{9\sqrt{2}g}{20} \times 2 \cdot 1932...$$

$$= 13 \cdot 6784...$$

Rhiannon's speed at the end of the slide is 13.68 m/s (correct to 2 decimal places).

(1 mark)

- f. Note that the “unrounded” value of 13.6784... from part e. should be carried through in these calculations.

Method 1

The only force acting on Rhiannon when she leaves the end of the slide is the gravitational force.

$$\underline{\underline{R}} = m \underline{\underline{a}}$$

$$\text{becomes } -40g \underline{\underline{j}} = 40 \underline{\underline{a}}$$

$$\underline{\underline{a}} = -g \underline{\underline{j}}$$

$$\frac{d\underline{\underline{v}}}{dt} = -g \underline{\underline{j}}$$

$$\underline{\underline{v}} = -gt \underline{\underline{j}} + \underline{\underline{c}} \quad - (1)$$



When $t = 0$, which is when Rhiannon leaves the slide,

$$\underline{\underline{v}} = 13 \cdot 6784 \cos(45^\circ) \underline{\underline{i}} - 13 \cdot 6784 \sin(45^\circ) \underline{\underline{j}}.$$

$$= \frac{13 \cdot 6784}{\sqrt{2}} \underline{\underline{i}} - \frac{13 \cdot 6784}{\sqrt{2}} \underline{\underline{j}}$$

$$\text{In (1)} \quad \frac{13 \cdot 6784}{\sqrt{2}} \underline{\underline{i}} - \frac{13 \cdot 6784}{\sqrt{2}} \underline{\underline{j}} = \underline{\underline{c}}$$

$$\text{So } \underline{\underline{v}} = \frac{13 \cdot 6784}{\sqrt{2}} \underline{\underline{i}} - \left(\frac{13 \cdot 6784}{\sqrt{2}} + gt \right) \underline{\underline{j}} \quad (1 \text{ mark})$$

Note: the above working can be shortened.
It is shown here for explanatory purposes.

$$\text{Now } \underline{r} = \frac{13 \cdot 6784}{\sqrt{2}} t \underline{i} - \left(\frac{13 \cdot 6784}{\sqrt{2}} t + \frac{gt^2}{2} \right) \underline{j} + \underline{c}_1$$

When $t = 0$, $\underline{r} = 0 \underline{i} + 0 \underline{j}$, taking the end of the slide as the origin of motion.

$$\text{So } \underline{c}_1 = \underline{0}$$

$$\text{So } \underline{r} = \frac{13 \cdot 6784}{\sqrt{2}} t \underline{i} - \left(\frac{13 \cdot 6784}{\sqrt{2}} t + \frac{gt^2}{2} \right) \underline{j}$$

(1 mark)

Rhiannon lands when the $\underline{j}$ component of $\underline{r}$ equals -3 (3 metres below the origin).

$$\text{So } \frac{13 \cdot 6784}{\sqrt{2}} t + \frac{gt^2}{2} = 3$$

$$t = 0.2725... \quad (t > 0)$$

(1 mark)

Rhiannon is in free fall for 0.27 secs (correct to 2 decimal places).

Method 2

Rhiannon's velocity at the end of the slide is $13.6784... \text{ m/s}$ at an angle of 45° downwards.

Taking the downwards direction as positive and considering the vertical component

$$\text{we have } u = \frac{13.6784}{\sqrt{2}}, a = 9.8 \text{ and } s = 3.$$

$$s = ut + \frac{1}{2} at^2$$

(1 mark)

$$3 = \frac{13.6784}{\sqrt{2}} t + \frac{9.8}{2} t^2$$

$$4.9t^2 + 9.6721t - 3 = 0$$

(1 mark)

$$t = \frac{-9.6721 \pm \sqrt{9.6721^2 - 4 \times 4.9 \times -3}}{9.8}$$

$$= 0.2725... \text{ since } t > 0$$

Rhiannon is in free fall for 0.27 secs (correct to 2 decimal places).

(1 mark)

- g.** Let the angle at which Rhiannon enters the water be θ .
Note again that the “unrounded” value of $t = 0.2725\dots$ together with the “unrounded” value of $13.6784\dots$ from part **e.** should be carried through in these calculations.

Method 1

The angle θ is the angle between $\underline{v}$ at $t = 0.2725\dots$ secs (i.e. when she hits the water) and the vector $\underline{i}$.

$$\text{So, } \underline{v} = \frac{13.6784}{\sqrt{2}} \underline{i} - \left(\frac{13.6784}{\sqrt{2}} + 0.2725g \right) \underline{j} \quad (1 \text{ mark})$$

$$\underline{v} \cdot \underline{i} = |\underline{v}| |\underline{i}| \cos(\theta)$$

$$\frac{13.6784}{\sqrt{2}} = \sqrt{\frac{13.6784^2}{2} + \left(\frac{13.6784}{\sqrt{2}} + 0.2725g \right)^2} \times 1 \times \cos(\theta)$$

$$\cos(\theta) = 0.6167\dots$$

$$\theta = 51^\circ 55' \text{ to the nearest minute}$$

(1 mark)

Method 2

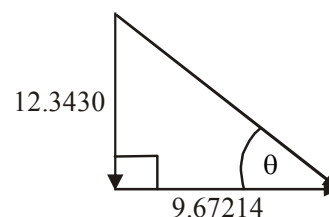
Calculate the vertical and horizontal velocities at the point of entry.
vertically:

$$\begin{aligned} v &= u + at \\ &= 9.67214 + 9.8 \times 0.27254 \\ &= 12.3430(\text{down}) \end{aligned}$$

horizontally:

$$v = 9.67214 \text{ (constant)}$$

(1 mark)



$$\theta = \tan^{-1} \left(\frac{12.3430}{9.67214} \right)$$

$$= 51^\circ 55' \text{ to the nearest minute}$$

(1 mark)

Total 14 marks

Question 3

a. i. $\dot{y}(t) = \int \frac{-t}{(a^2 - t^2)^{\frac{3}{2}}} dt, t > 0$ (1 mark)

where $u = a^2 - t^2$

$$\frac{du}{dt} = -2t$$

$$= \int \frac{1}{2} \frac{du}{dt} \cdot u^{-\frac{3}{2}} dt$$

$$= \frac{1}{2} \int u^{-\frac{3}{2}} du$$

(1 mark)

ii.

$$\dot{y}(t) = \frac{1}{2} \times u^{-\frac{1}{2}} \times -2 + c$$

$$= \frac{-1}{\sqrt{u}} + c$$

$$= \frac{-1}{\sqrt{a^2 - t^2}} + c$$

Since $\dot{y}(1) = \frac{-1}{\sqrt{a^2 - 1}}$

$$-\frac{1}{\sqrt{a^2 - 1}} = \frac{-1}{\sqrt{a^2 - 1}} + c$$

$$c = 0$$

$$\dot{y}(t) = \frac{-1}{\sqrt{a^2 - t^2}} \text{ as required}$$

(1 mark)

b.

$$y(t) = \int \frac{-1}{\sqrt{a^2 - t^2}} dt, t < a$$

$$= \arccos\left(\frac{t}{a}\right) + c$$

Now, $y\left(\frac{a}{2}\right) = \frac{\pi}{3}$

so $\frac{\pi}{3} = \arccos\left(\frac{1}{2}\right) + c$

$$\frac{\pi}{3} = \frac{\pi}{3} + c$$

$$c = 0$$

$$y(t) = \arccos\left(\frac{t}{a}\right)$$

(1 mark)

c.

$$x = -a \log_e(t)$$

$$\frac{dx}{dt} = \frac{-a}{t}$$

$$\frac{d^2x}{dt^2} = \frac{a}{t^2} \quad (1 \text{ mark})$$

Now, $t^2 \frac{d^2x}{dt^2} = a(1-t) - t^2 \frac{dx}{dt}$

$$LS = t^2 \times \frac{a}{t^2}$$

$$= a$$

$$RS = a(1-t) - t^2 \frac{dx}{dt}$$

$$= a - at - t^2 \times \frac{-a}{t}$$

$$= a - at + at$$

$$= a$$

$$= LS$$

Have verified.

(1 mark)

d. From part **b.**, $y(t) = \arccos\left(\frac{t}{a}\right)$.

From part **c.**, $\frac{d^2x}{dt^2} = \frac{a}{t^2} = \ddot{x}(t)$.

So a possible expression for $x(t) = -a \log_e(t) + c$.

(1 mark)

Given the condition $x(1) = 0$, we have

$$0 = -a \log_e(1) + c$$

$$0 = 0 + c$$

$$c = 0$$

So $-a \log_e(t)$ is the actual expression for $x(t)$.

So, $\underline{r} = -a \log_e(t) \underline{i} + \arccos\left(\frac{t}{a}\right) \underline{j}, \quad t > 0$

(1 mark)

e. $x = -a \log_e(t) \qquad y = \arccos\left(\frac{t}{a}\right)$

$$\frac{-x}{a} = \log_e(t) \qquad \frac{t}{a} = \cos(y)$$

$$e^{\frac{-x}{a}} = t \qquad t = a \cos(y)$$

So,

$$e^{\frac{-x}{a}} = a \cos(y)$$

$$\frac{1}{ae^{\frac{x}{a}}} = \cos(y)$$

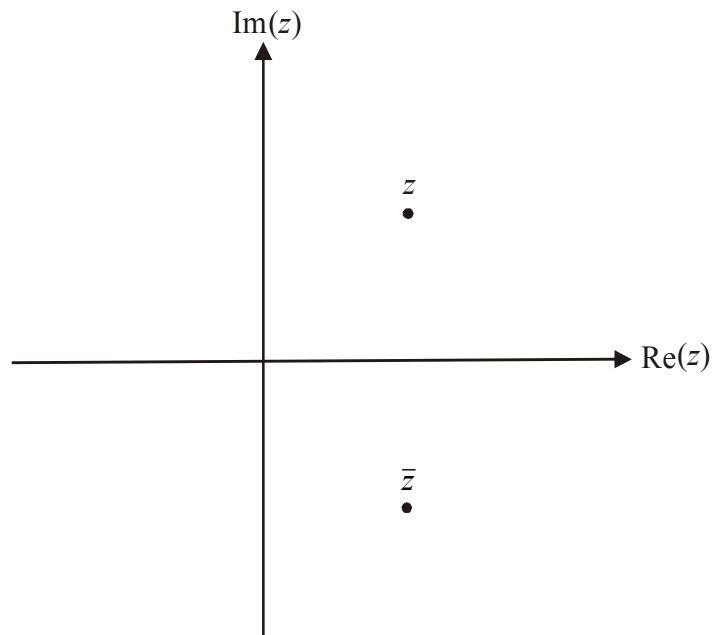
$$y = \arccos\left(\frac{1}{ae^{\frac{x}{a}}}\right)$$

(1 mark)

Total 9 marks

Question 4

a. i.



(1 mark)

ii. $\text{Arg}(z) = \theta$ so $\text{Arg}(\bar{z}) = -\theta$

(1 mark)

iii. $z = r\text{cis}(\theta)$ so $\bar{z} = r\text{cis}(-\theta)$

(1 mark)

iv. To Show $(\bar{z})^n = \overline{(z^n)}$

$$\begin{aligned} LS &= (\bar{z})^n \\ &= (r\text{cis}(-\theta))^n \\ &= r^n \text{cis}(-n\theta) \quad (\text{De Moivre}) \end{aligned}$$

(1 mark)

$$\begin{aligned} RS &= \overline{(z^n)} \\ &= \overline{(r\text{cis}(\theta))^n} \\ &= \overline{r^n \text{cis}(n\theta)} \quad (\text{De Moivre}) \\ &= r^n \text{cis}(-n\theta) \\ &= LS \end{aligned}$$

(1 mark)

Have shown.

b. To Show : $z^n + (\bar{z})^n = 2r^n \cos(n\theta)$

$$LS = z^n + (\bar{z})^n$$

$$= r^n \operatorname{cis}(n\theta) + r^n \operatorname{cis}(-n\theta)$$

(1 mark)

$$= r^n (\cos(n\theta) + i \sin(n\theta) + \cos(-n\theta) + i \sin(-n\theta))$$

$$= r^n (\cos(n\theta) + i \sin(n\theta) + \cos(n\theta) - i \sin(n\theta))$$

since $\cos(-\theta) = \cos(\theta)$

$$= r^n \times 2 \cos(n\theta)$$

and $\sin(-\theta) = -\sin(\theta)$

$$= 2r^n \cos(n\theta)$$

$$= RS$$

Have shown.

(1 mark)

c. Let $(1+i)^{12} + (1-i)^{12}$

$= z^n + (\bar{z})^n$ where $z = 1+i$ from part b. – because of “hence” this must be shown

$$= 2r^n \cos(n\theta) \quad (1 \text{ mark})$$

$$= 2(\sqrt{2})^{12} \cos\left(12 \times \frac{\pi}{4}\right) \quad \text{since if } z = 1+i$$

$$= 128 \cos(3\pi) \quad r = |z| = \sqrt{2}$$

$$= 128 \cos(\pi) \quad \theta = \tan^{-1}(1)$$

$$= -128$$

Have shown.

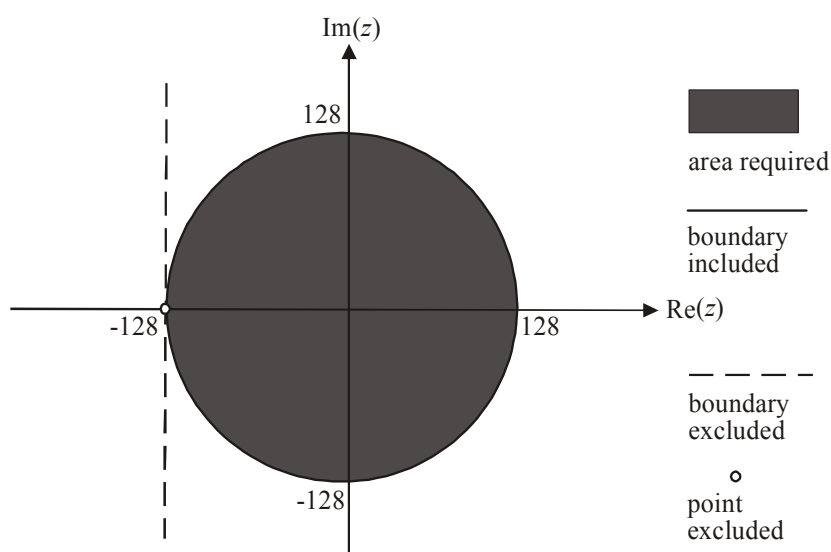
$$= \frac{\pi}{4}$$

Also, $n = 12$

(1 mark)

d. From part c. $(1+i)^{12} + (1-i)^{12} = -128$

So we require $\{z : \operatorname{Re}(z) > -128\} \cap \{z : |z| \leq 128\}$



Note that the complex number $z = -128$ is not included.

(1 mark) correct area

(1 mark) correct boundary

Total 11 marks

Question 5

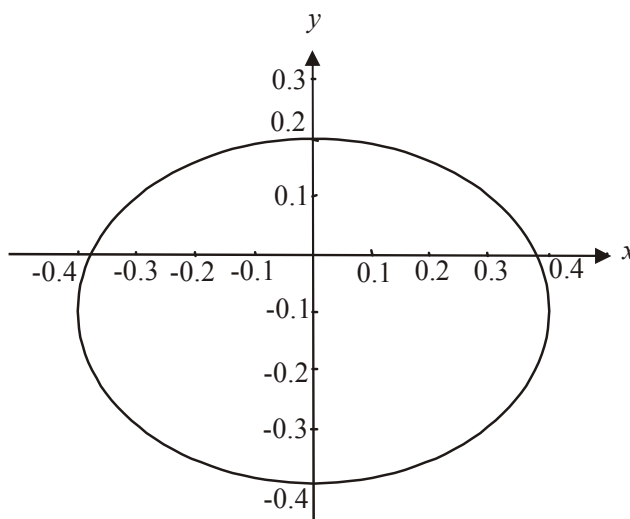
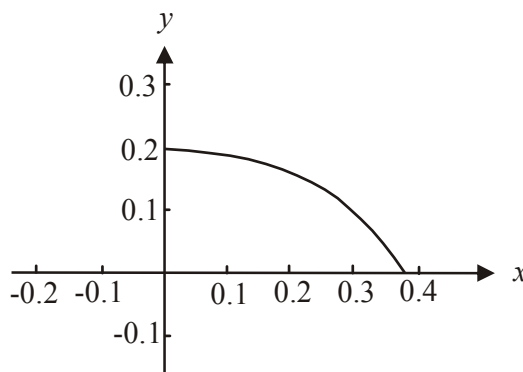
a. $16(y + 0.1)^2 + 9x^2 = 1.44$

$$\frac{(y + 0.1)^2}{9} + \frac{x^2}{16} = 0.01$$

$$\frac{100(y + 0.1)^2}{9} + \frac{100x^2}{16} = 1$$

$$\frac{x^2}{0.16} + \frac{(y + 0.1)^2}{0.09} = 1$$

We have an ellipse with centre at $(0, -0.1)$, semi-major axis length of 0.4 units and semi-minor axis length of 0.3 units.

(1 mark)**(1 mark)****b.**

$$\begin{aligned} \text{volume required} &= \pi \int_0^{0.2} x^2 dy \\ &= \frac{\pi}{9} \int_0^{0.2} (1.44 - 16(y + 0.1)^2) dy \end{aligned}$$

(1 mark) - correct terminals**(1 mark)** - correct integrand

- c. Using a graphics or CAS calculator and the definite integral above,
 volume = 0.0521m^3 (correct to 4 decimal places) **(1 mark)**

- d. i.

$$\frac{dS}{dt} = \text{rate of inflow of salt} - \text{rate of outflow of salt} \quad \textbf{(1 mark)}$$

$$= \frac{dS_{\text{inflow}}}{dl} \cdot \frac{dl}{dt} - \frac{dS_{\text{outflow}}}{dl} \cdot \frac{dl}{dt} \quad \text{where } l \text{ represents litres}$$

$$= 35 \times 0.5 - \frac{S}{50} \times 0.5$$

$$= 17.5 - \frac{S}{100}$$

$$= \frac{1750 - S}{100}$$

(1 mark)

- ii.

$$\frac{dt}{dS} = \frac{100}{1750 - S}$$

$$t = \int \frac{100}{1750 - S} dS$$

$$= -100 \log_e(1750 - S) + c$$

(1 mark)

Now, when $t = 0$, $S = 0$ because initially the base contained no sea water.

$$0 = -100 \log_e(1750) + c$$

$$c = 100 \log_e(1750)$$

$$t = 100 \log_e \left(\frac{1750}{1750 - S} \right)$$

(1 mark)

- iii. Method 1

The concentration of salt in the base will approach the concentration of the sea water being added which is 35g/l . **(1 mark)**

$$\text{Total mass} = \frac{35\text{g}}{l} \times 50\text{l} = 1750\text{g}. \quad \textbf{(1 mark)}$$

Method 2

$$t = 100 \log_e \left(\frac{1750}{1750 - S} \right)$$

$$\frac{t}{100} = \log_e \left(\frac{1750}{1750 - S} \right)$$

$$e^{\frac{t}{100}} = \frac{1750}{1750 - S}$$

$$1750 - S = 1750 e^{\frac{-t}{100}}$$

$$S = 1750 \left(1 - e^{\frac{-t}{100}} \right) \quad \textbf{(1 mark)}$$

As $t \rightarrow \infty$, $e^{\frac{-t}{100}} \rightarrow 0^+$ so $S \rightarrow 1750^-$. So in the long term the amount of salt in the base would approach 1750g . **(1 mark)**

Total 11 marks