

Question 1

a. Let $y = \arctan(\sqrt{2x-1}), x > \frac{1}{2}$

$$y = \arctan(u)$$

$$\frac{dy}{du} = \frac{1}{1+u^2}$$

Now, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ (chain rule) **(1 mark)**

$$= \frac{1}{1+u^2} \cdot \frac{1}{\sqrt{2x-1}}$$

$$= \frac{1}{1+2x-1} \cdot \frac{1}{\sqrt{2x-1}}$$

$$= \frac{1}{2x\sqrt{2x-1}} \text{ as required.}$$

where $u = \sqrt{2x-1}$

$$\frac{du}{dx} = \frac{1}{2}(2x-1)^{-\frac{1}{2}} \times 2$$

$$= \frac{1}{\sqrt{2x-1}}$$

(1 mark)

b. From a. $\frac{d}{dx}(\arctan(\sqrt{2x-1})) = \frac{1}{2x\sqrt{2x-1}}$

So, $\int_1^2 \frac{d}{dx}(\arctan(\sqrt{2x-1})) dx = \frac{1}{2} \int_1^2 \frac{1}{x\sqrt{2x-1}} dx$ **(1 mark)**

$$2[\arctan(\sqrt{2x-1})]_1^2 = \int_1^2 \frac{1}{x\sqrt{2x-1}} dx$$

$$2\{\arctan(\sqrt{3}) - \arctan(1)\} = \int_1^2 \frac{1}{x\sqrt{2x-1}} dx$$

$$\text{So } \int_1^2 \frac{1}{x\sqrt{2x-1}} dx = 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{6}$$

(1 mark)

Question 2

a. $2x^2y + y^2 - 5x = 3$

$$2x^2 \frac{dy}{dx} + 4xy + 2y \frac{dy}{dx} - 5 = 0$$

(1 mark)

$$\frac{dy}{dx}(2x^2 + 2y) = 5 - 4xy$$

$$\frac{dy}{dx} = \frac{5 - 4xy}{2x^2 + 2y}$$

(1 mark)

b. When $y = 0$,

$$2x^2y + y^2 - 5x = 3$$

becomes $-5x = 3$

$$x = -\frac{3}{5}$$

(1 mark)

So $\frac{dy}{dx} = \frac{5 - 4 \times -\frac{3}{5} \times 0}{2 \times \left(-\frac{3}{5}\right)^2 + 0}$

$$= 5 \div \frac{18}{25}$$

$$= \frac{125}{18}$$

$$= 6\frac{17}{18}$$

(1 mark)**Question 3**

$$\frac{dy}{dx} = (x-2)\sqrt{x-1}$$

$$\int \frac{dy}{dx} dx = \int (x-2)\sqrt{x-1} dx$$

$$y = \int (u-1)u^{\frac{1}{2}} \frac{du}{dx} dx$$

$$= \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}} \right) du$$

(1 mark)

$$= \frac{2u^{\frac{5}{2}}}{5} - \frac{2u^{\frac{3}{2}}}{3} + c$$

$$= \frac{2(x-1)^{\frac{5}{2}}}{5} - \frac{2(x-1)^{\frac{3}{2}}}{3} + c$$

(1 mark)

Let $u = x - 1$

$$\frac{du}{dx} = 1$$

and $u - 1 = x - 2$

When $x = 1, y = 0$

So $0 = 0 - 0 + c$

$c = 0$

So $y = \frac{2(x-1)^{\frac{5}{2}}}{5} - \frac{2(x-1)^{\frac{3}{2}}}{3}$

(1 mark)

Question 4

- a. $z^2 - z + 2.5 = 0$ is a quadratic equation so we can use the quadratic formula.

$$z = \frac{1 \pm \sqrt{1 - 4 \times 1 \times 2.5}}{2} \quad (1 \text{ mark})$$

$$= \frac{1 \pm \sqrt{-9}}{2}$$

$$= \frac{1 \pm 3i}{2} \text{ since } \sqrt{-1} = i$$

(1 mark)

- b. $z^3 + z^2 + 3z - 5 = 0$

The coefficients of all the terms are real so two of the three solutions form a conjugate pair and the third is a real solution.

$$\text{Let } p(z) = z^3 + z^2 + 3z - 5$$

$$p(1) = 1 + 1 + 3 - 5 = 0$$

$z - 1$ is a factor.

Method 1

$$\begin{array}{r} z^2 + 2z + 5 \\ z-1 \overline{) z^3 + z^2 + 3z - 5} \\ \underline{z^3 - z^2} \end{array}$$

$$2z^2 + 3z$$

$$\underline{2z^2 - 2z}$$

$$5z - 5$$

$$\underline{5z - 5}$$

$$p(z) = (z - 1)(z^2 + 2z + 5) \quad (1 \text{ mark})$$

Method 2

$$p(z) = z^3 + z^2 + 3z - 5$$

$$= (z - 1) \times \underline{\quad} + (z - 1) \times \underline{\quad} + (z - 1) \times \underline{\quad}$$

$$= (z - 1)z^2 + (z - 1) \times 2z + (z - 1) \times 5 \quad \text{by inspection}$$

$$= (z - 1)(z^2 + 2z + 5) \quad (1 \text{ mark})$$

$$\text{So } p(z) = (z - 1)(z^2 + 2z + 5)$$

$$= (z - 1)((z^2 + 2z + 1) - 1 + 5) \text{ completing the square}$$

$$= (z - 1)((z + 1)^2 + 4)$$

$$= (z - 1)((z + 1)^2 - 4i^2)$$

$$= (z - 1)(z + 1 - 2i)(z + 1 + 2i) \quad (1 \text{ mark})$$

So for $p(z) = 0$,

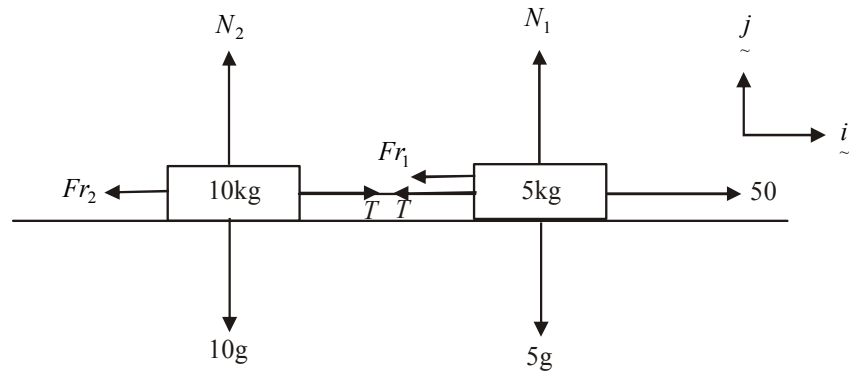
$$z = 1, \quad -1 \pm 2i$$

(1 mark)

Question 5

- a. Method 1 – resolving around each of the containers

Show all the forces on the diagram.



Around the 5kg container

$$\vec{R} = m \vec{a}$$

$$(50 - Fr_1 - T)\vec{i} + (N_1 - 5g)\vec{j} = 5 \times 2\vec{i}$$

$$\text{So, } 50 - Fr_1 - T = 10 \text{ and } N_1 - 5g = 0$$

$$- \mu N_1 - T = -40 \quad N_1 = 5g$$

$$- 5g\mu - T = -40$$

$$T = 40 - 5g\mu \quad - (1)$$

(1 mark)

Around the 10kg container

$$\vec{R} = m \vec{a}$$

$$(T - Fr_2)\vec{i} + (N_2 - 10g)\vec{j} = 10 \times 2\vec{i}$$

$$T - Fr_2 = 20 \text{ and } N_2 - 10g = 0$$

$$T - \mu N_2 = 20 \quad N_2 = 10g$$

$$T - 10g\mu = 20$$

$$T = 10g\mu + 20 \quad - (2)$$

(1 mark)

$$\text{From (1), } T = 40 - 5g\mu$$

$$\text{So } 40 - 5g\mu = 10g\mu + 20$$

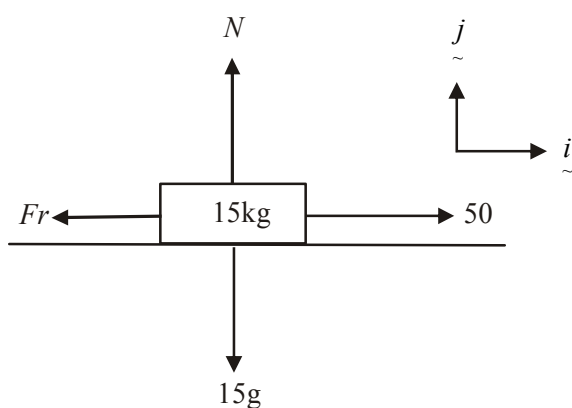
$$20 = 15g\mu$$

$$\mu = \frac{20}{15g}$$

$$\mu = \frac{4}{3g} \text{ as required.}$$

(1 mark)

Method 2 - Combining the containers into a single mass



(1 mark)

$$\vec{R} = m \vec{a}$$

$$(50 - Fr)\vec{i} + (N - 15g)\vec{j} = 15 \times 2\vec{i}$$

$$\text{So, } 50 - Fr = 30 \text{ and } N - 15g = 0$$

$$- \mu N = -20 \quad N = 15g$$

$$15g\mu = 20$$

$$\mu = \frac{4}{3g} \text{ as required}$$

(1 mark)

(1 mark)

b. Method 1 - following on from Method 1 in part a.

Substitute $\mu = \frac{4}{3g}$ into (1)

$$\begin{aligned} T &= 40 - 5g \times \frac{4}{3g} \\ &= 40 - \frac{20}{3} \\ &= \frac{100}{3} \text{ N} \end{aligned} \quad (1 \text{ mark})$$

$$\begin{aligned} (\text{Check in (2)}) T &= 10g\mu + 20 \\ &= 10g \times \frac{4}{3g} + 20 \\ &= \frac{40}{3} + 20 \\ &= \frac{100}{3} \text{ N} \end{aligned}$$

Method 2 - following on from Method 2 in part a.

Around the 10 kg container

$$\vec{R} = m \vec{a}$$

$$(T - Fr)\vec{i} + (N - 10g)\vec{j} = 10 \times 2\vec{i}$$

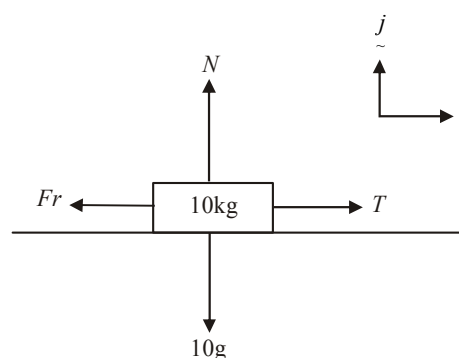
$$T - Fr = 20 \text{ and } N - 10g = 0$$

$$T - \mu N = 20 \quad N = 10g$$

$$T - \frac{4 \times 10g}{3g} = 20$$

$$T = \frac{100}{3} \text{ N}$$

(1 mark)



Question 6

Do a quick sketch.

$$y = \frac{1}{x^2 - 4x}$$

$$= \frac{1}{x(x-4)}$$

There are vertical asymptotes at $x = 0$ and $x = 4$.

The required area lies below the x -axis.

$$\text{Area} = -\int_1^3 \frac{1}{x^2 - 4x} dx$$

(1 mark)

$$\text{Let } \frac{1}{x(x-4)} \equiv \frac{A}{x} + \frac{B}{x-4}$$

$$\equiv \frac{A(x-4) + Bx}{x(x-4)}$$

$$\text{True iff } 1 \equiv A(x-4) + Bx$$

$$\text{Put } x = 4, 1 = 4B, B = \frac{1}{4}$$

$$\text{Put } x = 0, 1 = -4A, A = -\frac{1}{4}$$

$$\text{So } \frac{1}{x(x-4)} = \frac{-1}{4x} + \frac{1}{4(x-4)}$$

(1 mark)

$$\text{Area required} = -\int_1^3 \left(-\frac{1}{4x} + \frac{1}{4(x-4)} \right) dx$$

$$= -\left[-\frac{1}{4} \log_e |x| + \frac{1}{4} \log_e |x-4| \right]_1^3$$

$$= -\frac{1}{4} \left[\log_e \frac{|x-4|}{|x|} \right]_1^3$$

(1 mark)

$$= -\frac{1}{4} \left\{ \log_e \left(\frac{1}{3} \right) - \log_e \left(\frac{3}{1} \right) \right\}$$

$$= -\frac{1}{4} \left(\log_e \left(\frac{1}{3} \div 3 \right) \right)$$

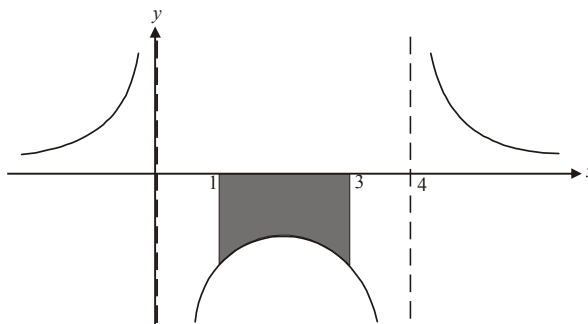
$$= -\frac{1}{4} \log_e \left(\frac{1}{9} \right)$$

(1 mark)

$$= -\frac{1}{4} \log_e (9^{-1})$$

$$= \frac{1}{4} \log_e (9) \text{ square units}$$

$$\text{So } a = \frac{1}{4} \text{ and } b = 9$$

(1 mark)

Question 7

The function is continuous for $x \in R$ and also $\frac{1}{\sqrt{4+x^2}} > 0$ for $x \in R$.

$$\begin{aligned} \text{Volume} &= \pi \int_0^2 y^2 dx \\ &= \pi \int_0^2 \frac{1}{4+x^2} dx \end{aligned} \quad (1 \text{ mark})$$

$$= \frac{\pi}{2} \int_0^2 \frac{2}{4+x^2} dx \quad (1 \text{ mark})$$

$$= \frac{\pi}{2} \left[\tan^{-1} \left(\frac{x}{2} \right) \right]_0^2$$

$$= \frac{\pi}{2} (\tan^{-1}(1) - \tan^{-1}(0))$$

$$= \frac{\pi}{2} \times \frac{\pi}{4} - 0$$

$$= \frac{\pi^2}{8} \text{ cubic units}$$

(1 mark)

Question 8

$$\begin{aligned} \int_0^3 \frac{4(x-1)}{\sqrt{9-x^2}} dx &= \int_0^3 \frac{4x-4}{\sqrt{9-x^2}} dx \\ &= \int_0^3 \frac{4x}{\sqrt{9-x^2}} dx - \int_0^3 \frac{4}{\sqrt{9-x^2}} dx \end{aligned} \quad (1 \text{ mark})$$

$$= \int_9^0 -2 \frac{du}{dx} u^{-\frac{1}{2}} dx - 4 \int_0^3 \frac{1}{\sqrt{9-x^2}} dx$$

$$= -2 \int_9^0 u^{-\frac{1}{2}} du - 4 \left[\sin^{-1} \left(\frac{x}{3} \right) \right]_0^3$$

$$= -2 \left[2u^{\frac{1}{2}} \right]_9^0 - 4 \left[\sin^{-1} \left(\frac{x}{3} \right) \right]_0^3$$

$$\begin{aligned} & \quad (1 \text{ mark}) \quad (1 \text{ mark}) \\ &= -2(0-6) - 4(\sin^{-1}(1) - \sin^{-1}(0)) \\ &= 12 - 4 \left(\frac{\pi}{2} - 0 \right) \\ &= 12 - 2\pi \end{aligned}$$

where $u = 9 - x^2$

$$\frac{du}{dx} = -2x$$

$$x = 3, \quad u = 0$$

$$x = 0, \quad u = 9$$

(1 mark)

Question 9

a. $\underline{v}(t) = (2 \sin^2(t) - 1)\underline{i} - \sin(2t)\underline{j} \quad t \geq 0$

speed = $|\underline{v}|$

$$\begin{aligned} &= \sqrt{(2 \sin^2(t) - 1)^2 + (-\sin(2t))^2} && \text{(1 mark)} \\ &= \sqrt{4 \sin^4(t) - 4 \sin^2(t) + 1 + (-2 \sin(t) \cos(t))^2} \\ &= \sqrt{4 \sin^4(t) - 4 \sin^2(t) + 1 + 4 \sin^2(t)(1 - \sin^2(t))} \\ &= \sqrt{4 \sin^4(t) - 4 \sin^2(t) + 1 + 4 \sin^2(t) - 4 \sin^4(t)} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

so speed is constant.

(1 mark)

b. $\underline{v}(t) = (2 \sin^2(t) - 1)\underline{i} - \sin(2t)\underline{j} \quad t \geq 0$

$\underline{v}(t) = -\cos(2t)\underline{i} - \sin(2t)\underline{j}$

$\underline{r}(t) = -\frac{1}{2}\sin(2t)\underline{i} + \frac{1}{2}\cos(2t)\underline{j} + \underline{c}$

(1 mark)

Now, when $t=0$, $\underline{r} = \frac{1}{2}\underline{j}$

so, $\frac{1}{2}\underline{j} = 0\underline{i} + \frac{1}{2}\underline{j} + \underline{c}$

$\underline{c} = 0$

$\underline{r}(t) = -\frac{1}{2}\sin(2t)\underline{i} + \frac{1}{2}\cos(2t)\underline{j}$

as required.

(1 mark)

c. $x = -\frac{1}{2}\sin(2t) \quad y = \frac{1}{2}\cos(2t)$

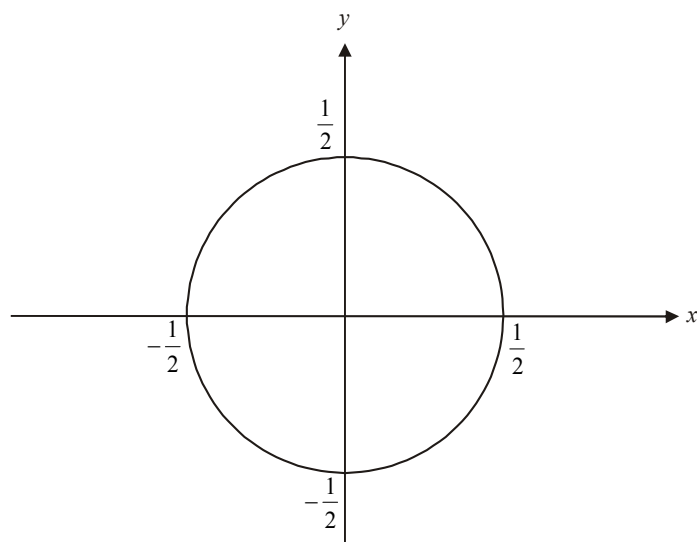
$x^2 = \frac{1}{4}\sin^2(2t) \quad y^2 = \frac{1}{4}\cos^2(2t)$

$x^2 + y^2 = \frac{1}{4}(\sin^2(2t) + \cos^2(2t))$

$x^2 + y^2 = \frac{1}{4}$

(1 mark)

d. i.



(1 mark)

ii. At $t=0$,

$$\vec{r} = 0\vec{i} + \frac{1}{2}\vec{j}$$

The particle starts at the point $\left(0, \frac{1}{2}\right)$.

(1 mark)

$$\text{At } t = \frac{\pi}{4}$$

$$\vec{r} = -\frac{1}{2}\vec{i} + 0\vec{j}$$

At $t = \frac{\pi}{4}$ seconds, the particle is at the point $\left(-\frac{1}{2}, 0\right)$.

So the particle moves around the circle indefinitely in an anticlockwise direction having started its motion at the point $\left(0, \frac{1}{2}\right)$.

(1 mark)