



The Mathematical Association of Victoria

2006 SPECIALIST MATHEMATICS

VCAA Sample Examination 1 and 2

Suggested Answers and Solutions

Examination 1

The examination will consist of short answer questions which are to be answered without the use of technology.

Examination 2

The examination will consist of two parts. Part 1 will be a multiple-choice section containing 22 questions and Part II will consist of extended answer questions, involving multi-stage solutions of increasing complexity.

The sample exams are found at:

<http://www.vcaa.vic.edu.au/vce/studies/mathematics/specialist/pastexams/2006/2006specmath.pdf>

These answers and solutions have been written and published to assist students in their preparations for the 2006 Mathematical Methods Examinations. The answers and solutions do not necessarily reflect the views of the Victorian Curriculum and Assessment Authority.

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VCAA Sample Questions 2006 – Specialist Mathematics

Written Examination 1

Suggested answers and solutions

Question 1

a. $F_R = \mu N$
 $= \frac{1}{7} \times 100 \times 9.8$
 $= 140$

The crate will move if $F > F_R$ ie. If $F > 140$

Since $F = 120$, the crate will not move.

b. $F - F_R = ma$
 $190 - 140 = 100a$
 $50 = 100a$
 $a = 0.5 \text{ m/s}^2$

Question 2

$\overline{BC} = \underline{a}$ and $\overline{AC} = \underline{b}$ since OABC is a parallelogram.

$\overline{OC} \cdot \overline{AB} = 0$ since $\overline{AB}$ is perpendicular to $\overline{OC}$.

$$\therefore (\underline{a} + \underline{b}) \cdot (\underline{b} - \underline{a}) = 0$$

$$\underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{b} - \underline{a} \cdot \underline{a} - \underline{a} \cdot \underline{b} = 0$$

$$\underline{b} \cdot \underline{b} - \underline{a} \cdot \underline{a} = 0$$

$$|\underline{b}|^2 - |\underline{a}|^2 = 0$$

$$|\underline{b}| = |\underline{a}| \text{ so all sides have the same length.}$$

Opposite sides are parallel, since it is a parallelogram.

Hence OABC is a rhombus.

Question 3

a. $f(x) = \arctan(x) + x \arctan(x)$
 $f'(x) = \frac{1}{1+x^2} + x \times \frac{1}{1+x^2} + 1 \times \arctan(x)$
 $= \frac{1}{1+x^2} + \frac{x}{1+x^2} + \arctan(x)$
 $= \arctan(x) + \frac{1+x}{1+x^2}$

b. Points of inflection occur when $f''(x) = 0$.

$$f''(x) = 0$$

$$\frac{1}{1+x^2} + \frac{(x^2+1) - 2x(x+1)}{(x^2+1)^2} = 0$$

$$x^2 + 1 + x^2 + 1 - 2x^2 - 2x = 0$$

$$2 - 2x = 0$$

$$x = 1$$

$$f(1) = \arctan(1) + \arctan(1)$$

$$= \frac{\pi}{4} + \frac{\pi}{4}$$

$$= \frac{\pi}{2}$$

Point of inflection at $\left(1, \frac{\pi}{2}\right)$

Question 4

a. Differentiating with respect to x :

$$\frac{d}{dx}(2y) - \frac{d}{dx}(xy^2) = \frac{d}{dx}(8)$$

$$2 \frac{dy}{dx} - \left(\frac{d(x)}{dx} y^2 + x \frac{d(y^2)}{dx} \right) = 0$$

$$2 \frac{dy}{dx} - y^2 - 2xy \frac{dy}{dx} = 0$$

$$(2 - 2xy) \frac{dy}{dx} = y^2$$

$$\frac{dy}{dx} = \frac{y^2}{2(1-xy)}$$

b. When $y = 2$

$$4 - 4x = 8$$

$$x = -1$$

$$\therefore \frac{dy}{dx} = \frac{2^2}{2(1 - 2 \times -1 \times 2)}$$

$$= \frac{2}{5}$$

Question 5

a. Substituting $z = 3i$ we have

$$\begin{aligned}(3i)^2 - 2(3i)^2 + 9(3i) - 18 \\ = -27i + 18 + 27i - 18 \\ = 0\end{aligned}$$

So $z = 3i$ is a solution.

b. As all coefficients are real and $z - 3i$ is a factor then $z + 3i$ is also a factor.

$$(z - 3i)(z + 3i)(z + a) = 0$$

$$(z^2 + 9)(z + a) = 0$$

$$\text{So } 9a = -18$$

$$\therefore a = -2$$

So the solutions are $-3i, 3i, 2$

Question 6

$$\begin{aligned}\text{a. } \frac{d}{dx}(\cos^{-1}(\sqrt{3x})) &= \frac{-1}{\sqrt{1 - (\sqrt{3x})^2}} \times \frac{1}{2}(3x)^{-\frac{1}{2}} \times 3 \\ &= \frac{-1}{\sqrt{1 - 3x}} \times \frac{3}{2\sqrt{3x}} \\ &= \frac{-1}{\sqrt{1 - 3x}} \times \frac{3}{2\sqrt{3}\sqrt{x}} \\ &= \frac{-\sqrt{3}}{2\sqrt{1 - 3x}\sqrt{x}} \\ &= \frac{-\sqrt{3}}{2\sqrt{x - 3x^2}}\end{aligned}$$

$$\text{b. } \int \frac{-\sqrt{3}}{2\sqrt{x - 3x^2}} dx = \cos^{-1}(\sqrt{3x})$$

$$\begin{aligned}\text{so } \frac{-\sqrt{3}}{2} \int \frac{1}{\sqrt{x - 3x^2}} dx &= \cos^{-1}(\sqrt{3x}) \\ \int \frac{1}{\sqrt{x - 3x^2}} dx &= \frac{-2}{\sqrt{3}} \cos^{-1}(\sqrt{3x})\end{aligned}$$

$$\begin{aligned}\int_{\frac{1}{6}}^{\frac{1}{4}} \frac{1}{\sqrt{x - 3x^2}} dx &= \left[\frac{-2}{\sqrt{3}} \cos^{-1}(\sqrt{3x}) \right]_{\frac{1}{6}}^{\frac{1}{4}} \\ &= \frac{-2}{\sqrt{3}} \left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \right) \\ &= \frac{-2}{\sqrt{3}} \left(\frac{\pi}{6} - \frac{\pi}{4} \right) \\ &= \frac{-2}{\sqrt{3}} \times \frac{-\pi}{12} \\ &= \frac{\sqrt{3}\pi}{18}\end{aligned}$$

Question 7

$$\begin{aligned}y &= \int \sin^3(x) \cos^2(x) dx \\ &= \int \sin(x) \sin^2(x) \cos^2(x) dx \\ &= \int \sin(x) (1 - \cos^2(x)) \cos^2(x) dx\end{aligned}$$

Let $u = \cos(x)$, so $\frac{du}{dx} = -\sin(x)$

$$dx = \frac{du}{\sin(x)}$$

Therefore, making the substitutions

$$\begin{aligned}y &= \int \sin(x) (1 - u^2) u^2 \frac{-du}{\sin(x)} \\ &= \int -(u^2 - u^4) du \\ &= \int (u^4 - u^2) du \\ &= \frac{u^5}{5} - \frac{u^3}{3} + c \\ &= \frac{1}{5} \cos^5(x) - \frac{1}{3} \cos^3(x) + c\end{aligned}$$

$$y(0) = 0$$

$$\frac{\cos^5(0)}{5} - \frac{\cos^3(0)}{3} + c = 0$$

$$\frac{1}{5} - \frac{1}{3} + c = 0$$

$$c = \frac{2}{15}$$

$$\therefore y = \frac{1}{5} \cos^5(x) - \frac{1}{3} \cos^3(x) + \frac{2}{15}$$

Question 8

$$V = \pi \int x^2 dy$$

$$y = \frac{16}{x^2 + 4} - 2$$

$$\therefore y + 2 = \frac{16}{x^2 + 4}$$

$$x^2 + 4 = \frac{16}{y + 2}$$

$$x^2 = \frac{16}{y + 2} - 4$$

$$V = \int_0^2 \left(\frac{16}{y + 2} - 4 \right) dy$$

$$= \pi [16 \log_e(y + 2) - 4y]_0^2$$

$$= \pi [(16 \log_e(4) - 8) - (16 \log_e(2) - 0)]$$

$$= \pi [16 \log_e(4) - 16 \log_e(2) - 8]$$

$$= \pi [16 \log_e(2) - 8]$$

$$= 8\pi [2 \log_e(2) - 1]$$

$$= 8\pi [\log_e(4) - 1] \text{ cubic units}$$

Question 9

- a. Vertical asymptotes occur when $x^2 + 4x + 5 = 0$

$$\Delta = (4)^2 - 4(1)(5)$$

$$= 16 - 20$$

$$= -4$$

As $\Delta < 0$, $x^2 + 4x + 5$ has no real solutions

So there are no vertical asymptotes.

- b.

$$\begin{aligned} \int \frac{3}{x^2 + 4x + 5} dx &= \int \frac{3}{(x + 2)^2 + 1} dx \\ &= 3 \tan^{-1}(x + 2) + c \end{aligned}$$

$$\therefore A = 3, B = 2$$

- c. Turning points occur when $\frac{dy}{dx} = 0$.

$$\frac{dy}{dx} = \frac{-3(2x + 4)}{(x^2 + 4x + 5)^2}$$

$$\text{When } \frac{dy}{dx} = 0$$

$$2x + 4 = 0$$

$$x = -2$$

$$\begin{aligned} \text{Area} &= \int_{-2}^{\sqrt{3}-2} \frac{3}{x^2 + 4x + 5} dx \\ &= [3 \tan^{-1}(x + 2)]_{-2}^{\sqrt{3}-2} \\ &= 3 \tan^{-1}(\sqrt{3}) - 3 \tan^{-1}(0) \\ &= \pi \end{aligned}$$

- d.

$$-\frac{\pi}{2} < \tan^{-1}(x + 7) < \frac{\pi}{2}$$

$$-\frac{5\pi}{2} < 5 \tan^{-1}(x + 7) < \frac{5\pi}{2}$$

$$-\frac{5\pi}{2} + c < 5 \tan^{-1}(x + 7) + c < \frac{5\pi}{2} + c$$

For $5 \tan^{-1}(x + 7) + c > 0$ then $-\frac{5\pi}{2} + c > 0$, so

$$c > \frac{5\pi}{2}.$$

VCAA Sample Questions 2006 – Specialist Mathematics

Written Examination 2 Section 1

Suggested answers and solutions

1	E	2	C	3	E	4	D	5	B
6	A	7	C	8	A	9	D	10	B
11	E	12	D	13	C	14	E	15	B
16	D	17	A	18	B	19	A	20	E
21	E	22	C						

Question 1

$$\begin{aligned}
 y &= \frac{-x^2 + 1}{2x} \\
 &= \frac{-x^2}{2x} + \frac{1}{2x} \\
 &= \frac{-x}{2} + \frac{1}{2x}
 \end{aligned}$$

Vertical asymptote at $x = 0$

$$\text{As } x \rightarrow \pm\infty, \frac{1}{2x} \rightarrow 0, \therefore y \rightarrow \frac{-x}{2}$$

Therefore $y = \frac{-x}{2}$ is also an asymptote

$\therefore$ E

Question 2

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Centre: $(1, -2)$

$$\therefore h = 1, k = -2$$

Also, the lengths of the major and minor axes give
 $a = 1, b = 2$

$$(x-1)^2 + \frac{(y+2)^2}{4} = 1$$

$\therefore$ C

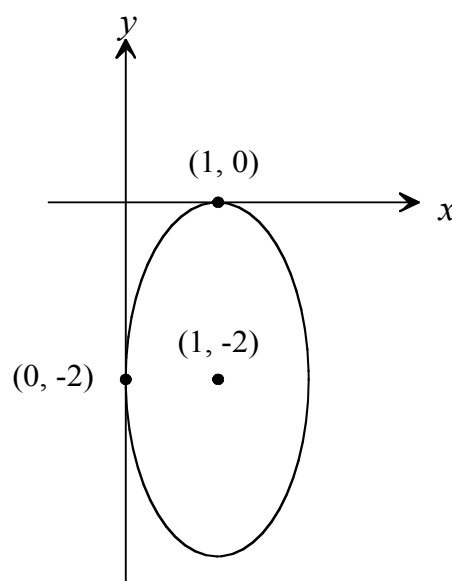


Diagram for Question 2

Question 3

$$\text{A. } \frac{\sin\left(\frac{\pi}{5}\right)}{\cos\left(\frac{\pi}{5}\right)} = \tan\left(\frac{\pi}{5}\right)$$

$$\begin{aligned}
 \text{B. } \frac{1}{\cot\left(\frac{\pi}{5}\right)} &= \frac{1}{\left(\frac{1}{\tan\left(\frac{\pi}{5}\right)}\right)} \\
 &= \tan\left(\frac{\pi}{5}\right)
 \end{aligned}$$

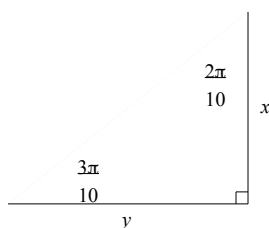
$$\begin{aligned}
 \text{C} \quad \cot\left(\frac{3\pi}{10}\right) &= \cot\left(\frac{\pi}{2} - \frac{\pi}{5}\right) \\
 &= \frac{\cos\left(\frac{\pi}{2} - \frac{\pi}{5}\right)}{\sin\left(\frac{\pi}{2} - \frac{\pi}{5}\right)} \\
 &= \frac{\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{5}\right) + \sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{5}\right)}{\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{5}\right) - \cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{5}\right)} \\
 &= \frac{\sin\left(\frac{\pi}{5}\right)}{\cos\left(\frac{\pi}{5}\right)} \\
 &= \tan\left(\frac{\pi}{5}\right)
 \end{aligned}$$

Alternatively

$$\cot\left(\frac{3\pi}{10}\right) = \frac{y}{x}$$

$$\tan\left(\frac{2\pi}{10}\right) = \frac{y}{x}$$

$$\begin{aligned}
 \therefore \cot\left(\frac{3\pi}{10}\right) &= \tan\left(\frac{2\pi}{10}\right) \\
 &= \tan\left(\frac{\pi}{5}\right)
 \end{aligned}$$



$$\begin{aligned}
 \text{D.} \quad \frac{2 \tan\left(\frac{\pi}{10}\right)}{1 - \tan^2\left(\frac{\pi}{10}\right)} &= \tan\left(2 \times \frac{\pi}{10}\right) \\
 &= \tan\left(\frac{\pi}{5}\right)
 \end{aligned}$$

$$\begin{aligned}
 \text{E.} \quad \frac{2 \tan\left(\frac{2\pi}{5}\right)}{1 - \tan^2\left(\frac{2\pi}{5}\right)} &= \tan\left(2 \times \frac{2\pi}{5}\right) \\
 &= \tan\left(\frac{4\pi}{5}\right) \\
 &= \tan\left(\pi - \frac{\pi}{5}\right) \\
 &= -\tan\left(\frac{\pi}{5}\right)
 \end{aligned}$$

Therefore option E is incorrect.

Question 4

The period of the graph is π . Therefore,

$$\begin{aligned}
 \frac{2\pi}{a} &= \pi \\
 \therefore a &= 2
 \end{aligned}$$

The graph is translated by $\frac{\pi}{4}$ units in the

positive x - direction. Therefore $b = \frac{\pi}{4}$

$\therefore$ D

Question 5

As the polynomial has real coefficients then the Complex Conjugate Rule applies. Hence, any complex roots must occur as a conjugate pair.

Hence, it is not possible to have an odd number of non-real roots. So, answer **B** must be false

Question 6

If $|w| = 1.5$ then $w = 1.5\text{cis}\theta$

Therefore,

$$\begin{aligned}
 w^{-1} &= 1.5^{-1}\text{cis}(-\theta) \\
 &= \frac{2}{3}\text{cis}(-\theta)
 \end{aligned}$$

So, P is the best representation

$\therefore$ A

Question 7

The shaded region corresponds to

$$\left\{z : \text{Arg}(z) > \frac{\pi}{3}\right\} \cap \left\{z : \text{Arg}(z) < \frac{\pi}{2}\right\}$$

$\therefore$ C

Question 8

$$\begin{aligned}\int_0^{\frac{\pi}{3}} \cos^2(x) \sin^3(x) dx &= \int_0^{\frac{\pi}{3}} \cos^2(x) \sin^2(x) \sin(x) dx \\ &= \int_0^{\frac{\pi}{3}} \cos^2(x) (1 - \cos^2(x)) \sin(x) dx\end{aligned}$$

Let $u = \cos(x)$

$$\frac{du}{dx} = -\sin(x)$$

$$dx = \frac{-du}{\sin(x)}$$

When $x = \frac{\pi}{3}$ $u = \cos\left(\frac{\pi}{3}\right)$
 $= \frac{1}{2}$
 $x = 0$ $u = \cos(0)$
 $= 1$

$$\begin{aligned}\int_0^{\frac{\pi}{3}} \cos^2(x) (1 - \cos^2(x)) \sin(x) dx \\ &= \int_1^{\frac{1}{2}} u^2 (1 - u^2) \sin(x) \frac{-du}{\sin(x)} \\ &= -\int_1^{\frac{1}{2}} u^2 (1 - u^2) du \\ &= \int_{\frac{1}{2}}^1 u^2 (1 - u^2) du\end{aligned}$$

$\therefore A$

Question 9

When $x = \frac{\pi}{2}$, $F'(x) = 0$

$0 < x < \frac{\pi}{2}$, $F'(x) > 0$

$\frac{\pi}{2} < x < \pi$, $F'(x) < 0$

Therefore a local maximum occurs at $x = \frac{\pi}{2}$

When $x = \pi$, $F'(x) = 0$

$\frac{\pi}{2} < x < \pi$, $F'(x) < 0$

$\pi < x < \frac{3\pi}{2}$, $F'(x) < 0$

Therefore a stationary point of inflexion occurs at $x = \pi$

When $x = \frac{3\pi}{2}$ $F'(x) = 0$

$\pi < x < \frac{3\pi}{2}$, $F'(x) < 0$

$\frac{3\pi}{2} < x < 2\pi$, $F'(x) > 0$

Therefore a local minimum occurs at $x = \frac{3\pi}{2}$

$\therefore D$

Question 10

$$x^2 + y^2 = 9$$

Differentiating both sides with respect to x

$$2x + 2y \frac{dy}{dx} = 0$$

$$2y \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

When $x = 1$, $y = \sqrt{8}$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{8}}$$

$$= \frac{-1}{2\sqrt{2}}$$

$\therefore B$

Question 11

$$\int_a^b \frac{1}{x} dx = [\log_e |x|]_a^b$$

$$= \log_e |b| - \log_e |a|$$

$\therefore E$

Question 12

$$y = \sqrt{x^2 - 9}$$

$$\therefore y^2 = x^2 - 9$$

$$\therefore x^2 = y^2 + 9$$

$$\begin{aligned} V &= \pi \int_0^4 5^2 dy - \pi \int_0^4 (y^2 + 9) dy \\ &= \pi \int_0^4 (25 - (y^2 + 9)) dy \\ &= \pi \int_0^4 (16 - y^2) dy \end{aligned}$$

$\therefore$ D

Question 13

A $|\underline{b}|^2 + |\underline{c}|^2 = |\underline{a}|^2$ by Pythagoras' Theorem

Therefore: **true**

B $\underline{b} + \underline{c} = \underline{a}$
 $\therefore \underline{a} - \underline{c} = \underline{b}$

So

$$\begin{aligned} \underline{b} \cdot (\underline{a} - \underline{c}) &= \underline{b} \cdot \underline{b} \\ &= |\underline{b}|^2 \end{aligned}$$

Therefore: **true**

C $\underline{b} + \underline{c} = \underline{a}$
 $\therefore \underline{a} - \underline{b} = \underline{c}$

So

$$\begin{aligned} \underline{b} \cdot (\underline{a} - \underline{b}) &= \underline{b} \cdot \underline{c} \\ &= |\underline{b}| |\underline{c}| \cos(90^\circ) \\ &= 0 \\ &\neq |\underline{b}| |\underline{c}| \end{aligned}$$

Therefore: **false**

D $\underline{a} \cdot \underline{b} = |\underline{a}| |\underline{b}| \cos(\theta)$ by dot product rule

Therefore: **true**

E $\underline{a} \cdot \underline{c} = |\underline{a}| |\underline{c}| \cos(90^\circ - \theta)$
 $= |\underline{a}| |\underline{c}| \sin(\theta)$

Therefore: **true**

Hence the correct answer is **C**

Question 14

$$\underline{r}(t) = 2 \sin(t) \underline{i} + \cos(t) \underline{j}$$

$$x = 2 \sin(t) \quad y = \cos(t)$$

$$\frac{x}{2} = \sin(t)$$

$$\sin^2(t) + \cos^2(t) = 1$$

$$\therefore \left(\frac{x}{2}\right)^2 + y^2 = 1$$

$$\frac{x^2}{4} + y^2 = 1$$

Now $0 \leq t \leq \pi$

When $t = 0$, $x = 0$

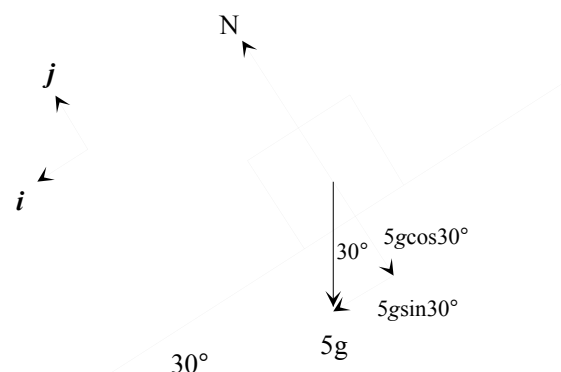
When $t = \frac{\pi}{2}$, $x = 2$

When $t = \pi$, $x = 0$

$$\therefore 0 \leq x \leq 2$$

$\therefore$ E

Question 15



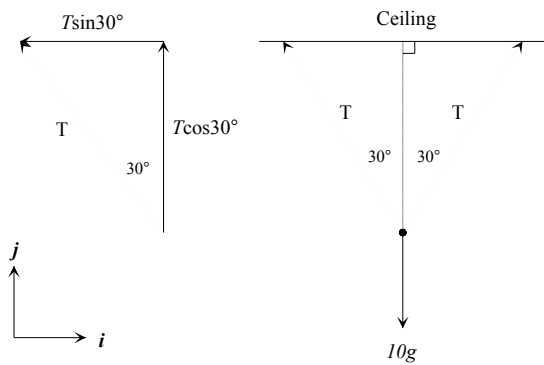
$$5g \sin 30\mathbf{i} + (N - 5g \cos 30)\mathbf{j} = 5a\mathbf{i}$$

$$5g \sin 30 = 5a$$

$$a = \frac{g}{2}$$

$\therefore$ B

Question 16



$$(T \sin 30 - T \sin 30)\mathbf{i} + (T \cos 30^\circ + T \cos 30^\circ - 10g)\mathbf{j} = 0$$

$$2T \cos 30 - 10g = 0$$

$$\frac{2T\sqrt{3}}{2} = 10g$$

$$T = \frac{10g}{\sqrt{3}}$$

$$= \frac{10g\sqrt{3}}{3}$$

$\therefore$ D

Question 17

$$\sum \mathbf{F} = 5\mathbf{a}$$

$$(2\mathbf{i} + \mathbf{j}) + (\mathbf{i} + 10\mathbf{j}) + (3\mathbf{i} - 3\mathbf{j}) = 5\mathbf{a}$$

$$6\mathbf{i} + 8\mathbf{j} = 5\mathbf{a}$$

$$|\mathbf{a}| = \frac{\sqrt{6^2 + 8^2}}{5}$$

$$= 2$$

$\therefore$ A

Question 18

$$u = -21, t = 10, a = 9.8$$

$$s = ut + \frac{1}{2}at^2$$

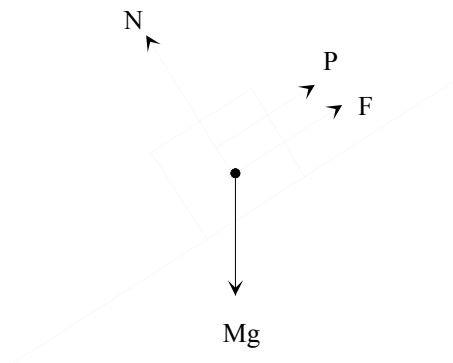
$$s = -21 \times 10 + \frac{1}{2} \times 9.8 \times 10^2$$

$$= 280$$

$$\therefore h = 280 \text{ m}$$

$\therefore$ B

Question 19



As the body is on the point of sliding down the plane, the friction force acts up the plane.

$\therefore$ A

Question 20

A. $\frac{dy}{dx} = \frac{-1}{x^2}$

B. $\frac{dy}{dx} = \frac{-1}{x}$

C. $\frac{dy}{dx} = -e^{-x}$

D. $\frac{dy}{dx} = \frac{-2}{x^3}$

E. $\frac{dy}{dx} = \frac{1}{x}$

As the slope field has the form of a hyperbola in the form $y = \frac{a}{x}$, for $a > 0$, this corresponds to answer **E**

Question 21

$$\frac{dy}{dt} = -k(y - 4), \quad t = 0, y = 20$$

$\therefore$ E

Question 22

$$\frac{dy}{dx} = \cos\left(\frac{x}{2}\right)$$

$$\begin{aligned} y_1 &= y_0 + hf'(x_0) \\ &= 2 + 0.2 \cos(0) \\ &= 2.2 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + hf'(x_1) \\ &= 2.2 + 0.2 f'(0.2) \\ &= 2.2 + 0.2 \cos\left(\frac{0.2}{2}\right) \\ &= 2.2 + 0.2 \cos(0.1) \end{aligned}$$

$\therefore$ C

VCAA Sample Questions 2006 – Specialist Mathematics

Written Examination 2 Section 2

Suggested answers and solutions

Question 1

a. $\vec{r} \cdot \vec{s}$

$$\begin{aligned} &= (2 - 2\cos(t))(\sin(t)) + (1 + \sin(t))(2\cos(t)) \\ &= 2\sin(t) - 2\cos(t)\sin(t) + 2\cos(t) + 2\sin(t)\cos(t) \\ &= 2\sin(t) + 2\cos(t) \end{aligned}$$

The particles will be at right angles to each other when $\vec{r} \cdot \vec{s} = 0$

Hence,

$$\begin{aligned} 2\sin(t) + 2\cos(t) &= 0 \\ 2\sin(t) &= -2\cos(t) \\ \tan(t) &= -1 \end{aligned}$$

$$\therefore t = \frac{3\pi}{4}$$

b.

$$\begin{aligned} x &= 2 - 2\cos(t) & y &= 1 + \sin(t) \\ \cos(t) &= \frac{2-x}{2} & \sin(t) &= 1-y \end{aligned}$$

$$\cos^2(t) + \sin^2(t) = 1$$

$$\left(\frac{2-x}{2}\right)^2 + (1-y)^2 = 1$$

$$\frac{(x-2)^2}{4} + (y-1)^2 = 1$$

Now $t \geq 0 \therefore x \geq 0$

$$\therefore \frac{(x-2)^2}{4} + (y-1)^2 = 1, x \geq 0$$

c. Velocity of particle R:

$$\dot{\vec{r}} = 2\sin(t)\vec{i} + \cos(t)\vec{j}$$

Speed of particle R:

$$\begin{aligned} |\dot{\vec{r}}| &= \sqrt{(2\sin(t))^2 + (\cos(t))^2} \\ &= \sqrt{4\sin^2(t) + \cos^2(t)} \end{aligned}$$

Velocity of particle S:

$$\dot{\vec{s}} = \cos(t)\vec{i} - 2\sin(t)\vec{j}$$

Speed of particle S:

$$\begin{aligned} |\dot{\vec{s}}| &= \sqrt{(\cos(t))^2 + (2\sin(t))^2} \\ &= \sqrt{\cos^2(t) + 4\sin^2(t)} \end{aligned}$$

Therefore particles R and S move at the same speed at any given time

Velocities will be equal when

$$2\sin(t)\vec{i} + \cos(t)\vec{j} = \cos(t)\vec{i} - 2\sin(t)\vec{j}$$

Therefore:

$$2\sin(t) = \cos(t) \quad \text{and} \quad \cos(t) = -2\sin(t)$$

$$\tan(t) = \frac{1}{2} \quad \tan(t) = \frac{-1}{2}$$

As $\tan(t)$ cannot equal both $\frac{1}{2}$ and $\frac{-1}{2}$

simultaneously, the particles can never have the same velocity.

Question 2

a. i. $u = 0, s = 400, t = 8$

$$s = ut + \frac{1}{2}at^2$$

$$400 = 0 \times 8 + \frac{1}{2} \times a \times 64$$

$$400 = 32a$$

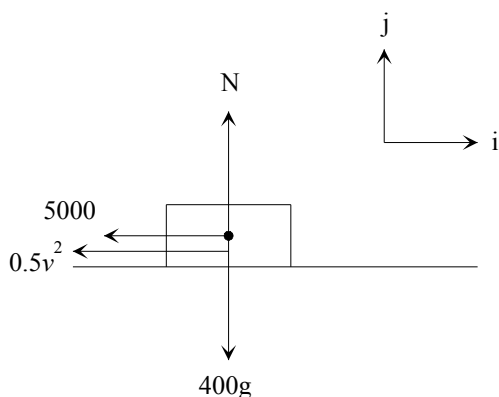
$$a = 12.5 \text{ m/s}^2$$

ii. $v = u + at$

$$v = 0 + 12.5 \times 8$$

$$= 100 \text{ m/s}$$

b. i.



$$\sum F = ma$$

$$(-5000 - 0.5v^2)\underline{i} + (N - 400g)\underline{j} = 400a\underline{j}$$

$$\therefore 400a = -5000 - 0.5v^2$$

$$a = \frac{-5000 - 0.5v^2}{400}$$

ii.

$$a = v \frac{dv}{dx}$$

$$v \frac{dv}{dx} = \frac{-5000 - 0.5v^2}{400}$$

$$= \frac{-(10000 + v^2)}{800}$$

$$\frac{dv}{dx} = -\frac{(10^4 + v^2)}{800v}$$

iii.

$$\frac{dv}{dx} = -\frac{(10^4 + v^2)}{800v}$$

$$\frac{dx}{dv} = -\frac{800v}{(10^4 + v^2)}$$

When $x = 0, v = 100$ (from a. ii.)

$$x = \int_{100}^0 -\frac{800v}{(10^4 + v^2)} dv$$

Let $u = 10^4 + v^2$

When $v = 100$ $u = 2 \times 10^4$

$v = 0$ $u = 10^4$

$$\frac{du}{dv} = 2v$$

$$dv = \frac{du}{2v}$$

$$x = \int_{2 \times 10^4}^{10^4} -\frac{800v}{u} \times \frac{du}{2v}$$

$$= \int_{2 \times 10^4}^{10^4} -\frac{400}{u} du$$

$$= \int_{10^4}^{2 \times 10^4} \frac{400}{u} du$$

$$= 400 [\log_e u]_{10^4}^{2 \times 10^4}$$

$$= 400 [\log_e (2 \times 10^4) - \log_e (10^4)]$$

$$= 400 \log_e (2)$$

$$= 277 \text{ m}$$

c.

$$a = -\frac{(10^4 + v^2)}{800}$$

$$\frac{dv}{dT} = -\frac{(10^4 + v^2)}{800}$$

$$\frac{dT}{dv} = -\frac{800}{(10^4 + v^2)}$$

$$T = -800 \int_{100}^0 \frac{1}{10^4 + v^2} dv$$

$$= 800 \int_0^{100} \frac{1}{10^4 + v^2} dv$$

$$= 6.28 \text{ s}$$

```

Plot1 Plot2 Plot3
\Y1=800/(10000+X
2)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=

```

```

WINDOW
Xmin=-5
Xmax=110
Xscl=20
Ymin=-.01
Ymax=.09
Yscl=.05
Xres=1

```



Alternatively:

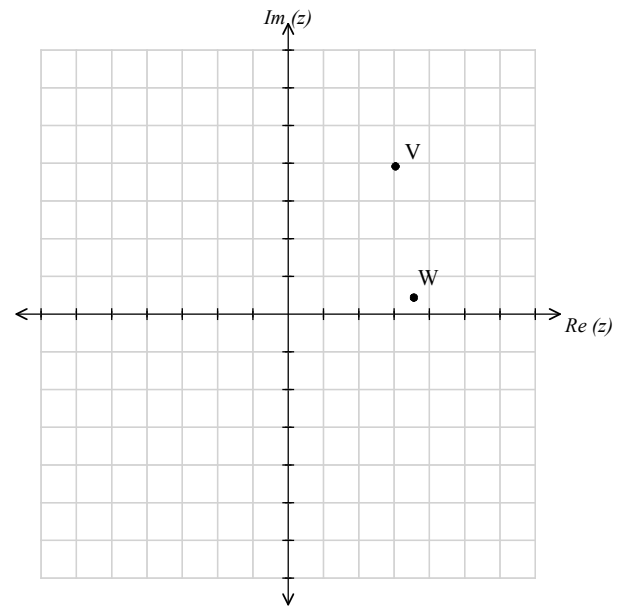
```

fnInt(800/(10000
+X^2),X,0,100)
6.283185307

```

Question 3

a.



b.

i.

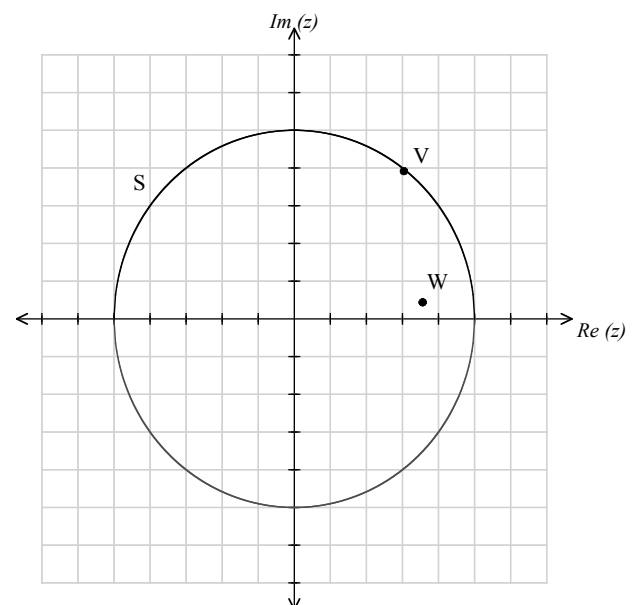
$$|v| = \sqrt{6^2 + 8^2}$$

$$= \sqrt{100}$$

$$= 10$$

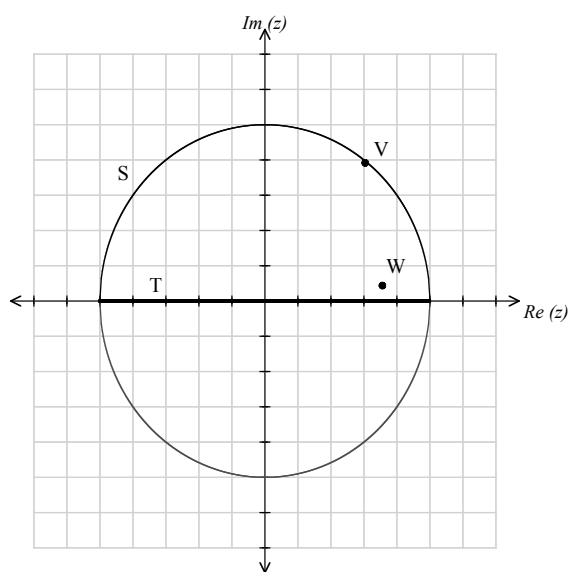
$$\therefore v \in S$$

ii.



$$\begin{aligned} \text{c.} \quad u + i\bar{w} &= \bar{w} \\ u &= \bar{w} + i\bar{w} \\ &= 7 - i - 7i - 1 \\ u &= 6 - 8i \end{aligned}$$

$$\begin{aligned} \text{d.} \quad |z - u| &= |z - v| \\ |x + iy - 6 + 8i| &= |x + iy - 6 - 8i| \\ \sqrt{(x-6)^2 + (y+8)^2} &= \sqrt{(x-6)^2 + (y-8)^2} \\ (y+8)^2 &= (y-8)^2 \\ \therefore y &= 0 \end{aligned}$$



$$\begin{aligned} \text{e.} \quad \overrightarrow{OV} &= 6\hat{i} + 8\hat{j} \\ \overrightarrow{OW} &= 7\hat{i} + \hat{j} \\ \therefore \overrightarrow{WO} &= -7\hat{i} - \hat{j} \\ \overrightarrow{WV} &= \overrightarrow{WO} + \overrightarrow{OV} \\ &= -\hat{i} + 7\hat{j} \end{aligned}$$

$$\begin{aligned} \overrightarrow{WO} \cdot \overrightarrow{WV} &= |\overrightarrow{WO}| |\overrightarrow{WV}| \cos \theta \quad \text{where } \theta \text{ is } \angle OWV \\ (-7)(-1) + (-1)(7) &= \sqrt{49+1} \times \sqrt{49+1} \cos \theta \\ 0 &= 2\sqrt{50} \cos \theta \\ \cos \theta &= 0 \\ \therefore \theta &= 90^\circ \end{aligned}$$

$\therefore \angle OWV$ is a right angle.

Question 4

$$\begin{aligned} \text{a.} \quad x &\geq 0, \quad 1 - x^2 > 0 \\ -1 &< x < 1 \\ \therefore x &\in [0, 1) \end{aligned}$$

$$\begin{aligned} \text{b.} \quad D &= 2 \times f(0.5) \\ &= 2 \times 2(0.5)^{\frac{1}{2}} (1 - 0.5^2)^{\frac{1}{4}} + \frac{1}{(1 - 0.5^2)^{\frac{1}{4}}} \\ &= 3.71 \end{aligned}$$

$$\text{c.} \quad \tan \theta = f'(0.5) \text{ where } \theta \text{ is the angle } f(x) \text{ makes with the } x\text{-axis.}$$

Note that the angle the platform makes with the surface is $\frac{\pi}{2} - \theta$.

```
Plot1 Plot2 Plot3
Y1=2X^.5*(1-X^2)
^.25+1/(1-X^2)^.2
5
Y2=
Y3=
Y4=
Y5=
```

```
nDeriv(Y1,X,.5
1.235573158
```

$$f'(0.5) = 1.23557$$

$$\tan \theta = 1.23557$$

$$\theta = 51.015^\circ$$

So, the angle the platform makes with the surface is 39° .

d. i.

$$(f(x))^2 = \left(2x^{\frac{1}{2}}(1-x^2)^{\frac{1}{4}} + \frac{1}{(1-x^2)^{\frac{1}{4}}} \right)^2$$

The middle term is given by

$$\begin{aligned} 2 \times 2x^{\frac{1}{2}}(1-x^2)^{\frac{1}{4}} \times \frac{1}{(1-x^2)^{\frac{1}{4}}} \\ = 4x^{\frac{1}{2}} \\ = 4\sqrt{x} \end{aligned}$$

ii. $V = \pi \int y^2 dx$

$$= \pi \int_0^{0.5} \left(4x\sqrt{1-x^2} + 4\sqrt{x} + \frac{1}{\sqrt{1-x^2}} \right) dx$$

iii. The volume of the platform is $1.9\pi \text{ m}^3$

Therefore 7 packages are needed.

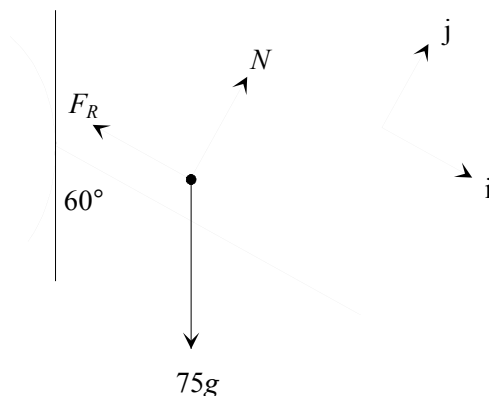
Alternative solution:

$$\begin{aligned} V_x &= \pi \int y^2 dx \\ &= \pi \int_0^{\frac{1}{2}} \left(4x\sqrt{1-x^2} + \frac{1}{\sqrt{1-x^2}} + 4\sqrt{x} \right) dx \\ &= \pi \left[-\frac{4}{3}(1-x^2)^{\frac{3}{2}} + \sin^{-1}(x) + \frac{8}{3}x^{\frac{3}{2}} \right]_0^{\frac{1}{2}} \\ &= \pi \left[\frac{4}{3} + \frac{\pi}{6} + \frac{2\sqrt{2}}{3} - \frac{\sqrt{3}}{2} \right] \\ &= 6.075 = 6.1 \text{ m}^3 \end{aligned}$$

so they need 7 packages.

Question 5

a.



b. $\sum F = ma$
 $(75g \sin 30^\circ - F_R)\underline{i} + (N - 75g \cos 30^\circ)\underline{j} = 75a\underline{i}$

$$75g \sin 30^\circ - F_R = 75a \quad (1)$$

$$N = 75g \cos 30^\circ \quad (2)$$

$$\begin{aligned} F_R &= \mu N \\ &= \frac{1}{5} \times 75g \times \frac{\sqrt{3}}{2} \end{aligned}$$

Sub in (1)

$$75a = 75g \times \frac{1}{2} - \frac{1}{5} \times 75g \times \frac{\sqrt{3}}{2}$$

$$a = \frac{g}{2} - \frac{1}{5} \times g \times \frac{\sqrt{3}}{2}$$

$$a = \frac{g}{2} \left(1 - \frac{\sqrt{3}}{5} \right)$$

c. i. $u = 0, a = \frac{g}{2} \left(1 - \frac{\sqrt{3}}{5} \right), s = 6$

$$s = ut + \frac{1}{2}at^2$$

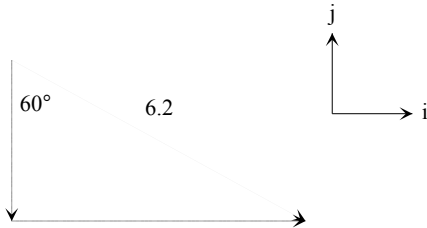
$$6 = \frac{1}{2} \times \frac{g}{2} \left(1 - \frac{\sqrt{3}}{5} \right) t^2$$

$$t = 1.94 \text{ s}$$

ii.

$$\begin{aligned}v^2 &= u^2 + 2as \\&= 0 + 2 \times \frac{g}{2} \left(1 - \frac{\sqrt{3}}{5} \right) \times 6 \\v^2 &= 38.431 \\v &= 6.2 \text{ m/s}\end{aligned}$$

d.



$$\begin{aligned}\underline{v}(t) &= 6.2 \sin 60^\circ \underline{i} - 6.2 \cos 60^\circ \underline{j} - g t \underline{j} \\&= 3.1\sqrt{3}\underline{i} - (3.1 + gt)\underline{j} \\ \underline{r}(t) &= 3.1\sqrt{3}t\underline{i} - \left(3.1t + \frac{gt^2}{2} \right) \underline{j} + \underline{c}\end{aligned}$$

Jay hits the ground when

$$\begin{aligned}3.1t + \frac{gt^2}{2} &= 2 \\4.9t^2 + 3.1t - 2 &= 0 \\t &= 0.3966 \text{ s}\end{aligned}$$

Therefore the horizontal distance is given by

$$\begin{aligned}3.1\sqrt{3} \times 0.3966 \\&= 2.1 \text{ m}\end{aligned}$$

$$\begin{aligned}\text{e. } \underline{v}(t) &= 3.1\sqrt{3}\underline{i} - (3.1 + gt)\underline{j} \\ \underline{v}(0.3966) &= 3.1\sqrt{3}\underline{i} - (3.1 + 9.8 \times 0.3966)\underline{j} \\&= 5.369\underline{i} - 6.987\underline{j} \\|\underline{v}| &= 8.812 \\p &= mv \\p &= 75 \times 8.812 \\&= 661 \text{ kg m/s}\end{aligned}$$