

SPECIALIST MATHEMATICS EXAM 2: SOLUTIONS**Section 1: Multiple Choice****Answers**

- | | | | | |
|-------|-------|-------|-------|-------|
| 1. A | 2. E | 3. D | 4. B | 5. A |
| 6. C | 7. C | 8. A | 9. E | 10. B |
| 11. E | 12. E | 13. C | 14. B | 15. D |
| 16. B | 17. A | 18. C | 19. B | 20. E |
| 21. D | 22. D | | | |

Question 1**Answer A**

$$(x+1)^2 - (y+2)^2 = 1$$

Hyperbola of the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ where $a = 1$, $b = 1$, $h = -1$, $k = -2$

Asymptotes have equations $y = \pm \frac{b}{a}(x-h) + k$, $\Rightarrow y = \pm 1(x+1) - 2$

x-intercepts occur where $y = 0$

$$0 = \pm(x+1) - 2$$

$$\begin{array}{ll} -x-1=2 & \text{or} \quad x+1=2 \\ x=-3 & \text{or} \quad x=1 \end{array}$$

Question 2**Answer E**

$$u + \bar{v} = (-1+i) + (1+i)$$

$$= 2i \quad \text{not real}$$

$$u - v = (-1+i) - (1-i)$$

$$= -2 + 2i \quad \text{not real}$$

$$uv = (-1+i)(1-i)$$

$$= -1 + i + i - i^2$$

$$= 2i \quad \text{not real}$$

$$\frac{1}{v} = \frac{1}{1-i}$$

$$= \frac{1}{1-i} \times \frac{1+i}{1+i}$$

$$= \frac{1+i}{2} \quad \text{not real}$$

$$\frac{u}{v} = \frac{-1+i}{1-i}$$

$$= \frac{(-1+i)(1+i)}{(1-i)(1+i)}$$

$$= \frac{-1-i+i+i^2}{1-i^2}$$

$$= \frac{-2}{2}$$

$$= -1 \quad \text{real number}$$

Question 3 **Answer D**

The roots of a complex number are equally spaced around the circumference of a circle. The angle between each fourth root will be $\frac{2\pi}{4} = \frac{\pi}{2}$.

Given the argument of one root is $\frac{\pi}{3}$, the arguments of the three other roots will be:

$$\theta = \frac{\pi}{3} + \frac{2k\pi}{4} \text{ where } k = 1, -1, -2$$

$$\begin{aligned} \theta &= \frac{\pi}{3} + \frac{\pi}{2} & \theta &= \frac{\pi}{3} - \frac{\pi}{2} & \theta &= \frac{\pi}{3} - \pi \\ &= \frac{5\pi}{6} & &= -\frac{\pi}{6} & &= -\frac{2\pi}{3} \end{aligned}$$

The modulus of each root must be the same, therefore $r = 16$

The other roots of z are $16 \operatorname{cis}\left(\frac{5\pi}{6}\right)$, $16 \operatorname{cis}\left(-\frac{\pi}{6}\right)$ and $16 \operatorname{cis}\left(-\frac{2\pi}{3}\right)$.

Question 4 **Answer B**

Equation is $(\bar{z} - i)(z + i) = 4$

$$(x - iy - i)(x + iy + i) = 4$$

$$(x - i(y + 1))(x + i(y + 1)) = 4$$

$$x^2 - i^2(y + 1)^2 = 4$$

$$x^2 + (y + 1)^2 = 4$$

Circle centre $(0, -1)$, radius 2

Question 5 **Answer A**

$f(x) = \arccos(x)$ has domain $[-1, 1]$

This is translated right m units and upwards n units to give $f(x) = \arccos(x - m) + n$

Since $-1 \leq x - m \leq 1$

then $m - 1 \leq x \leq m + 1$

The domain is $[m - 1, m + 1]$

Question 6 **Answer C**

Graph is of the form $a \operatorname{cosec}(nx) + k$

$$\begin{aligned} \text{Period is } \pi : \quad \frac{2\pi}{n} &= \pi \\ n &= 2 \end{aligned}$$

$$\text{minimum at } \left(\frac{\pi}{4}, 4\right), \text{ maximum at } \left(\frac{3\pi}{4}, 0\right)$$

Graph has been translated 2 units vertically, $k = 2$

$$a = 2$$

Equation of curve is $y = 2 \operatorname{cosec}(2x) + 2$

Question 7 **Answer C**

Graphs of the form $f(x) = x^m + x^{-n}$ where $m \in \{1, 2\}$ will have a vertical asymptote at $x = 0$.

Graphs A, B, D and E have vertical asymptote at $x = 0$, but graph C does not.

Question 8**Answer A**

$x = 2\sin(t)$ is translated 1 unit in the negative x direction $\Rightarrow x = 2\sin(t) - 1 \dots(1)$

$y = \cos(t)$ is translated 2 units in the positive y direction $\Rightarrow y = \cos(t) + 2 \dots(2)$

From (1) and (2)

$$\sin(t) = \frac{x+1}{2} \quad \text{and} \quad \cos(t) = y - 2$$

$$\Rightarrow \sin^2(t) = \frac{(x+1)^2}{4} \dots(3) \quad \Rightarrow \cos^2(t) = (y-2)^2 \dots(4)$$

Add (3) and (4)

$$\frac{(x+1)^2}{4} + (y-2)^2 = \sin^2(t) + \cos^2(t)$$

$$\frac{(x+1)^2}{4} + (y-2)^2 = 1$$

$$(x+1)^2 + 4(y-2)^2 = 4$$

Question 9**Answer E**

$$x = 2\sqrt{t+1}$$

$$y = t^2 + 1$$

$$\frac{dx}{dt} = 2 \times \frac{1}{2}(t+1)^{-\frac{1}{2}} \times 1$$

$$\frac{dy}{dt} = 2t$$

$$\frac{dx}{dt} = \frac{1}{\sqrt{t+1}}$$

$$\frac{dt}{dx} = \sqrt{t+1}$$

Using the chain rule $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$$\frac{dy}{dx} = 2t \times \sqrt{t+1}$$

When $t = 3$ $\frac{dy}{dx} = 2 \times 3 \times \sqrt{3+1}$

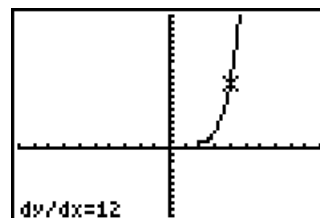
$$\frac{dy}{dx} = 12$$

Can also be done on the calculator using parametric mode

```

Plot1 Plot2 Plot3
X1T 2√(T+1)
Y1T 1+T²
X2T =
Y2T =
X3T =
Y3T =
X4T =

```



Question 10**Answer B**

$$2y - x^2 e^y + 1 = 0$$

Using implicit differentiation

$$2 \frac{dy}{dx} - \left(2x e^y + x^2 \frac{dy}{dx} e^y \right) + 0 = 0$$

$$2 \frac{dy}{dx} - x^2 \frac{dy}{dx} e^y = 2x e^y$$

$$\frac{dy}{dx} (2 - x^2 e^y) = 2x e^y$$

$$\frac{dy}{dx} = \frac{2x e^y}{2 - x^2 e^y}$$

Finding the gradient of the tangent at the point (1, 0)

$$\frac{dy}{dx} = \frac{2 \times 1 \times e^0}{2 - 1^2 e^0} = \frac{2}{1} = 2$$

Equation of tangent

$$y - 0 = 2(x - 1)$$

$$y - 2x + 2 = 0$$

Question 11**Answer E**

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^3(2x) dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos(2x) (1 - \sin^2(2x)) dx \quad \text{Let } u = \sin(2x) \Rightarrow \frac{du}{dx} = 2 \cos(2x)$$

$$= \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 - \sin^2(2x)) 2 \cos(2x) dx$$

Terminals:

$$\text{When } x = \frac{\pi}{2}, \quad u = \sin\left(2 \times \frac{\pi}{2}\right) \\ u = 0$$

$$\text{When } x = \frac{\pi}{4}, \quad u = \sin\left(2 \times \frac{\pi}{4}\right) \\ u = 1$$

$$= \frac{1}{2} \int_1^0 (1 - u^2) du$$

$$= -\frac{1}{2} \int_0^1 (1 - u^2) du$$

$$= \frac{1}{2} \int_0^1 (u^2 - 1) du$$

Question 12**Answer E**Rotating shaded region is around the x -axis

$$V = \pi \int_0^2 (y_1)^2 dx - \pi \int_0^2 (y_2)^2 dx \quad \text{where } y_1 = 1 \text{ and } y_2 = e^{x-2}$$

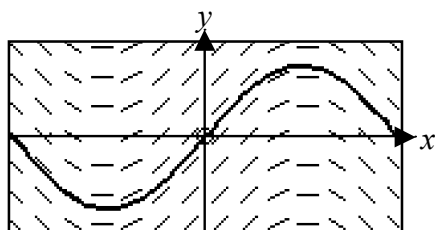
$$V = \pi \int_0^2 1^2 dx - \pi \int_0^2 (e^{x-2})^2 dx$$

$$V = \pi [x]_0^2 - \pi \int_0^2 (e^{x-2})^2 dx$$

$$V = \pi [2 - 0] - \pi \int_0^2 (e^{x-2}) (e^{x-2}) dx$$

$$V = 2\pi - \pi \int_0^2 e^{2x-4} dx$$

$$V = 2\pi - \pi \int_0^2 e^{2(x-2)} dx$$

Question 13**Answer C**

$$y = \sin(x)$$

Question 14**Answer B**

$$y = \int \frac{1}{\log_e(x+1)} dx$$

$$= f(x) + c$$

When $x = 1, y = -2$

$$-2 = f(1) + c$$

$$c = -f(1) - 2 \quad (1)$$

When $x = 2, y = f(2) + c \dots \quad (2)$

Substitute (1) into (2)

$$y = f(2) - f(1) - 2$$

$$y = \int_1^2 \frac{1}{\log_e(x+1)} dx - 2$$

Numerical solution using calculator

```
fnInt(1/ln(X+1),
X,1,2)-2
-.8815751855
```

Question 15**Answer D**

$$a = v \frac{dv}{dx}$$

$$= \frac{v^2 + 1}{8}$$

$$\frac{dv}{dx} = \frac{v^2 + 1}{8v}$$

$$\frac{dx}{dv} = \frac{8v}{v^2 + 1}$$

$$x = \int \frac{8v}{v^2 + 1} dv$$

Integral can be evaluated numerically on calculator

$$x = 4 \int \frac{2v}{v^2 + 1} dv$$

$$x = 4 \log_e(v^2 + 1) + C$$

$$\text{When } x = 0 \quad v = 1 \quad \Rightarrow \quad 0 = 4 \log_e(1^2 + 1) + C$$

$$C = -4 \log_e 2$$

$$x = 4 \log_e(v^2 + 1) - 4 \log_e 2$$

$$x = 4 \log_e \left(\frac{v^2 + 1}{2} \right)$$

$$\text{When } v = 3 \quad x = 4 \log_e \left(\frac{3^2 + 1}{2} \right)$$

$$x = 4 \log_e 5$$

$$x = 6.4 \text{ metres}$$

```
fnInt(8X/(X^2+1),
X,1,3)+0
6.43775165
```

Question 16**Answer B**

Area of trapezium above horizontal axis

$$A = \frac{1}{2}(10 + 20) \times 20$$

$$= 300$$

Area of trapezium below horizontal axis

$$A = \frac{1}{2}(40 + 10) \times -10$$

$$= -250$$

After 60 seconds the particle is $300 - 250 = 50$ metres from its starting point.

Question 17**Answer A**

Toy is travelling under constant acceleration

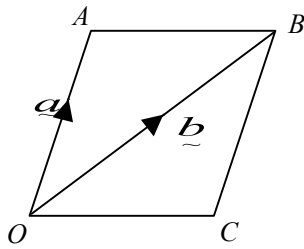
$$u = 4, s = 3, t = 1.2$$

$$s = \frac{1}{2}(u + v)t$$

$$3 = \frac{1}{2}(4 + v) \times 1.2$$

$$\frac{3}{0.6} = 4 + v$$

$$v = 1 \text{ m/s}$$

Question 18**Answer C**

The diagonals of a rhombus are perpendicular, therefore $\vec{AC} \cdot \vec{OB} = 0$

Find a vector expression for $\vec{AC}$ given $\vec{OA} = \underline{a}$ and $\vec{OB} = \underline{b}$.

$$\begin{aligned}\vec{AC} &= \vec{AO} + \vec{OC} \\ &= -\underline{a} + (\underline{b} - \underline{a}) \\ &= \underline{b} - 2\underline{a}\end{aligned}$$

$$\vec{AC} \cdot \vec{OB} = (\underline{b} - 2\underline{a}) \cdot \underline{b}$$

$$\therefore (\underline{b} - 2\underline{a}) \cdot \underline{b} = 0$$

Question 19**Answer B**

$$\begin{aligned}\underline{p} \cdot \hat{\underline{q}} &= \frac{(\underline{i} + 0\underline{j} + 2\underline{k}) \cdot (5\underline{i} + \underline{j} - \underline{k})}{\sqrt{5^2 + 1^2 + 1^2}} \\ &= \frac{5 + 0 - 2}{\sqrt{27}} \\ &= \frac{3}{3\sqrt{3}} \\ &= \frac{\sqrt{3}}{3}\end{aligned}$$

Question 20**Answer E**

$$\underline{F} = m \underline{a}$$

$$(3\underline{i} - 2\underline{j}) + (\underline{i} + 5\underline{j}) = 0.5\underline{a}$$

$$4\underline{i} + 3\underline{j} = 0.5\underline{a}$$

$$\underline{a} = \frac{4\underline{i} + 3\underline{j}}{0.5}$$

$$\underline{a} = 8\underline{i} + 6\underline{j}$$

Question 21**Answer D**

The forces acting on the decoration are in equilibrium.

Therefore the resultant force is zero.

$$\underline{T} + \underline{P} + \underline{W} = \underline{0}$$

$$\underline{T} + \underline{P} = -\underline{W}$$

Question 22**Answer D**

Show forces on diagram

Let T be the tension in the string and a the acceleration of the system

Resolve 3 kg mass horizontally:

$$T = 3a$$

$$a = \frac{T}{3} \dots (1)$$

System is moving downwards

Resolve 4 kg mass vertically:

$$4g - T = 4a \dots (2)$$

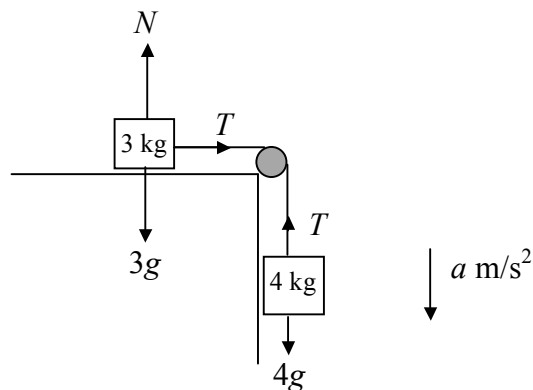
Substitute (1) into (2)

$$4g - T = 4\left(\frac{T}{3}\right)$$

$$4g = \frac{4T}{3} + T$$

$$4g = \frac{7T}{3}$$

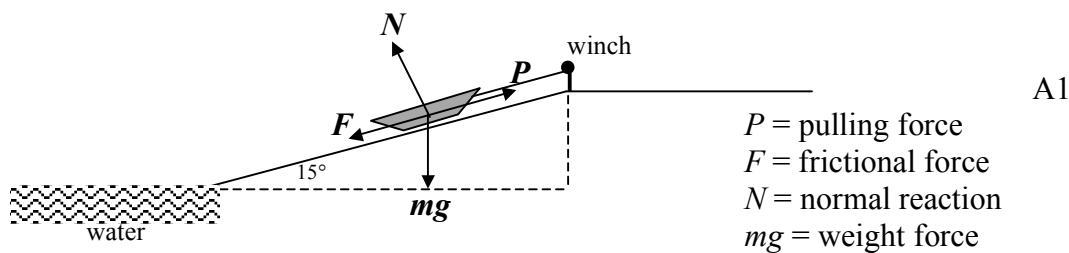
$$T = \frac{12g}{7}$$



Section 2: Extended Answer

Question 1

a.



A1

b. Resolving forces perpendicular to the ramp

$$N = mg \cos(15^\circ) \quad \dots(1)$$

$$\text{Friction } F = \mu N$$

$$\Rightarrow F = \mu mg \cos(15^\circ)$$

Boat is moving upwards

Resolving forces parallel to the ramp

$$P - mg \sin(15^\circ) - F = ma \quad \dots(2)$$

$$P - mg \sin(15^\circ) - \mu mg \cos(15^\circ) = ma$$

$$P - 1000g \sin(15^\circ) - 0.25 \times 1000g \cos(15^\circ) = 1000 \times 0.1$$

$$P = 100 + 1000g \sin(15^\circ) + 0.25 \times 1000g \cos(15^\circ)$$

$$P = 100 + 1000g \sin(15^\circ) + 250g \cos(15^\circ)$$

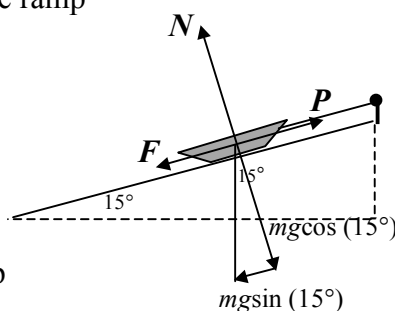
$$P = 100 + 2536.4 + 2366.5$$

$$P = 5003 \text{ newtons}$$

A1

M1

A1



c. Boat slides down the ramp under the force of gravity.

$$mg \sin(15^\circ) - F = ma$$

$$mg \sin(15^\circ) - \mu mg \cos(15^\circ) = ma$$

$$a = g \sin(15^\circ) - 0.25g \cos(15^\circ)$$

$$a = 0.17 \text{ m/s}^2$$

M1

A1

A1

d. $u = 0, \quad s = 6, \quad a = 0.17$

$$v^2 = u^2 + 2as$$

$$v^2 = 0^2 + 2 \times 0.17 \times 6$$

$$v = \sqrt{2.04}$$

$$v = 1.43 \text{ m/s}$$

M1

A1

e. $v = u + at$

$$1.43 = 0 + 0.17t$$

$$t = \frac{1.43}{0.17}$$

$$t = 8.4 \text{ seconds}$$

A1

Question 2

a. $\underline{v} = 20 \cos(\theta) \underline{i} + (20 \sin(\theta) - 9.8t) \underline{j}$

When $t = 0$, $\underline{v} = 20 \cos(\theta) \underline{i} + 20 \sin(\theta) \underline{j}$

$$|\underline{v}| = \sqrt{(20 \cos(\theta))^2 + (20 \sin(\theta))^2}$$

$$|\underline{v}| = 20 \sqrt{\cos^2(\theta) + \sin^2(\theta)}$$

$$|\underline{v}| = 20 \text{ m/s}$$

The initial speed of the stone is 20m/s

A1

b. $\underline{r} = \int (20 \cos(\theta) \underline{i} + (20 \sin(\theta) - 9.8t) \underline{j}) dt$

$$\underline{r} = 20t \cos(\theta) \underline{i} + (20t \sin(\theta) - 4.9t^2) \underline{j} + C$$

C is constant

A1

When $t = 0$, $\underline{r} = 0 \underline{i} + 50 \underline{j} \Rightarrow C = 50 \underline{j}$

$$\underline{r} = 20t \cos(\theta) \underline{i} + (20t \sin(\theta) - 4.9t^2) \underline{j} + 50 \underline{j}$$

$$\underline{r} = 20t \cos(\theta) \underline{i} + (20t \sin(\theta) - 4.9t^2 + 50) \underline{j}$$

A1

c. Given $\theta = 60^\circ$.

- i. Stone reaches its greatest height when the $\underline{j}$ component of the velocity is zero.

$$20 \sin(60^\circ) - 9.8t = 0$$

$$t = \frac{20 \sin 60^\circ}{9.8}$$

$$t = 1.8 \text{ seconds}$$

A1

Greatest height is given by the $\underline{j}$ component of $\underline{r}$ when $t = 1.8$ seconds

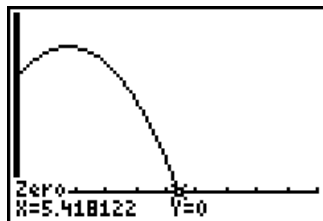
$$20 \times 1.8 \times \sin(60^\circ) - 4.9(1.8)^2 + 50 = 65.3 \text{ metres}$$

A1

- ii. Stone is at ground level when $\underline{j}$ component of $\underline{r}$ is zero

$$20t \sin(60^\circ) - 4.9t^2 + 50 = 0$$

Solve for t using calculator



$$t = 5.42 \text{ seconds}$$

M1

Distance from base of building is given by the $\underline{i}$ component of $\underline{r}$ when $t = 5.42$

$$20 \times 5.42 \times \cos(60^\circ) = 54.2 \text{ metres}$$

A1

- d. Landing place of stone $\underline{r} = 25\underline{i} + 30\underline{j}$

Solve $20t \cos(\theta)\underline{i} + (20t \sin(\theta) - 4.9t^2 + 50)\underline{j} = 25\underline{i} + 30\underline{j}$ to find θ M1

Equating $\underline{i}$ and $\underline{j}$ components

$$20t \cos(\theta) = 25$$

$$20t \sin(\theta) - 4.9t^2 + 50 = 30 \quad \dots(2)$$

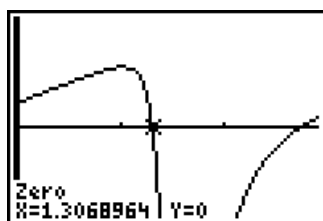
$$t = \frac{5}{4 \cos(\theta)} \quad \dots(1)$$

Substitute (1) into (2) to eliminate t from the equation

$$20 \left(\frac{5}{4 \cos(\theta)} \right) \sin(\theta) - 4.9 \left(\frac{5}{4 \cos(\theta)} \right)^2 + 50 = 30 \quad \text{A1}$$

$$25 \tan(\theta) - \frac{245}{32 \cos^2(\theta)} + 20 = 0$$

Solve for θ using calculator



$\theta = 1.31$ radian.

The angle of projection is 74.9° .

A1

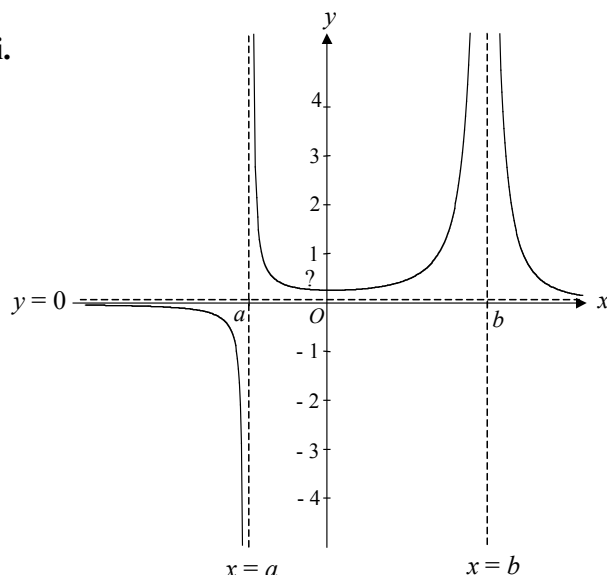
Question 3

- a. Curve is of the form $y = k(x-a)(x-b)^2$ where k is the dilation factor. M1
 $(0, 4)$ is a point on the curve $\Rightarrow 4 = k(0-a)(0-b)^2$

$$k = -\frac{4}{ab^2}$$

Equation of curve is $f(x) = -\frac{4}{ab^2}(x-a)(x-b)^2$ A1

- b. i.



Shape and y-intercept A1
 Asymptotes A1

- ii. Domain: $R \setminus \{a, b\}$, Range: $R \setminus \{0\}$ A1

- c. $f(x)$ is the derivative or gradient of $g(x)$
 At $x = a$ and $x = b$ the gradient is zero ($f(x) = 0$)
 $\therefore g(x)$ has stationary points at $x = a$ and $x = b$

When $x < a$ the gradient is negative ($f(x) < 0$)

When $a < x < b$ the gradient is positive ($f(x) > 0$)

$\therefore x = a$ is a minimum turning point on $g(x)$.

The y-coordinate of this point cannot be found accurately from graph of $f(x)$.

When $a < x < b$ the gradient is positive ($f(x) > 0$)

When $x < b$ the gradient is positive ($f(x) > 0$)

$\therefore x = b$ is a stationary point of inflexion on $g(x)$.

The y-coordinate of this point cannot be found accurately from graph of $f(x)$.

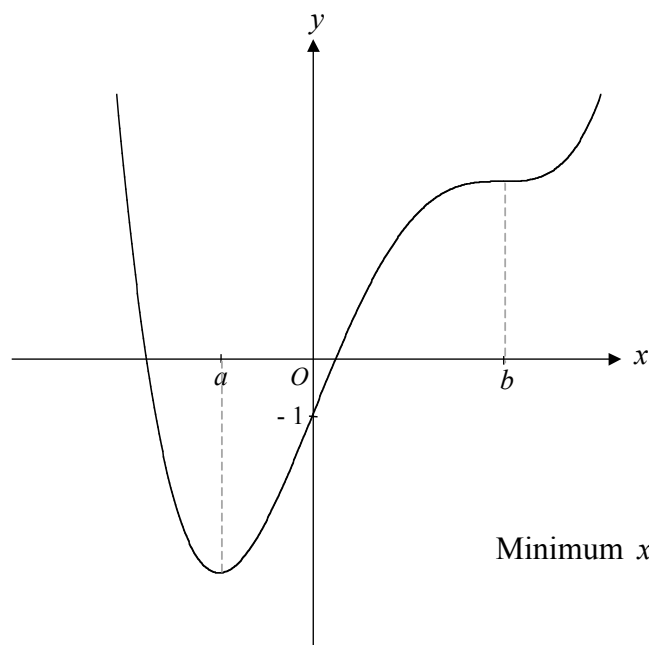
The graph of $f(x)$ shows a local maximum at the point $(0, 4)$.

This means $x = 0$ is a non stationary point of inflexion on $g(x)$.

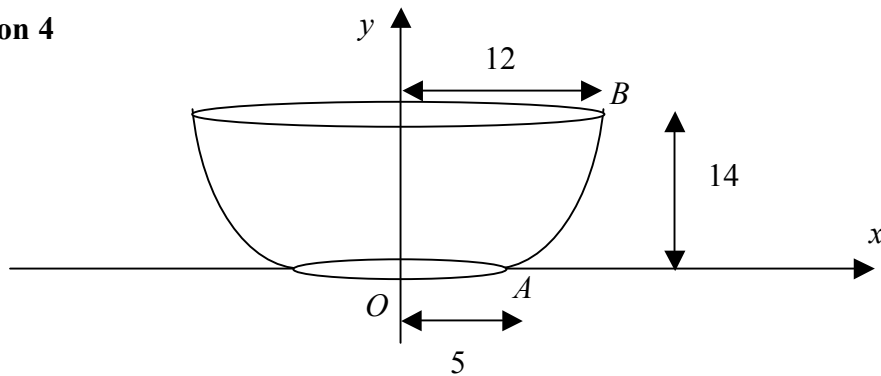
Given $g(0) = -1$, then the graph of $g(x)$ will pass through the point $(0, -1)$.

This is the point of maximum gradient over the interval $a < x < b$.

$\therefore$ The gradient of $g(x)$ is 4 at point $(0, -1)$. (It is not possible to show this accurately on the scale that is given.)



Shape A1
y-intercept A1
Minimum $x = a$, Inflexion $x = b$ A1

Question 4

a. $A(5,0)$ $B(12,14)$

Since the parabola is symmetrical about the y -axis $b = 0$

Substituting the points A and B into $y = ax^2 + c$

(1) $0 = 25a + c$

M1

(2) $14 = 144a + c$

Subtracting gives $119a = 14$ so that $a = \frac{14}{119} = \frac{2}{17}$

and $c = -25a$ $c = -\frac{50}{17}$ so that $y = \frac{2}{17}(x^2 - 25)$

A1

b. $V = \pi \int_a^b x^2 dy$ now $\frac{17y}{2} = x^2 - 25$ $x^2 = \frac{17y}{2} + 25 = \frac{1}{2}(17y + 50)$

$$V = \pi \int_0^{14} \frac{1}{2}(17y + 50) dy$$

M1

$$V = \frac{\pi}{2} \left[\frac{17y^2}{2} + 50y \right]_0^{14}$$

A1

$$V = \frac{\pi}{2} \left[\frac{17(14)^2}{2} + 50 \times 14 - 0 \right] = 1183\pi$$

$$V = 3716.5 \text{ cm}^3$$

A1

c. $V = \pi \int_0^h \frac{1}{2}(17y + 50) dy$

M1

$$V = \frac{\pi}{2} \left[\frac{17y^2}{2} + 50y \right]_0^h$$

$$V = \frac{\pi}{2} \left[\frac{17h^2}{2} + 50h - 0 \right]$$

$$V = \frac{\pi h}{4}(17h + 100) \quad \text{for } 0 \leq h \leq 14$$

A1

- d. Given $\frac{dV}{dt} = 2000 \text{ cm}^3/\text{min}$ we need to find $\frac{dh}{dt}$ when $h = 7$.

$$V = \frac{\pi h}{4}(17h + 100) = \frac{\pi}{4}(17h^2 + 100h)$$

$$\frac{dV}{dh} = \frac{\pi}{4}(34h + 100) = \frac{\pi}{2}(17h + 50) \text{ and by the chain rule} \quad \text{M1}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \frac{dV}{dt} = \frac{4000}{\pi(17h + 50)} \quad \text{A1}$$

$$\left. \frac{dh}{dt} \right|_{h=7} = \frac{4000}{\pi(17 \times 7 + 50)} = 7.53 \text{ cm/min} \quad \text{A1}$$

- e. $\frac{dV}{dt} = -28\sqrt{h} \text{ cm}^3/\text{min}$

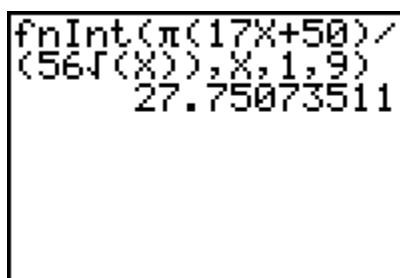
$$\frac{dh}{dt} = \frac{dh}{dV} \frac{dV}{dt} = \frac{-28\sqrt{h}}{\frac{\pi}{2}(17h + 50)} = \frac{-56\sqrt{h}}{\pi(17h + 50)} \text{ inverting}$$

$$\frac{dt}{dh} = \frac{\pi(17h + 50)}{-56\sqrt{h}} \text{ integrating with respect to } h \quad \text{M1}$$

$$t = \int_9^1 \frac{\pi(17h + 50)}{-56\sqrt{h}} dh = \int_1^9 \frac{\pi(17h + 50)}{56\sqrt{h}} dh \quad \text{A1}$$

- f. $= \int_1^9 \frac{\pi(17h + 50)}{56\sqrt{h}} dh = \frac{53\pi}{6} \approx 27.75$

The time for the height of the water to fall from 9 cm to 1 cm is 28 minutes



A1

Question 5

a. i. $P(-2, -1) \quad Q(-2 - 2\sqrt{3}, 5) \quad R(-2 + 2\sqrt{3}, 5) \quad C(-2, 3)$

$$\overrightarrow{OP} = -2\hat{i} - \hat{j} \quad \overrightarrow{OQ} = (-2 - 2\sqrt{3})\hat{i} + 5\hat{j}$$

$$\overrightarrow{OR} = (-2 + 2\sqrt{3})\hat{i} + 5\hat{j} \quad \overrightarrow{OC} = -2\hat{i} + 3\hat{j}$$

$$\overrightarrow{CP} = \overrightarrow{CO} + \overrightarrow{OP}$$

$$\overrightarrow{CQ} = \overrightarrow{CO} + \overrightarrow{OQ}$$

$$\overrightarrow{CP} = (2\hat{i} - 3\hat{j}) + (-2\hat{i} - \hat{j}) \quad \overrightarrow{CQ} = (2\hat{i} - 3\hat{j}) + (-2 - 2\sqrt{3})\hat{i} + 5\hat{j}$$

$$\overrightarrow{CP} = -4\hat{j}$$

$$\overrightarrow{CQ} = -2\sqrt{3}\hat{i} + 2\hat{j}$$

$$|\overrightarrow{CP}| = 4$$

$$|\overrightarrow{CQ}| = \sqrt{12 + 4} = 4$$

M1

$$\overrightarrow{CP} \cdot \overrightarrow{CQ} = -8$$

M1

$$\cos(\theta) = \frac{\overrightarrow{CP} \cdot \overrightarrow{CQ}}{|\overrightarrow{CP}| |\overrightarrow{CQ}|} = -\frac{8}{16} = -\frac{1}{2}$$

$$\theta = 120^\circ$$

A1

ii. $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$

$$\overrightarrow{PQ} = (2\hat{i} + \hat{j}) + (-2 - 2\sqrt{3})\hat{i} + 5\hat{j}$$

$$\overrightarrow{PQ} = -2\sqrt{3}\hat{i} + 6\hat{j}$$

$$|\overrightarrow{PQ}| = \sqrt{12 + 36}$$

$$|\overrightarrow{PQ}| = 4\sqrt{3}$$

$$\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR}$$

$$\overrightarrow{PR} = (2\hat{i} + \hat{j}) + (-2 + 2\sqrt{3})\hat{i} + 5\hat{j}$$

$$\overrightarrow{PR} = 2\sqrt{3}\hat{i} + 6\hat{j}$$

$$|\overrightarrow{PR}| = \sqrt{12 + 36} = 4\sqrt{3}$$

$$\overrightarrow{QR} = \overrightarrow{QO} + \overrightarrow{OR}$$

$$\overrightarrow{QR} = (2 + 2\sqrt{3})\hat{i} - 5\hat{j} + (-2 + 2\sqrt{3})\hat{i} + 5\hat{j}$$

$$\overrightarrow{QR} = 4\sqrt{3}\hat{i}$$

$$|\overrightarrow{QR}| = 4\sqrt{3}$$

M1

since $|\overrightarrow{PQ}| = |\overrightarrow{PR}| = |\overrightarrow{QR}|$ PQR forms an equilateral triangle.

A1

b. i. $S = \{ z : |z - c| = 4 \}$ where $c = -2 + 3i$ $|z - (-2 + 3i)| = 4$

When $z = p = -2 - i$

$$|(-2 - i) - (-2 + 3i)| = |-4i| = 4 \quad \text{A1}$$

When $z = q = -2 - 2\sqrt{3} + 5i$

$$|(-2 - 2\sqrt{3} + 5i) - (-2 + 3i)| = |-2\sqrt{3} + 2i| = \sqrt{12 + 4} = \sqrt{16} = 4 \quad \text{A1}$$

When $z = r = -2 + 2\sqrt{3} + 5i$

$$|(-2 + 2\sqrt{3} + 5i) - (-2 + 3i)| = |2\sqrt{3} + 2i| = \sqrt{12 + 4} = \sqrt{16} = 4 \quad \text{A1}$$

so that p , q and r all belong to S

ii. $|z - c| = 4$ $c = -2 + 3i$ $z = x + yi$

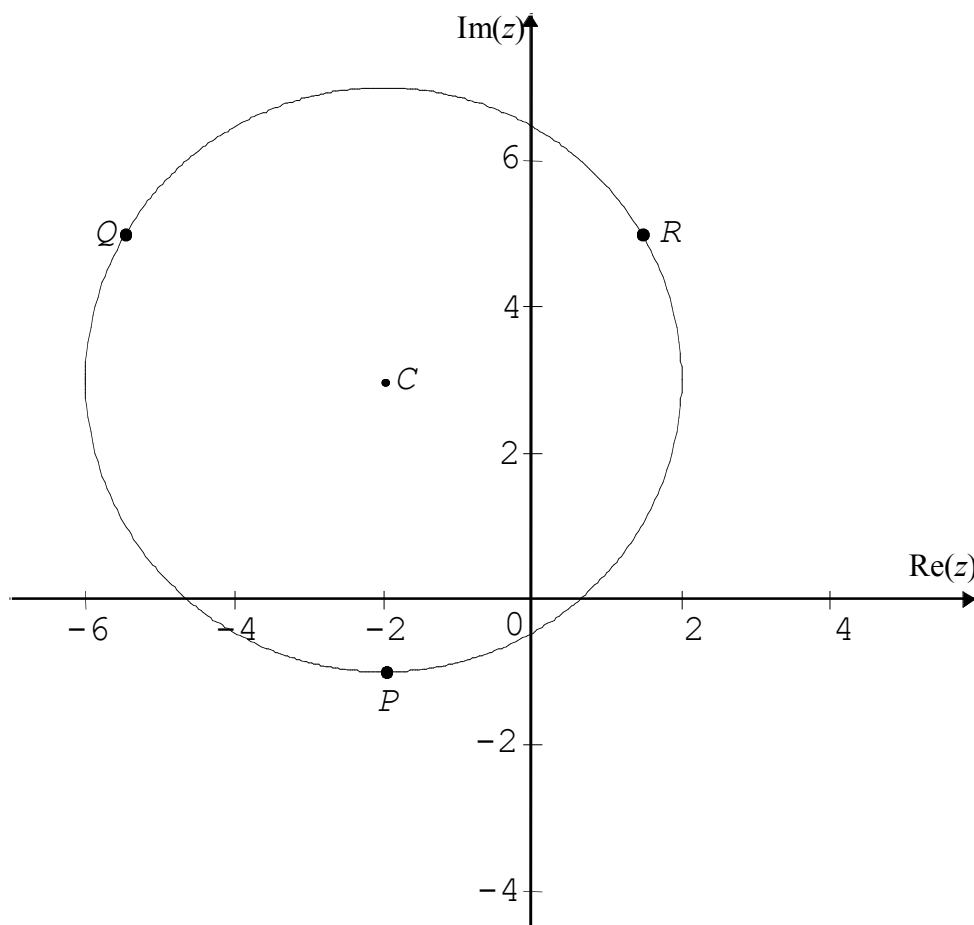
$$|(x + yi) - (-2 + 3i)| = 4$$

$$|(x + 2) + i(y - 3)| = 4$$

$$\sqrt{(x + 2)^2 + (y - 3)^2} = 4 \quad \text{squaring both sides}$$

$$(x + 2)^2 + (y - 3)^2 = 16$$

S is a circle centre at $C(-2, 3)$ radius 4 A1



For the graph circle and points.

A1

- c. i.** $(z - c)^3 = 64i$ expanding
 $z^3 - 3z^2c + 3zc^2 - c^3 = 64i$
 $z^3 - 3z^2(-2 + 3i) + 3z(-2 + 3i)^2 - (-2 + 3i)^3 = 64i$ A1
 $z^3 - 3z^2(-2 + 3i) + 3z(-5 - 12i) - (46 + 9i) = 64i$
 $z^3 + z^2(6 - 9i) - z(15 + 36i) - (46 + 9i) = 64i$
 $z^3 + (6 - 9i)z^2 - (15 + 36i)z - (46 + 73i) = 0$ M1
- ii.** $z^3 + (6 - 9i)z^2 - (15 + 36i)z - (46 + 73i) = 0$
 $(z - c)^3 = 64i$
 $(z - c)^3 = 64\text{cis}\left(\frac{\pi}{2} + 2k\pi\right)$ M1
 $(z - c) = 64^{\frac{1}{3}}\text{cis}\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)$
 $z - (-2 + 3i) = 4\text{cis}\left(\frac{\pi}{6}\right) = 2\sqrt{3} + 2i$ when $k = 0$ A1
 $z - (-2 + 3i) = 4\text{cis}\left(\frac{5\pi}{6}\right) = -2\sqrt{3} + 2i$ when $k = 1$ A1
 $z - (-2 + 3i) = 4\text{cis}\left(-\frac{\pi}{2}\right) = -4i$ when $k = -1$ A1
so $z_R = -2 + 2\sqrt{3} + 5i$ $z_Q = -2 - 2\sqrt{3} + 5i$ $z_P = -2 - i$
These are the points P , Q and R . These points lie on the circle S of radius 4 and are equally spaced around the circle.
The points P , Q and R form an equilateral triangle. A1