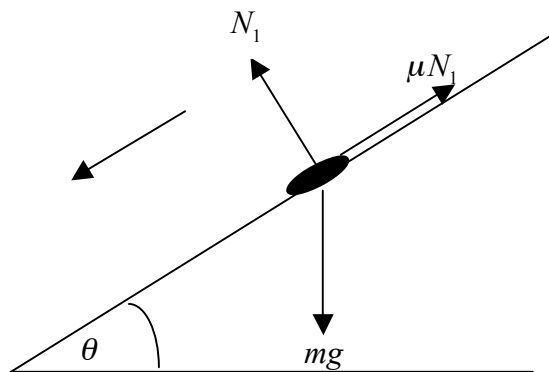


SPECIALIST MATHEMATICS EXAM 1

SOLUTIONS

Question 1

a. i.

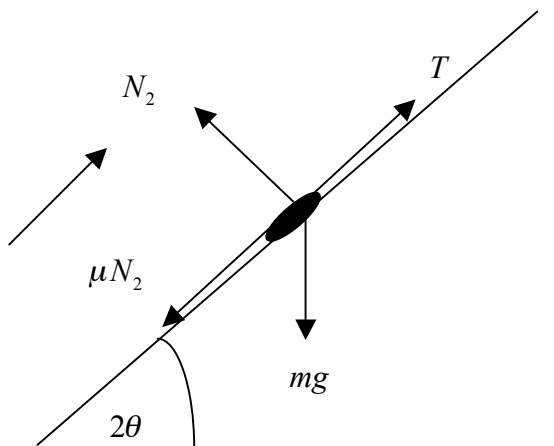


A1

- ii. Resolving down and parallel to the plane $mg \sin(\theta) - \mu N_1 = 0$ (1)
 Resolving perpendicular to the plane $N_1 - mg \cos(\theta) = 0$ (2)
 from (2) $N_1 = mg \cos(\theta)$ into (1) $mg \sin(\theta) = \mu mg \cos(\theta)$
 so that $\mu = \tan(\theta)$ shown

A1

b. i.



A1

- ii. Resolving up and parallel to the plane $T - mg \sin(2\theta) - \mu N_2 = 0$ (3)
 Resolving perpendicular to the plane $N_2 - mg \cos(2\theta) = 0$ (4)
 from (4) $N_2 = mg \cos(2\theta)$ into (3) $T = \mu mg \cos(2\theta) + mg \sin(2\theta)$
 but $\mu = \tan(\theta)$

$$T = mg (\sin(2\theta) + \tan(\theta) \cos(2\theta)) \quad \text{M1}$$

$$T = mg \left(\sin(2\theta) + \frac{\sin(\theta)}{\cos(\theta)} \cos(2\theta) \right) \quad \text{A1}$$

$$T = mg \left(\frac{\sin(2\theta) \cos(\theta) + \sin(\theta) \cos(2\theta)}{\cos(\theta)} \right) \quad \text{by addition theorems}$$

$$T = \frac{mg \sin(3\theta)}{\cos(\theta)} \quad \text{shown} \quad \text{A1}$$

Question 2

$y = \cos(x^2)$ using the chain rule

$$\frac{dy}{dx} = -2x \sin(x^2) \quad \text{now differentiating using the product rule} \quad \text{A1}$$

$$\frac{d^2y}{dx^2} = -2 \sin(x^2) - 4x^2 \cos(x^2) \quad \text{A1}$$

substituting into $x \frac{d^2y}{dx^2} + a \frac{dy}{dx} + b x^3 y = 0$

$$x(-2 \sin(x^2) - 4x^2 \cos(x^2)) - 2a x \sin(x^2) + b x^3 \cos(x^2) = 0 \quad \text{M1}$$

$$(b-4)x^3 \cos(x^2) - 2x(a+1) \sin(x^2) = 0$$

$$\text{so that } b = 4 \quad a = -1 \quad \text{A1}$$

Question 3

$2x^2 + 12x + y^2 - 8y + 22 = 0$ using implicit differentiation

$$\frac{d}{dx}(2x^2) + \frac{d}{dx}(12x) + \frac{d}{dx}(y^2) - \frac{d}{dx}(8y) + \frac{d}{dx}(22) = 0$$

$$4x + 12 + 2y \frac{dy}{dx} - 8 \frac{dy}{dx} = 0 \quad \text{M1}$$

$$4x + 12 = (8 - 2y) \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{4x + 12}{8 - 2y} = \frac{2(x + 3)}{4 - y} \quad \text{A1}$$

$$\text{when } \frac{dy}{dx} = 0 \quad x = -3 \quad \text{A1}$$

Question 4

a. $P(z) = z^4 + pz^2 - 8$

$$P(2i) = (2i)^4 + p(2i)^2 - 8 = 0$$

$$P(2i) = 16 - 4p - 8 = 0$$

$$4p = 8$$

$$\text{so that } p = 2 \quad \text{A1}$$

b. $P(z) = z^4 + 2z^2 - 8$

$$P(z) = (z^2 + 4)(z^2 - 2) = 0 \quad \text{M1}$$

$$z = \pm 2i \quad \pm \sqrt{2} \quad \text{A1}$$

Question 5

a. $y = \tan^{-1}\left(\frac{3x^2}{4}\right) = \tan^{-1}\left(\frac{u}{4}\right)$ where $u = 3x^2$

$$\frac{dy}{du} = \frac{4}{16+u^2} \qquad \frac{du}{dx} = 6x \qquad \text{M1}$$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{24x}{16+u^2}$$

$$\frac{dy}{dx} = \frac{24x}{9x^4+16} \qquad \text{A1}$$

b. $\int \frac{24x}{9x^4+16} dx = \tan^{-1}\left(\frac{3x^2}{4}\right)$

$$\int_0^{\frac{2\sqrt{3}}{3}} \frac{x}{9x^4+16} dx$$

$$= \left[\frac{1}{24} \tan^{-1}\left(\frac{3x^2}{4}\right) \right]_0^{\frac{2\sqrt{3}}{3}} \qquad \text{M1}$$

$$= \frac{1}{24} \left(\tan^{-1}\left(\frac{3}{4} \times \frac{12}{9}\right) - \tan^{-1}(0) \right)$$

$$= \frac{1}{24} (\tan^{-1}(1) - \tan^{-1}(0))$$

$$= \frac{\pi}{96} \qquad \text{A1}$$

Question 6

a. $y = \frac{x}{\sqrt{2x-3}} = \frac{u}{v}$ quotient rule

$$u = x \qquad v = \sqrt{2x-3} = (2x-3)^{\frac{1}{2}}$$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \frac{1}{2} \times 2(2x-3)^{-\frac{1}{2}} = \frac{1}{\sqrt{2x-3}} \qquad \text{M1}$$

$$\frac{dy}{dx} = \frac{\sqrt{2x-3} - \frac{x}{\sqrt{2x-3}}}{2x-3} \qquad \text{M1}$$

$$\frac{dy}{dx} = \frac{\left[\frac{(2x-3) - (x)}{\sqrt{2x-3}} \right]}{2x-3}$$

$$\frac{dy}{dx} = \frac{\left[\frac{x-3}{\sqrt{2x-3}} \right]}{2x-3} = \frac{x-3}{\sqrt{(2x-3)^3}}$$

Therefore $a = 1$ $b = -3$ A1

b. $A = \int_2^6 \frac{x}{\sqrt{2x-3}} dx$

let $u = 2x - 3 \quad \frac{du}{dx} = 2 \quad dx = \frac{du}{2} \quad 2x = u + 3 \quad x = \frac{1}{2}(u + 3)$

terminals when $x = 6 \quad u = 9$ and when $x = 2 \quad u = 1$

M1

$$A = \frac{1}{4} \int_1^9 \frac{u+3}{\sqrt{u}} du$$

$$A = \frac{1}{4} \int_1^9 \left(u^{\frac{1}{2}} + 3u^{-\frac{1}{2}} \right) du$$

$$A = \frac{1}{4} \left[\frac{2}{3} u^{\frac{3}{2}} + 6u^{\frac{1}{2}} \right]_1^9$$

A1

$$A = \frac{1}{4} \left(\left(\frac{2}{3} \times 9^{\frac{3}{2}} + 6\sqrt{9} \right) - \left(\frac{2}{3} \times 1^{\frac{3}{2}} + 6\sqrt{1} \right) \right)$$

$$A = \frac{1}{4} \left(18 + 18 - \frac{2}{3} - 6 \right)$$

$$A = 7\frac{1}{3} \text{ square units}$$

A1

Question 7

a. $\vec{r}(t) = (3 + 4\cos(2t))\vec{i} + (-2 + 3\sin(2t))\vec{j} \text{ for } t \geq 0$

$$x = 3 + 4\cos(2t) \quad \cos(2t) = \frac{x-3}{4}$$

M1

$$y = -2 + 3\sin(2t) \quad \sin(2t) = \frac{y+2}{3}$$

$$\sin^2(2t) + \cos^2(2t) = 1$$

$$\frac{(x-3)^2}{16} + \frac{(y+2)^2}{9} = 1$$

A1

b. $\dot{\vec{r}}(t) = -8\sin(2t)\vec{i} + 6\cos(2t)\vec{j}$

M1

$$|\dot{\vec{r}}(t)| = \sqrt{64\sin^2(2t) + 36\cos^2(2t)}$$

$$|\dot{\vec{r}}(t)| = \sqrt{64\sin^2(2t) + 36(1 - \sin^2(2t))}$$

$$|\dot{\vec{r}}(t)| = \sqrt{36 + 28\sin^2(2t)}$$

when $\sin(2t) = 1 \quad |\dot{\vec{r}}(t)|_{\max} = \sqrt{64} = 8$

A1

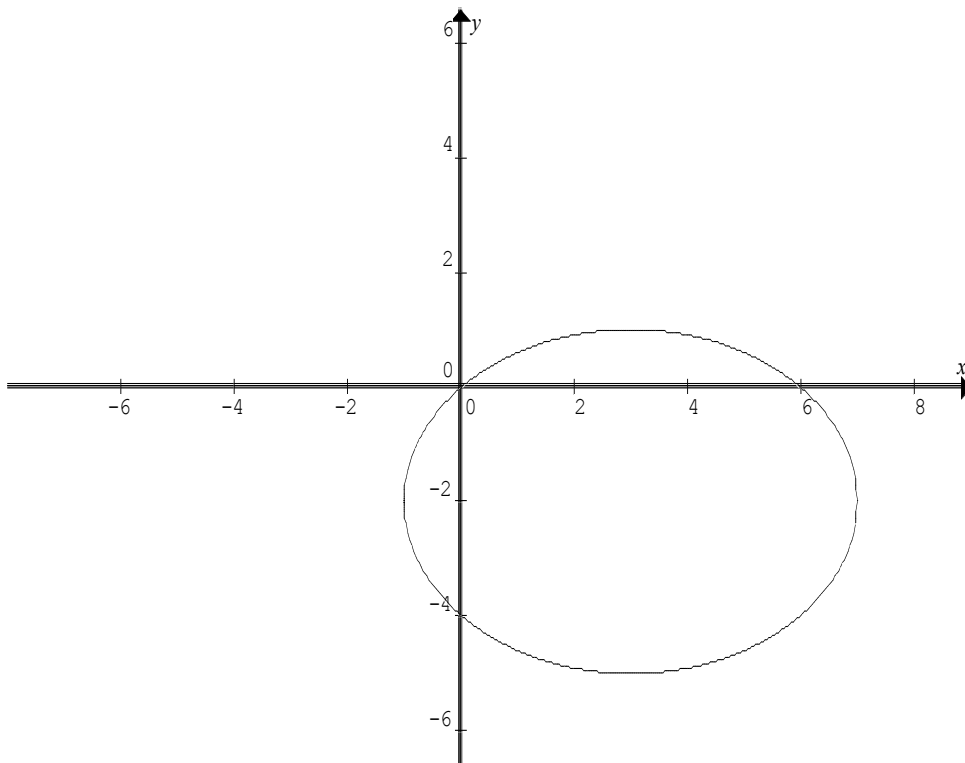
when $\sin(2t) = 0 \quad |\dot{\vec{r}}(t)|_{\min} = \sqrt{36} = 6$

A1

Question 8

a. The domain is $[-1, 7] = [c - 4, c + 4]$ so that $c = 3$ A1

b. $\frac{(x-3)^2}{16} + \frac{(y+2)^2}{9} = 1$
 ellipse centre $(3, -2)$ semi-major axes 4, semi-minor 3 A1



Correct graph A1

Question 9

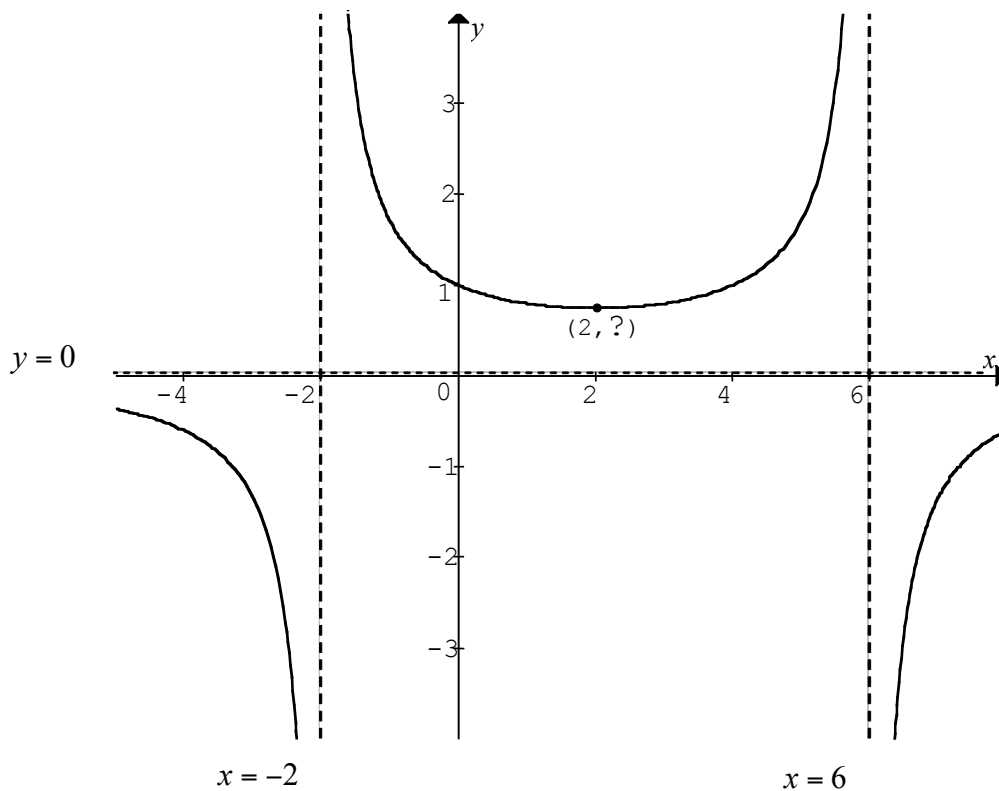
a. $y = \frac{12}{12 + 4x - x^2} = \frac{12}{(x+2)(6-x)}$

vertical asymptotes at $x = -2$ and $x = 6$ A1

horizontal asymptote at $y = 0$

turning point of $12 + 4x - x^2$ occurs when $4 - 2x = 0$ $x = 2$ $f(2) = \frac{12}{16} = \frac{3}{4}$

turning point at $\left(2, \frac{3}{4}\right)$ A1



Correct graph A1

b. the area $A = \int_0^3 \frac{12}{12+4x-x^2} dx = \int_0^3 \frac{12}{(x+2)(6-x)} dx$

by partial fractions $\frac{12}{12+4x-x^2} = \frac{A}{x+2} + \frac{B}{6-x} = \frac{12}{(x+2)(6-x)}$

$$\frac{A(6-x) + B(x+2)}{(x+2)(6-x)} = \frac{x(B-A) + 6A + 2B}{12+4x-x^2}$$

(1) $B - A = 0$ and (2) $6A + 2B = 12$

from (1) $A = B$ into (2)

$8B = 12$ so that $A = B = \frac{3}{2}$ Note that alternative methods are possible

M1

$$A = \frac{3}{2} \int_0^3 \left(\frac{1}{x+2} + \frac{1}{6-x} \right) dx$$

$$A = \frac{3}{2} \left[\log_e |x+2| - \log_e |6-x| \right]_0^3$$

$$A = \frac{3}{2} \left[\log_e \left| \frac{x+2}{6-x} \right| \right]_0^3$$

M1

$$A = \frac{3}{2} \left(\log_e \left(\frac{5}{3} \right) - \log_e \left(\frac{1}{3} \right) \right)$$

$$A = \frac{3}{2} \log_e (5) = \log_e (\sqrt{5^3}) = \log_e (\sqrt{125}) = \log_e (\sqrt{p})$$

so $p = 125$

A1