Year 2006

VCE

Specialist Mathematics Solutions Trial Examination 1

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Question 1

a. $\underset{\sim}{b} = -2\underset{\sim}{i} + y\,\underset{\sim}{j} + 4\underset{\sim}{k}$

$|\underset{\sim}{b}| = \sqrt{(-2)^2 + y^2 + 4^2} = \sqrt{20 + y^2} = 5$ squaring both sides M1

$20 + y^2 = 25$

$y^2 = 5$

$y = \pm\sqrt{5}$ A1

b. $\cos\left(150^0\right) = -\dfrac{\sqrt{3}}{2}$ $\cos(\theta) = \dfrac{y}{|\underset{\sim}{b}|}$

$-\dfrac{\sqrt{3}}{2} = \dfrac{y}{\sqrt{20 + y^2}}$ $y < 0$ M1

$-\sqrt{3}\sqrt{20 + y^2} = 2y$ squaring both sides

$3\left(20 + y^2\right) = 4y^2$

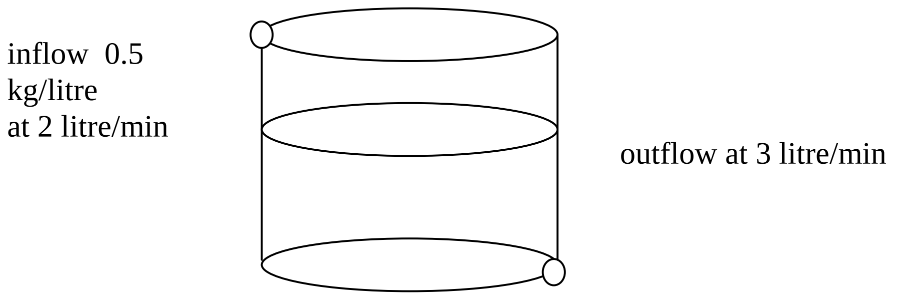
$60 + 3y^2 = 4y^2$

$y^2 = 60 = 4\text{x}15$

$y = -2\sqrt{15}$ since $y < 0$ A1

Question 2

a.



Now $\dfrac{dQ}{dt} = \text{inflow} - \text{outflow}$

and the volume $V(t)$ of the tank at a time *t*. $V(t) = 200 + (2-3)t = 200 - t$

$\dfrac{dQ}{dt} = 2\text{x}0.5 - \dfrac{3Q}{200-t} = 1 - \dfrac{3Q}{200-t}$ A1

b. $Q = \frac{1}{2}(200-t) + C(200-t)^n$

differentiating LHS $\frac{dQ}{dt} = -\frac{1}{2} - nC(200-t)^{n-1}$ M1

RHS $1 - \frac{3Q}{200-t} = 1 - \frac{3}{200-t}\left(\frac{1}{2}(200-t) + C(200-t)^n\right)$

$= -\frac{1}{2} - 3C(200-t)^{n-1}$ therefore $n = 3$ A1

Question 3

a. Let $y = \cos^{-1}\left(\sqrt{\frac{3}{x}}\right) = \cos^{-1}(u)$ where $u = \sqrt{\frac{3}{x}} = \sqrt{3}\,x^{-\frac{1}{2}}$

$\frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}}$ $\frac{du}{dx} = -\frac{\sqrt{3}}{2}x^{-\frac{3}{2}} = \frac{-\sqrt{3}}{2\sqrt{x^3}}$ M1

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{\sqrt{3}}{2\sqrt{x^3}\sqrt{1-\frac{3}{x}}} = \frac{\sqrt{3}}{2\sqrt{x^3}\sqrt{\frac{x-3}{x}}}$

$\frac{dy}{dx} = \frac{\sqrt{3}}{2x\sqrt{x-3}}$ for $x > 3$

So shown $\frac{d}{dx}\left(\cos^{-1}\left(\sqrt{\frac{3}{x}}\right)\right) = \frac{\sqrt{3}}{2x\sqrt{x-3}}$ for $x > 3$ A1

b. $\int_4^{12} \frac{1}{x\sqrt{x-3}}\,dx$

$= \frac{2}{\sqrt{3}}\left[\cos^{-1}\left(\sqrt{\frac{3}{x}}\right)\right]_4^{12}$ M1

$= \frac{2}{\sqrt{3}}\left(\cos^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$

$= \frac{2\sqrt{3}}{3}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$

$= \frac{\sqrt{3}\pi}{9}$ A1

Question 4

a. $y = \dfrac{x}{\sqrt{x-3}}$ using the quotient rule

$u = x \qquad v = \sqrt{x-3}$

$\dfrac{du}{dx} = 1 \qquad \dfrac{dv}{dx} = \dfrac{1}{2}(x-3)^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x-3}}$ M1

$$\frac{dy}{dx} = \frac{\sqrt{x-3} - \dfrac{x}{2\sqrt{x-3}}}{x-3} = \frac{\dfrac{2(x-3)-x}{2\sqrt{x-3}}}{x-3}$$

$\dfrac{dy}{dx} = \dfrac{x-6}{2\sqrt{(x-3)^3}}$ therefore $a = \dfrac{1}{2} \qquad b = -3$ A1

b. the area $A = \displaystyle\int_3^4 \frac{x\,dx}{\sqrt{x-3}}$ let $u = x-3 \quad \dfrac{du}{dx} = 1 \quad x = u+3$

terminals when $x = 4 \quad u = 1$ and when $x = 3 \quad u = 0$

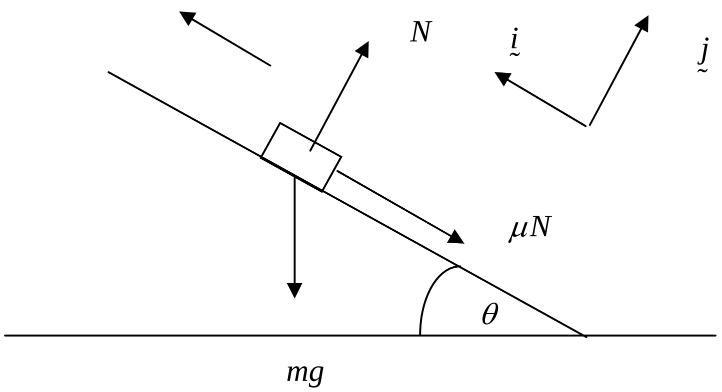
$A = \displaystyle\int_0^1 \frac{u+3}{\sqrt{u}}\,du = \int_0^1 \left(u^{\frac{1}{2}} + 3u^{-\frac{1}{2}}\right) du$ M1

$A = \left[\dfrac{2}{3}u^{\frac{3}{2}} + 6u^{\frac{1}{2}}\right]_0^1 = \left(\dfrac{2}{3}(1) + 6\sqrt{1} - 0\right)$

$A = 6\frac{2}{3}$ A1

Question 5

a.



A1

b. resolving up parallel to the slide in the $\underset{\sim}{i}$ direction.

(1) $ma = -mg\sin(\theta) - \mu N$

resolving perpendicular to the slide in the $\underset{\sim}{j}$ direction.

(2) $N - mg\cos(\theta) = 0$ (2) $\Rightarrow N = mg\cos(\theta)$ into (1) M1

$ma = -mg\sin(\theta) - \mu mg\cos(\theta) = -mg(\sin(\theta) + \mu\cos(\theta))$

$a = -g(\sin(\theta) + \mu\cos(\theta))$ A1

Using constant acceleration formulae $v^2 = u^2 + 2as$ with $u = U$ $v = 0$ $s = D = ?$

$0 = U^2 - 2g(\sin(\theta) + \mu\cos(\theta))D$

$D = \dfrac{U^2}{2g(\sin(\theta) + \mu\cos(\theta))}$ A1

Question 6

$$V = \pi\int_a^b \left(y_1^2 - y_2^2\right)dx$$

$$V = \pi\int_0^{\frac{\pi}{6}} \left(\cos^2(2x) - \sin^2(x)\right)dx \qquad \text{M1}$$

$$V = \pi\int_0^{\frac{\pi}{6}} \left(\left(\frac{1}{2} + \frac{1}{2}\cos(4x)\right) - \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)\right)dx \qquad \text{M1}$$

$$V = \frac{\pi}{2}\int_0^{\frac{\pi}{6}} \left(\cos(4x) + \cos(2x)\right)dx \qquad \text{A1}$$

$$V = \frac{\pi}{2}\left[\frac{1}{4}\sin(4x) + \frac{1}{2}\sin(2x)\right]_0^{\frac{\pi}{6}} = \frac{\pi}{2}\left[\left(\frac{1}{4}\sin\left(\frac{2\pi}{3}\right) + \frac{1}{2}\sin\left(\frac{\pi}{3}\right)\right) - 0\right]$$

$$V = \frac{\pi}{2}\left(\frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{4}\right)$$

$$V = \frac{3\pi\sqrt{3}}{16} \qquad \text{A1}$$

Question 7

a. $z^4 + z^2 - 12 = 0$

$\left(z^2 - 3\right)\left(z^2 + 4\right) = 0$ M1

$\left(z^2 - \left(\sqrt{3}\right)^2\right)\left(z^2 - 4i^2\right) = 0$

$\left(z + \sqrt{3}\right)\left(z - \sqrt{3}\right)\left(z - 2i\right)\left(z + 2i\right) = 0$

$z = \pm\sqrt{3} \quad z = \pm 2i$ A1

b. $z^2 = \left(a + bi\right)^2 = -1 - 4\sqrt{3}\,i$ where $a, b \in R$

$z^2 = \left(a^2 + 2ab\,i + b^2 i^2\right) = \left(a^2 - b^2\right) + 2ab\,i = -1 - 4\sqrt{3}\,i$

equating real and imaginary parts

real (1) $a^2 - b^2 = -1$ M1

imaginary (2) $2ab = -4\sqrt{3}$ from (2) $b = -\dfrac{2\sqrt{3}}{a}$ substitute into (1)

$a^2 - \left(\dfrac{-2\sqrt{3}}{a}\right)^2 = -1$ M1

$a^2 - \dfrac{12}{a^2} + 1 = 0$ multiply by a^2

$a^4 + a^2 - 12 = 0$ from **i.** since a is real

$a = \pm\sqrt{3}$ and $b = \mp 2$ so $\left(\pm\left(\sqrt{3} - 2i\right)\right)^2 = -1 - 4\sqrt{3}\,i$

therefore if $z^2 = -1 - 4\sqrt{3}\,i$ then $z = \pm\left(\sqrt{3} - 2i\right)$ A1

c. $z^2 + \sqrt{3}\,z + \left(1 + \sqrt{3}\,i\right) = 0$

using the quadratic formulae with $a = 1 \quad b = \sqrt{3} \quad c = 1 + \sqrt{3}\,i$

$\Delta = b^2 - 4ac = 3 - 4\left(1 + \sqrt{3}\,i\right) = -1 - 4\sqrt{3}\,i$ A1

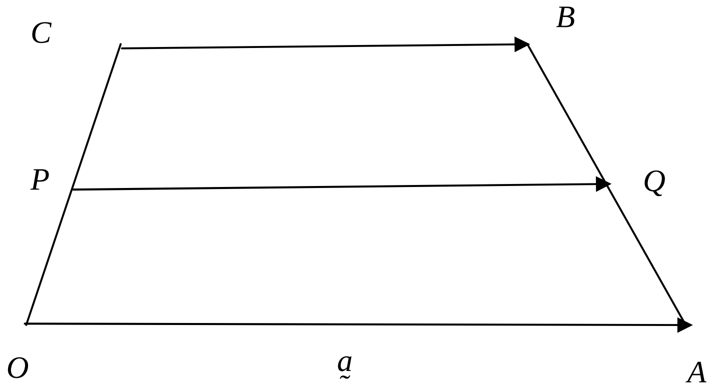
so $\sqrt{\Delta} = \sqrt{\left(-1 - 4\sqrt{3}\,i\right)} = \sqrt{3} - 2i$

$$z = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$$z = \frac{-\sqrt{3} \pm \left(\sqrt{3} - 2i\right)}{2}$$ M1

$z = -i$ and $-\sqrt{3} + i$ A1

Question 8



a. $\overrightarrow{OA} = \underset{\sim}{a} = \lambda\overrightarrow{CB}$ $\quad \overrightarrow{CB} = \frac{1}{\lambda}\underset{\sim}{a}$

since P is the midpoint of $\overrightarrow{OC}$ $\quad \overrightarrow{OP} = \overrightarrow{PC} = \frac{1}{2}\overrightarrow{OC}$

since Q is the midpoint of $\overrightarrow{AB}$ $\quad \overrightarrow{AQ} = \overrightarrow{QB} = \frac{1}{2}\overrightarrow{AB}$

$\overrightarrow{PQ} = \overrightarrow{PC} + \overrightarrow{CB} + \overrightarrow{BQ} \quad (1)$

$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OA} + \overrightarrow{AQ} \quad (2)$ M1

adding these two equations gives

$2\overrightarrow{PQ} = \overrightarrow{PC} + \overrightarrow{PO} + \overrightarrow{CB} + \overrightarrow{OA} + \overrightarrow{BQ} + \overrightarrow{AQ}$ M1

$2\overrightarrow{PQ} = \overrightarrow{CB} + \overrightarrow{OA}$

$2\overrightarrow{PQ} = \left(\frac{1}{\lambda} + 1\right)\underset{\sim}{a}$

$\overrightarrow{PQ} = \left(\frac{\lambda + 1}{2\lambda}\right)\underset{\sim}{a}$ A1

b. the length of $\overrightarrow{PQ} = \left|\overrightarrow{PQ}\right|$ is equal to $\frac{\lambda+1}{2\lambda}$ times the length of $\underset{\sim}{a}$

so the ratio of the length is $\frac{\lambda+1}{2\lambda}$ A1

Question 9

a. vertical asymptotes at $x=-4$ and $x=-2$ so the denominator is

$(x+4)(x+2)=x^2+6x+8$ therefore $b=6 \quad c=8$ A1

the turning point is $(-3,-2)$

when $x=-3 \quad y=-2$ so $-2=\dfrac{a}{(-3)^2-18+8}=\dfrac{a}{-1}$

therefore $a=2$ A1

b. the area $=\displaystyle\int_0^2 \frac{2}{x^2+6x+8}\,dx=\int_0^2 \frac{2}{(x+4)(x+2)}\,dx$

by partial fractions $\dfrac{2}{x^2+6x+8}=\dfrac{2}{(x+4)(x+2)}=\dfrac{A}{x+4}+\dfrac{B}{x+2}$

$\dfrac{A}{x+4}+\dfrac{B}{x+2}$ Now adding the partial fractions

$$=\frac{A(x+2)+B(x+4)}{(x+4)(x+2)}=\frac{x(A+B)+2A+4B}{x^2+6x+8}$$

$(1)\quad A+B=0$ and $(2)\quad 2A+4B=2$ from $(1)\ A=-B$ into (2)

$2B=2 \quad B=1$ and $A=-1$ A1

area $=\displaystyle\int_0^2 \left(\frac{1}{x+2}-\frac{1}{x+4}\right)dx$

$=\left[\log_e\left(\dfrac{x+2}{x+4}\right)\right]_0^2=\left(\log_e\left(\dfrac{4}{6}\right)-\log_e\left(\dfrac{2}{4}\right)\right)$ A1

$=\log_e\left(\dfrac{4}{3}\right)$ therefore $p=\dfrac{4}{3}$ A1

END OF SUGGESTED SOLUTIONS