

Year 2006

VCE

Specialist Mathematics

Solutions

Trial Examination 1



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Question 1

a. $\underline{b} = -2\underline{i} + y\underline{j} + 4\underline{k}$

$$|\underline{b}| = \sqrt{(-2)^2 + y^2 + 4^2} = \sqrt{20 + y^2} = 5 \quad \text{squaring both sides} \quad \text{M1}$$

$$20 + y^2 = 25$$

$$y^2 = 5$$

$$y = \pm\sqrt{5} \quad \text{A1}$$

b. $\cos(150^\circ) = -\frac{\sqrt{3}}{2} \quad \cos(\theta) = \frac{y}{|\underline{b}|}$

$$-\frac{\sqrt{3}}{2} = \frac{y}{\sqrt{20 + y^2}} \quad y < 0 \quad \text{M1}$$

$$-\sqrt{3}\sqrt{20 + y^2} = 2y \quad \text{squaring both sides}$$

$$3(20 + y^2) = 4y^2$$

$$60 + 3y^2 = 4y^2$$

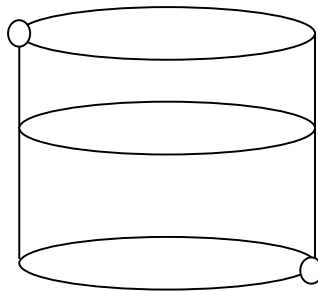
$$y^2 = 60 = 4 \times 15$$

$$y = -2\sqrt{15} \quad \text{since } y < 0 \quad \text{A1}$$

Question 2

a.

inflow 0.5
kg/litre
at 2 litre/min



outflow at 3 litre/min

Now $\frac{dQ}{dt} = \text{inflow} - \text{outflow}$

and the volume $V(t)$ of the tank at a time t . $V(t) = 200 + (2 - 3)t = 200 - t$

$$\frac{dQ}{dt} = 2 \times 0.5 - \frac{3Q}{200 - t} = 1 - \frac{3Q}{200 - t} \quad \text{A1}$$

b. $Q = \frac{1}{2}(200-t) + C(200-t)^n$

differentiating LHS $\frac{dQ}{dt} = -\frac{1}{2} - nC(200-t)^{n-1}$ M1

RHS $1 - \frac{3Q}{200-t} = 1 - \frac{3}{200-t} \left(\frac{1}{2}(200-t) + C(200-t)^n \right)$
 $= -\frac{1}{2} - 3C(200-t)^{n-1}$ therefore $n = 3$ A1

Question 3

a. Let $y = \cos^{-1}\left(\sqrt{\frac{3}{x}}\right) = \cos^{-1}(u)$ where $u = \sqrt{\frac{3}{x}} = \sqrt{3}x^{-\frac{1}{2}}$

$\frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}}$ $\frac{du}{dx} = -\frac{\sqrt{3}}{2}x^{-\frac{3}{2}} = \frac{-\sqrt{3}}{2\sqrt{x^3}}$ M1

$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{\sqrt{3}}{2\sqrt{x^3} \sqrt{1-\frac{3}{x}}} = \frac{\sqrt{3}}{2\sqrt{x^3} \sqrt{\frac{x-3}{x}}}$

$\frac{dy}{dx} = \frac{\sqrt{3}}{2x\sqrt{x-3}}$ for $x > 3$

So shown $\frac{d}{dx}\left(\cos^{-1}\left(\sqrt{\frac{3}{x}}\right)\right) = \frac{\sqrt{3}}{2x\sqrt{x-3}}$ for $x > 3$ A1

b. $\int_4^{12} \frac{1}{x\sqrt{x-3}} dx$

$= \frac{2}{\sqrt{3}} \left[\cos^{-1}\left(\sqrt{\frac{3}{x}}\right) \right]_4^{12}$ M1

$= \frac{2}{\sqrt{3}} \left(\cos^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \right)$

$= \frac{2\sqrt{3}}{3} \left(\frac{\pi}{3} - \frac{\pi}{6} \right)$

$= \frac{\sqrt{3}\pi}{9}$ A1

Question 4

a. $y = \frac{x}{\sqrt{x-3}}$ using the quotient rule

$$u = x \quad v = \sqrt{x-3}$$

$$\frac{du}{dx} = 1 \quad \frac{dv}{dx} = \frac{1}{2}(x-3)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x-3}} \quad \text{M1}$$

$$\frac{dy}{dx} = \frac{\sqrt{x-3} - \frac{x}{2\sqrt{x-3}}}{x-3} = \frac{2(x-3) - x}{2\sqrt{x-3}(x-3)}$$

$$\frac{dy}{dx} = \frac{x-6}{2\sqrt{(x-3)^3}} \quad \text{therefore} \quad a = \frac{1}{2} \quad b = -3 \quad \text{A1}$$

b. the area $A = \int_3^4 \frac{x dx}{\sqrt{x-3}}$ let $u = x-3$ $\frac{du}{dx} = 1$ $x = u+3$

terminals when $x = 4$ $u = 1$ and when $x = 3$ $u = 0$

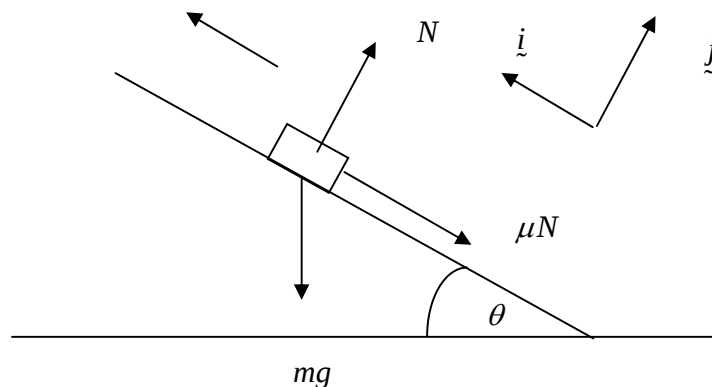
$$A = \int_0^1 \frac{u+3}{\sqrt{u}} du = \int_0^1 \left(u^{\frac{1}{2}} + 3u^{-\frac{1}{2}} \right) du \quad \text{M1}$$

$$A = \left[\frac{2}{3} u^{\frac{3}{2}} + 6u^{\frac{1}{2}} \right]_0^1 = \left(\frac{2}{3}(1) + 6\sqrt{1} - 0 \right)$$

$$A = 6\frac{2}{3} \quad \text{A1}$$

Question 5

a.



A1

b. resolving up parallel to the slide in the $\hat{i}$ direction.

$$(1) ma = -mg \sin(\theta) - \mu N$$

resolving perpendicular to the slide in the $\hat{j}$ direction.

$$(2) N - mg \cos(\theta) = 0 \quad (2) \Rightarrow N = mg \cos(\theta) \text{ into (1)} \quad \text{M1}$$

$$ma = -mg \sin(\theta) - \mu mg \cos(\theta) = -mg (\sin(\theta) + \mu \cos(\theta))$$

$$a = -g (\sin(\theta) + \mu \cos(\theta)) \quad \text{A1}$$

Using constant acceleration formulae $v^2 = u^2 + 2as$ with $u = U$ $v = 0$ $s = D = ?$

$$0 = U^2 - 2g (\sin(\theta) + \mu \cos(\theta)) D$$

$$D = \frac{U^2}{2g (\sin(\theta) + \mu \cos(\theta))} \quad \text{A1}$$

Question 6

$$V = \pi \int_a^b (y_1^2 - y_2^2) dx$$

$$V = \pi \int_0^{\frac{\pi}{6}} (\cos^2(2x) - \sin^2(x)) dx \quad \text{M1}$$

$$V = \pi \int_0^{\frac{\pi}{6}} \left(\left(\frac{1}{2} + \frac{1}{2} \cos(4x) \right) - \left(\frac{1}{2} - \frac{1}{2} \cos(2x) \right) \right) dx \quad \text{M1}$$

$$V = \frac{\pi}{2} \int_0^{\frac{\pi}{6}} (\cos(4x) + \cos(2x)) dx \quad \text{A1}$$

$$V = \frac{\pi}{2} \left[\frac{1}{4} \sin(4x) + \frac{1}{2} \sin(2x) \right]_0^{\frac{\pi}{6}} = \frac{\pi}{2} \left[\left(\frac{1}{4} \sin\left(\frac{2\pi}{3}\right) + \frac{1}{2} \sin\left(\frac{\pi}{3}\right) \right) - 0 \right]$$

$$V = \frac{\pi}{2} \left(\frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{4} \right)$$

$$V = \frac{3\pi\sqrt{3}}{16} \quad \text{A1}$$

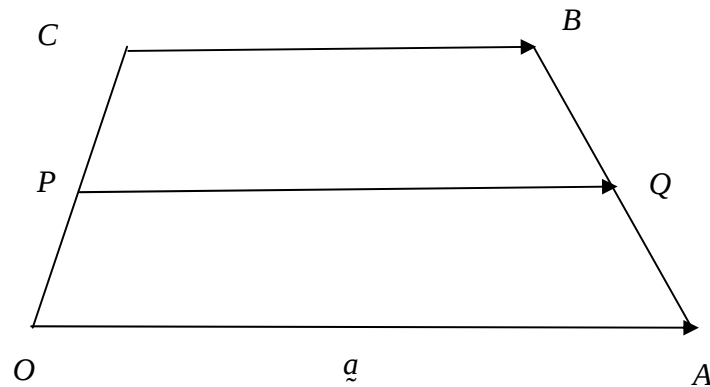
Question 7

a. $z^4 + z^2 - 12 = 0$
 $(z^2 - 3)(z^2 + 4) = 0$ M1
 $(z^2 - (\sqrt{3})^2)(z^2 - 4i^2) = 0$
 $(z + \sqrt{3})(z - \sqrt{3})(z - 2i)(z + 2i) = 0$
 $z = \pm\sqrt{3} \quad z = \pm 2i$ A1

b. $z^2 = (a + bi)^2 = -1 - 4\sqrt{3}i$ where $a, b \in R$
 $z^2 = (a^2 + 2abi + b^2i^2) = (a^2 - b^2) + 2abi = -1 - 4\sqrt{3}i$
equating real and imaginary parts
real (1) $a^2 - b^2 = -1$ M1
imaginary (2) $2ab = -4\sqrt{3}$ from (2) $b = -\frac{2\sqrt{3}}{a}$ substitute into (1)
 $a^2 - \left(\frac{-2\sqrt{3}}{a}\right)^2 = -1$ M1
 $a^2 - \frac{12}{a^2} + 1 = 0$ multiply by a^2
 $a^4 + a^2 - 12 = 0$ from **i.** since a is real
 $a = \pm\sqrt{3}$ and $b = \mp 2$ so $(\pm(\sqrt{3} - 2i))^2 = -1 - 4\sqrt{3}i$
therefore if $z^2 = -1 - 4\sqrt{3}i$ then $z = \pm(\sqrt{3} - 2i)$ A1

c. $z^2 + \sqrt{3}z + (1 + \sqrt{3}i) = 0$
using the quadratic formulae with $a = 1$ $b = \sqrt{3}$ $c = 1 + \sqrt{3}i$
 $\Delta = b^2 - 4ac = 3 - 4(1 + \sqrt{3}i) = -1 - 4\sqrt{3}i$ A1
so $\sqrt{\Delta} = \sqrt{(-1 - 4\sqrt{3}i)} = \sqrt{3} - 2i$
 $z = \frac{-b \pm \sqrt{\Delta}}{2a}$
 $z = \frac{-\sqrt{3} \pm (\sqrt{3} - 2i)}{2}$ M1
 $z = -i$ and $-\sqrt{3} + i$ A1

Question 8



a. $\overrightarrow{OA} = q = \lambda \overrightarrow{CB} \quad \overrightarrow{CB} = \frac{1}{\lambda} q$

since P is the midpoint of $\overrightarrow{OC}$ $\overrightarrow{OP} = \overrightarrow{PC} = \frac{1}{2} \overrightarrow{OC}$

since Q is the midpoint of $\overrightarrow{AB}$ $\overrightarrow{AQ} = \overrightarrow{QB} = \frac{1}{2} \overrightarrow{AB}$

$$\overrightarrow{PQ} = \overrightarrow{PC} + \overrightarrow{CB} + \overrightarrow{BQ} \quad (1)$$

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OA} + \overrightarrow{AQ} \quad (2)$$

M1

adding these two equations gives

$$2 \overrightarrow{PQ} = \overrightarrow{PC} + \overrightarrow{PO} + \overrightarrow{CB} + \overrightarrow{OA} + \overrightarrow{BQ} + \overrightarrow{AQ}$$

M1

$$2 \overrightarrow{PQ} = \overrightarrow{CB} + \overrightarrow{OA}$$

$$2 \overrightarrow{PQ} = \left(\frac{1}{\lambda} + 1 \right) q$$

$$\overrightarrow{PQ} = \left(\frac{\lambda + 1}{2\lambda} \right) q$$

A1

b. the length of $\overrightarrow{PQ} = |\overrightarrow{PQ}|$ is equal to $\frac{\lambda + 1}{2\lambda}$ times the length of q

so the ratio of the length is $\frac{\lambda + 1}{2\lambda}$

A1

Question 9

- a. vertical asymptotes at $x = -4$ and $x = -2$ so the denominator is
 $(x+4)(x+2) = x^2 + 6x + 8$ therefore $b = 6$ $c = 8$ A1
 the turning point is $(-3, -2)$

$$\text{when } x = -3 \quad y = -2 \quad \text{so} \quad -2 = \frac{a}{(-3)^2 - 18 + 8} = \frac{a}{-1}$$

$$\text{therefore } a = 2 \quad \text{A1}$$

b. the area $= \int_0^2 \frac{2}{x^2 + 6x + 8} dx = \int_0^2 \frac{2}{(x+4)(x+2)} dx$

$$\text{by partial fractions } \frac{2}{x^2 + 6x + 8} = \frac{2}{(x+4)(x+2)} = \frac{A}{x+4} + \frac{B}{x+2}$$

$$\frac{A}{x+4} + \frac{B}{x+2} \quad \text{Now adding the partial fractions}$$

$$= \frac{A(x+2) + B(x+4)}{(x+4)(x+2)} = \frac{x(A+B) + 2A + 4B}{x^2 + 6x + 8}$$

$$(1) \quad A+B=0 \quad \text{and} \quad (2) \quad 2A+4B=2 \quad \text{from } (1) \quad A=-B \quad \text{into } (2)$$

$$2B=2 \quad B=1 \quad \text{and} \quad A=-1 \quad \text{A1}$$

$$\text{area} = \int_0^2 \left(\frac{1}{x+2} - \frac{1}{x+4} \right) dx$$

$$= \left[\log_e \left(\frac{x+2}{x+4} \right) \right]_0^2 = \left(\log_e \left(\frac{4}{6} \right) - \log_e \left(\frac{2}{4} \right) \right) \quad \text{A1}$$

$$= \log_e \left(\frac{4}{3} \right) \quad \text{therefore } p = \frac{4}{3} \quad \text{A1}$$

END OF SUGGESTED SOLUTIONS